

# Effect of relaxation time on generalized double porosity thermoelastic medium with diffusion

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**Abstract.** This paper studies the effect of the relaxation time on a two-dimensional thermoelastic medium which has a doubly porous structure in the presence of diffusion and gravity. The normal mode analysis is used to obtain the analytic expressions of the physical quantities, which we take the solution form in the exponential image. We have discussed a homogeneous thermoelastic half-space with double porosity with the effect of diffusion and gravity. The equations of generalized thermoelastic material with double porosity structure with one relaxation time have been developed. Moreover, the expressions of many physical quantities are explained. The general solutions, under specific boundary conditions of the problem, were found in some detail. In addition, numerical results are computed.

**Keywords:** double porosity; homogeneous; L-S theory; normal mode method; thermoelastic diffusion

## 1. Introduction

Lord-Shulman (1967) discussed a thermoelasticity generalized theory, (L-S) theory, which includes unique relaxation time for a thermoelastic process. The basis of the model introduces a new physical concept, called relaxation time, to modify Fourier's law of the heat conduction equation. In addition; the theory suggests a wave type for the heat equations to ensure finite velocities of propagation of the elastic and heat waves. In this head, the double porosity functions are added to the heat equation to yield Iesan and Quintanilla (2010). Most authors have studied the behavior of the solution through two groups. The first is concerned with studying the behavior of the solution using a method called "normal mode". This method is based on the general form that the authors propose to visualize the solution. More, some users of this road assumed that time was imaginary, which led to these problems giving me precise physical meanings.

The second team linked the study of the behavior of the solution using the "Laplace Integral Method". This method is given a good solution to solve when it is easy to find the inverse of Laplace transform. The solution is to be studied in this way, if each the Laplace transform is difficult to generate. Anya and Khan (2019), Lata and Kaur (2019, 2020) have applied different theories to study the plane harmonic waves in the context of generalized thermoelasticity.

The study of the diffusion phenomenon is of major

degree of interest due to its many applications in industrial applications and geophysics. Diffusion can be defined as the random walk, of an ensemble of particles, from the regions of high concentration to regions of lower concentration. Diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate in integrated circuit fabrication. A study of the phenomenon of diffusion is used to improve the conditions of oil extractions. Oil companies these days, are interested in the process of thermoelastic diffusion for more efficient extraction of oil from oil deposits. In this theory, the coupled thermoelastic model is used. A new theory of thermo-diffusion in elastic solids using the methodology of fractional calculus was derived Ezzat and Fayik (2011). The effect of magnetic field and thermal relaxation on the 2-D problem of generalized thermoelastic diffusion was discussed by Othman *et al.* (2013).

In this paper, the work is focused on the linear theory of elastic materials with double porosity as Barenblatt *et al.* (1960), Barenblatt and Zheltov (1960), Khalili and Valliappan (1996), Masters *et al.* (2000), Khalili and Selvadurai (2003), Zhao and Chen Wang (2006), Mohamed, *et al.* (2008), Othman *et al.* (2009). Svanadze (2010), Ainouz (2011), Abo-Dahab and Abbas, (2011), Zenkour and Abbas (2013), Kumar *et al.* (2013), Straughan (2013), Abbas (2014a, b), Abbas *et al.* (2016), Marin, *et al.* (2021), Fahmy (2022a, b), Fahmy and Alsulami (2022), Fahmy *et al.* (2022), Fahmy and Almeahmadi (2022). Here, the authors devised analytical solutions for this type of problem by looking for a time function that corresponds to all changes in the presented problem. In addition, the authors discussed the equations of generalized thermoelastic diffusion material with double porosity structure with one relaxation time using a separation of variable methods.

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**2. Formulation of the problem and basic equations**

We consider a homogenous isotropic elastic solid with generalized thermoelastic diffusion with double porosity in a half-space using the L-S theory. We have studied this problem in a plane strain- $xy$  with the displacement components  $u, v$  such that  $u = u(x, y, t), v = v(x, y, t)$ .

**2.1 Stress-strain equation**

Follow Abdou *et al.* (2020a)

$$\sigma_{ij} = \lambda \delta_{ij} e_{rr} + 2\mu e_{ij} + b \delta_{ij} \Phi + d \delta_{ij} \Psi - \beta \delta_{ij} (T - T_0) - \gamma^* C \delta_{ij}; \quad (i, j = 1, 2). \tag{1}$$

**2.2 Equations of double porosity Abdou et al. (2018)**

$$\sigma_i = \alpha \Phi_i + b_1 \Psi_i \tag{2}$$

$$\tau_i = b_1 \Phi_i + \gamma \Psi_i. \tag{3}$$

**2.3 Equations of motion**

$$\begin{aligned} \mu \nabla^2 u + (\lambda + \mu) \frac{\partial e}{\partial x} + b \frac{\partial \Phi}{\partial x} + d \frac{\partial \Psi}{\partial x} - \beta \frac{\partial T}{\partial x} - \gamma^* \frac{\partial C}{\partial x} \\ + \rho g \frac{\partial v}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{4}$$

$$\begin{aligned} \mu \nabla^2 v + (\lambda + \mu) \frac{\partial e}{\partial y} + b \frac{\partial \Phi}{\partial y} + d \frac{\partial \Psi}{\partial y} - \beta \frac{\partial T}{\partial y} - \gamma^* \frac{\partial C}{\partial y} \\ - \rho g \frac{\partial u}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2}. \end{aligned} \tag{5}$$

**2.4 Equation of heat Abdou et al. (2020b)**

$$\begin{aligned} K \nabla^2 T = (1 + \tau_q \frac{\partial}{\partial t})(\rho C^* T_{,t} + a T_0 C_{,t} + \beta T_0 e_{,t} \\ + \gamma_1 T_0 \Phi_{,t} + \gamma_2 T_0 \Psi_{,t}). \end{aligned} \tag{6}$$

**2.5 Double porosity equations**

$$\alpha \nabla^2 \Phi + b_1 \nabla^2 \Psi - b e - \alpha_1 \Phi - \alpha_3 \Psi + \gamma_1 T + \mathcal{G} C = K_1 \Phi_{,tt}, \tag{7}$$

$$b_1 \nabla^2 \Phi + \gamma \nabla^2 \Psi - d e - \alpha_3 \Phi - \alpha_2 \Psi + \gamma_2 T + m C = K_2 \Psi_{,tt}. \tag{8}$$

**2.6 Diffusion equation**

$$\begin{aligned} \beta_1 \gamma^* \nabla^2 e_{kk} + a \beta_1 \nabla^2 T + C_{,t} + \tau^0 C_{,tt} - \beta_1 \beta_2 \nabla^2 C \\ + \beta_1 \mathcal{G} \nabla^2 \Phi + \beta_1 m \nabla^2 \Psi = 0. \end{aligned} \tag{9}$$

**2.7 Potential equation for diffusion**

$$p = -\gamma^* e_{kk} + \beta_2 C - a (T - T_0) - \mathcal{G} \Phi - m \Psi. \tag{10}$$

Where  $\sigma_{ij}$  are the stress components;  $\lambda, \mu$  are elastic constants,  $\gamma = (3\lambda + 2\mu)\alpha_t$   $\alpha_t$  is the thermal expansion coefficient,  $\delta_{ij}$  is Kronecker delta. In addition,  $T$  is the temperature above the reference temperature  $T_0$ ,  $K$  is the thermal conductivity,  $u_i$  the vector of displacement,  $\rho$  the mass density,  $c_e$  the specific heat at constant strain, Here,  $\Psi, \Phi$  the volume fraction fields. Here,  $k_1$  and  $k_2$  the coefficients of equilibrated inertia,  $\tau_q$  the relaxation time,  $b, d, b_1, \gamma, \gamma_1, \gamma_2, m, \mathcal{G}$  the constitutive coefficients of double porosity,  $p$  the chemical potential,  $C$  concentration distribution,  $a$  coefficient describing the measure of thermoelastic diffusion effect,  $\gamma^*$  coefficient of diffusion thermal expansion,  $\beta_1$  thermoelastic diffusion constant,  $\beta_2$  coefficient describing the measure of diffusive effect,  $\tau^0$  diffusion relaxation time,  $\sigma_i, \tau_i$  the stresses of equilibrium and  $\beta$  is the coefficient of linear thermal expansion

$\beta = (3\lambda + 2\mu)\alpha_t, e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ .  
For the aim of numerical evaluation, we illustrate dimensionless variables

$$\begin{aligned} (x', y') &= \frac{w_1}{c_1}(x, y), \quad (u', v') = \frac{w_1}{c_1}(u, v), \quad g' = \frac{g}{c_1 w_1}, \\ (\sigma'_1, \tau'_1) &= \frac{c_1}{\alpha w_1}(\sigma_1, \tau_1), \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad P' = \frac{P}{\rho \beta_2}, \quad C' = \frac{C}{\rho}, \\ w_1 &= \frac{\rho c_e c_1^2}{K}, \quad \nabla^2 = \frac{w_1^2}{c_1^2} \nabla'^2, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\beta T_0}, \quad T' = \frac{T}{T_0}, \\ (t', \tau'_q, \tau'^0) &= w_1(t, \tau_q, \tau^0), \quad (\Phi', \Psi') = \frac{K_1 w_1^2}{\alpha_1}(\Phi, \Psi). \end{aligned}$$

Using the above dimensionless quantities then, Eqs. (4) - (10) become

$$\begin{aligned} a_1 \nabla^2 u + a_2 \frac{\partial e}{\partial x} + a_3 \frac{\partial \Phi}{\partial x} + a_4 \frac{\partial \Psi}{\partial x} - a_5 \frac{\partial T}{\partial x} - a_6 \frac{\partial C}{\partial x} \\ + a_7 \frac{\partial v}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{11}$$

$$\begin{aligned} a_1 \nabla^2 v + a_2 \frac{\partial e}{\partial y} + a_3 \frac{\partial \Phi}{\partial y} + a_4 \frac{\partial \Psi}{\partial y} - a_5 \frac{\partial T}{\partial y} - a_6 \frac{\partial C}{\partial y} \\ - a_7 \frac{\partial u}{\partial x} = \frac{\partial^2 v}{\partial t^2}. \end{aligned} \tag{12}$$

$$\nabla^2 T = (1 + \tau_q \frac{\partial}{\partial t})(T_{,t} + a_8 C_{,t} + a_9 e_{,t} + a_{10} \Phi_{,t} + a_{11} \Psi_{,t}) \tag{13}$$

$$a_{12} \nabla^2 \Phi + a_{13} \nabla^2 \Psi - \frac{b}{\alpha_1} e_{,tt} - a_{14} \Phi - a_{15} \Psi + a_{16} T + a_{17} C = \Phi_{,tt}, \tag{14}$$

$$a_{18} \nabla^2 \Phi + a_{19} \nabla^2 \Psi - a_{20} e - a_{21} \Phi - a_{22} \Psi + a_{23} T + a_{24} C = \Psi_{,tt}, \tag{15}$$

$$a_{29}\nabla^2 e_{kk} + a_{30}\nabla^2 T + a_{31}C_{,t} + a_{32}C_{,tt} - a_{33}\nabla^2 C + \mu\nabla^2 \Phi + \nu\nabla^2 \Psi = 0, \quad (16)$$

$$p = -a_{25}e_{kk} + C - a_{26}T - a_{27}\Phi - a_{28}\Psi. \quad (17)$$

The dimensionless variables of the stress components take the form

$$\sigma_{xx} = a_{40} \frac{\partial v}{\partial y} + a_{41} \frac{\partial u}{\partial x} + a_{42}\Phi + a_{43}\Psi - T - a_{44}C, \quad (18)$$

$$\sigma_{yy} = a_{40} \frac{\partial u}{\partial x} + a_{41} \frac{\partial v}{\partial y} + a_{42}\Phi + a_{43}\Psi - T - a_{44}C, \quad (19)$$

$$\sigma_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \quad (20)$$

Define displacement potentials  $q$  and  $p_1$  which relate to displacement components  $u$  and  $v$  as,

$$u = q_{,x} - p_{1,y}, \quad v = q_{,y} + p_{1,x}. \quad (21)$$

Using Eq. (21) in Eqs. (11)-(16), we obtain

$$(a_1 + a_2)\nabla^2 q + a_3\Phi + a_4\Psi - a_5T - a_6C + a_7p_{1,x} = q_{,tt}, \quad (22)$$

$$a_1\nabla^2 p_{1,x} - a_7q_{,x} = p_{1,tt}, \quad (23)$$

$$\nabla^2 T = (1 + \tau_q \frac{\partial}{\partial t})(T_{,t} + a_8C_{,t} + a_9e_{,t} + a_{10}\Phi_{,t} + a_{11}\Psi_{,t}), \quad (24)$$

$$a_{12}\nabla^2 \Phi + a_{13}\nabla^2 \Psi - \frac{b}{a_1}e - a_{14}\Phi - a_{15}\Psi + a_{16}T + a_{17}C = \Phi_{,tt}, \quad (25)$$

$$a_{18}\nabla^2 \Phi + a_{19}\nabla^2 \Psi - a_{20}e - a_{21}\Phi - a_{22}\Psi + a_{23}T + a_{24}C = \Psi_{,tt}, \quad (26)$$

$$a_{29}\nabla^2 e_{kk} + a_{30}\nabla^2 T + a_{31}C_{,t} + a_{32}C_{,tt} - a_{33}\nabla^2 C + \mu\nabla^2 \Phi + \nu\nabla^2 \Psi = 0, \quad (27)$$

$$p = -a_{25}e_{kk} + C - a_{26}T - a_{27}\Phi - a_{28}\Psi. \quad (28)$$

### 3. Normal mode analysis

The solution of the considered physical variable can be decomposed in terms of normal mode as the following form Abdou *et al.* (2020a)

$$[q, p_1, e, T, \Phi, \Psi, \tau_{ij}](x, y, t) = [q^*, p_1^*, e^*, T^*, \Phi^*, \Psi^*, \tau_{ij}^*](y) \exp(\omega t + iax). \quad (29)$$

where,  $\omega$  is the complex time constant (frequency),  $i$  is the imaginary unit, and  $a$  is the wave number in  $x$ -direction.

Using (29) in Eqs. (22)-(28), we obtain

$$(a_{34}D^2 - s_1)q + a_{35}p_1 + a_3\Phi + a_4\Psi - a_5T - a_6C = 0, \quad (30)$$

$$-a_{35}q + (a_1D^2 - s_2)p_1 = 0, \quad (31)$$

$$(B_1D^2 - s_3)q + B_2\Phi + B_3\Psi - (D^2 + s_4)T + B_4C = 0, \quad (32)$$

$$(B_5D^2 + s_5)q + (a_{12}D^2 - s_6)\Phi + (a_{13}D^2 - s_7)\Psi + a_{16}T + a_{17}C = 0, \quad (33)$$

$$(-a_{20}D^2 + s_8)q + (a_{18}D^2 - s_9)\Phi + (a_{19}D^2 - s_{10})\Psi + a_{23}T + a_{24}C = 0, \quad (34)$$

$$(a_{29}D^4 - B_6D^2 + B_7)q + (\mu D^2 - s_{11})\Phi + (\nu D^2 - s_{12})\Psi + (a_{30}D^2 - s_{13})T + (-a_{33}D^2 + s_{14})C = 0, \quad (35)$$

$$p = -a_{25}(D^2 - a^2)q - a_{27}\Phi - a_{28}\Psi - a_{26}T + C. \quad (36)$$

Put the above Eqs. (30), (32), (33), (34), (35) in the matrix, we find that the differential equation takes the form

$$[D^{12} - A D^{10} + B D^8 - E D^6 + F D^4 - G D^2 + I]\{q, p_1, \Phi, \Psi, T, C\}(y) = 0. \quad (37)$$

where, A, B, E, F, G and I constant of matrix.

The solution of Eq. (36) has the form

$$q = \sum_{n=1}^6 M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (38)$$

$$p_1 = \sum_{n=1}^6 H_{1n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (39)$$

$$\Phi = \sum_{n=1}^6 H_{2n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (40)$$

$$\Psi = \sum_{n=1}^6 H_{3n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (41)$$

$$T = \sum_{n=1}^6 H_{4n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (42)$$

$$C = \sum_{n=1}^6 H_{5n} M_n e^{-k_n y} e^{(\omega t + iax)}. \quad (43)$$

To get the displacements, substituting Eqs. (37), (38) in (21) we get

$$u = \sum_{i=1}^6 H_{6n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (44)$$

$$v = \sum_{i=1}^6 H_{7n} M_n e^{-k_n y} e^{(\omega t + iax)}. \quad (45)$$

To get the stresses displacement, substituting from Eqs. (39)-(44) in (18)-(20) we get

$$\sigma_{xx} = \sum_{i=1}^6 H_{8n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (46)$$

$$\sigma_{yy} = \sum_{i=1}^6 H_{9n} M_n e^{-k_n y} e^{(\omega t + iax)}, \quad (47)$$

$$\sigma_{xy} = \sum_{i=1}^6 H_{10n} M_n e^{-k_n y} e^{(\omega t + iax)}. \quad (48)$$

Dimensionless variables for the components of  $\sigma_i, \tau_i$

$$\sigma_3 = \eta_1 \Phi_{,y} + \eta_2 \Psi_{,y}, \tag{49}$$

$$\tau_3 = \eta_2 \Phi_{,y} + \eta_3 \Psi_{,y}. \tag{50}$$

To get the solution of  $\sigma_3$  and  $\tau_3$ , substituting from Eqs. (39), (40) in (46) and (47)

$$\sigma_3 = \sum_{n=1}^6 H_{11n} M_n e^{-k_n y} e^{i(\omega t + \alpha x)}, \tag{51}$$

$$\tau_3 = \sum_{n=1}^6 H_{12n} M_n e^{-k_n y} e^{i(\omega t + \alpha x)}. \tag{52}$$

In Appendix  $a_i, (i = 1, 2, \dots, 44)$  and  $H_{in}$

### 4. Boundary conditions

We apply six boundary conditions for present problem at the plane surface  $y = 0$ .

$$\sigma_{xx}(x, 0, t) = \sigma_{xy}(x, 0, t) = 0, \quad T = P_1 e^{i(\omega t + \alpha x)}, \quad \frac{\partial \Phi}{\partial y} = 0, \tag{53}$$

$$\frac{\partial \Psi}{\partial y} = 0, \quad \frac{\partial C}{\partial y} = P_2 e^{i(\omega t + \alpha x)}.$$

Applying Eqs. (52) in (39)-(42) and (45), (47) we get

$$\sum_{n=1}^6 -k_n H_{2n} M_n = 0, \tag{54}$$

$$\sum_{n=1}^6 -k_n H_{3n} M_n = 0, \tag{55}$$

$$\sum_{n=1}^6 H_{4n} M_n = p_1, \tag{56}$$

$$\sum_{n=1}^6 -k_n H_{5n} M_n = p_2, \tag{57}$$

$$\sum_{n=1}^6 H_{8n} M_n = 0, \tag{58}$$

$$\sum_{n=1}^6 H_{10n} M_n = 0. \tag{59}$$

To get  $M_1, M_2, \dots, M_6$ , we can put Eqs. (53)-(58) in the matrix

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{pmatrix} = \begin{pmatrix} -k_1 H_{21} & -k_2 H_{22} & -k_3 H_{23} & -k_4 H_{24} & -k_5 H_{25} & -k_6 H_{26} \\ -k_1 H_{31} & -k_2 H_{32} & -k_3 H_{33} & -k_4 H_{34} & -k_5 H_{35} & -k_6 H_{36} \\ H_{41} & H_{42} & H_{43} & H_{44} & H_{45} & H_{46} \\ -k_1 H_{51} & -k_2 H_{52} & -k_3 H_{53} & -k_4 H_{54} & -k_5 H_{55} & -k_6 H_{56} \\ H_{81} & H_{82} & H_{83} & H_{84} & H_{85} & H_{86} \\ H_{101} & H_{102} & H_{103} & H_{104} & H_{105} & H_{106} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

### 5. Numerical results

We now present some numerical results. For this purpose, copper is taken as a material of thermoelastic for which we take the following values of temperature (Abdou *et al.* 2020)

$$\lambda = 7.7 \times 10^{10} N.M^{-2}, \quad \mu = 3.86 \times 10^{10} N.M^{-2}, \quad T_0 = 293K,$$

$$K = 3.86 \times 10^3 N.s^{-1}.K^{-1}, \quad \alpha_t = 1.78 \times 10^{-5} K^{-1}, \quad \tau_0 = 1.2,$$

$$\rho = 8954 Kg.m^{-3}, \quad C^* = 383.1 J.Kg^{-1}.K^{-1}, \quad x = 0.5, \quad a = 0.1,$$

$$\omega_0 = 1.1, \quad \xi = -0.6, \quad \omega = \omega_0 + i\xi.$$

The double porous parameters are taken as Khalili (2003)

$$\alpha = 1.3 \times 10^{-5} N, \quad b_1 = 0.12 \times 10^{-5} N, \quad \gamma = 1.1 \times 10^{-5} N.m^{-2},$$

$$\gamma_1 = 0.16 \times 10^5 N.m^{-2}, \quad \gamma_2 = 0.219 \times 10^5 N.m^{-2},$$

$$\alpha_1 = 2.3 \times 10^{10} N.m^{-2}, \quad \alpha_2 = 2.4 \times 10^{10} N.m^{-2},$$

$$\alpha_3 = 2.5 \times 10^{10} N.m^{-2}, \quad d = 0.1 \times 10^{10} N.m^{-2},$$

$$b = 0.9 \times 10^{10} N.m^{-2},$$

$$K_2 = 0.1546 \times 10^{-12} N.m^{-2}, \quad K_1 = 0.1456 \times 10^{-12} N.m^{-2}.$$

distribution of the real part of the temperature  $T$  the displacement components  $u, v$  and the stress components  $\sigma_{yy}, \sigma_{xy}$ .

Figs. 1 and 2 show the comparison of the displacement  $u, v$  in the presence of double porosity at  $g=9.8$  and  $t=0.5, t = 1.5$ . We find that in Fig. 1 describes the variation of the horizontal displacement  $u$  against  $y$  in the frames of the

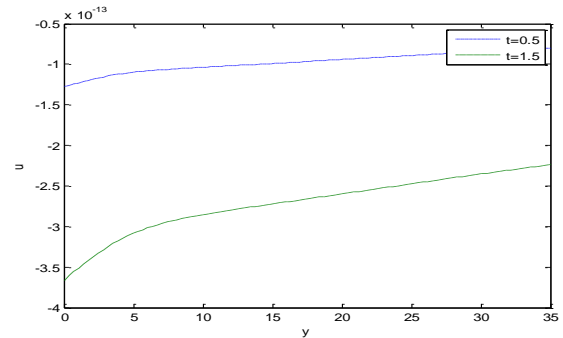


Fig. 1 Distribution of the displacement  $u$

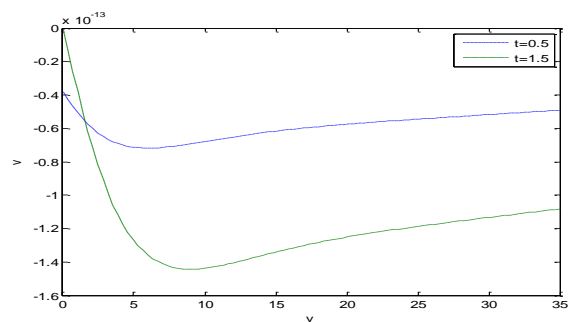


Fig. 2 Distribution of the displacement  $v$

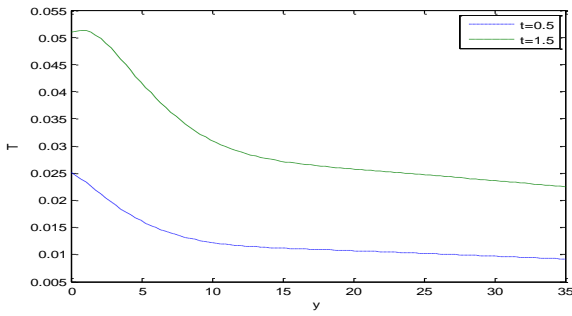


Fig. 3 Distribution of the temperature  $T$

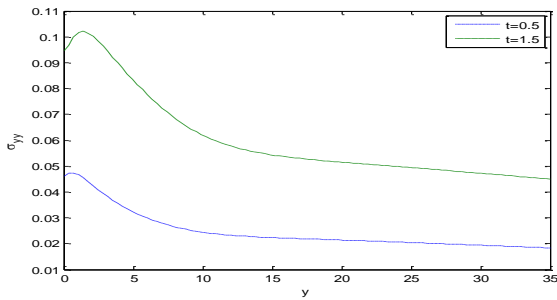


Fig. 4 Distribution of the stress  $\sigma_{yy} T$

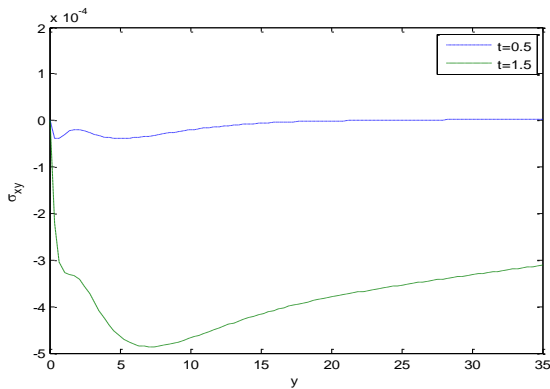


Fig. 5 Distribution of the stress  $\sigma_{xy}$

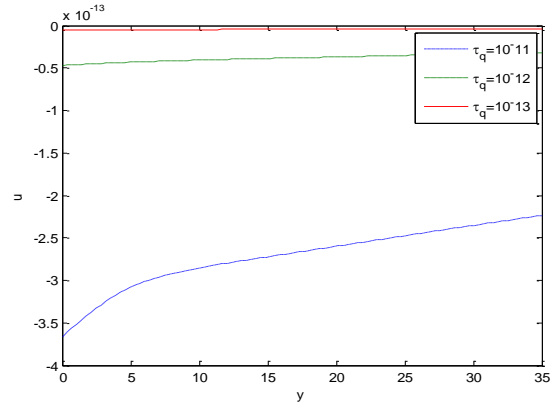


Fig. 6 Distribution of the displacement

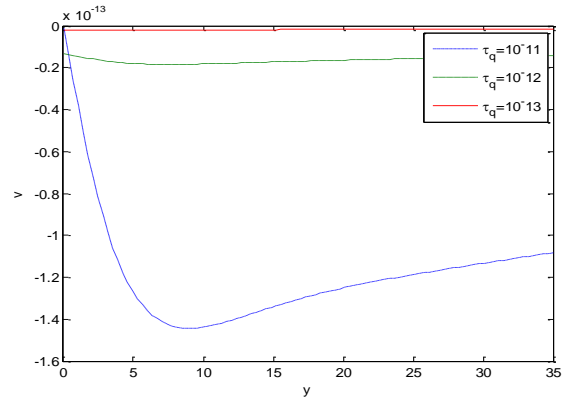


Fig. 7 Distribution of the displacement  $v$

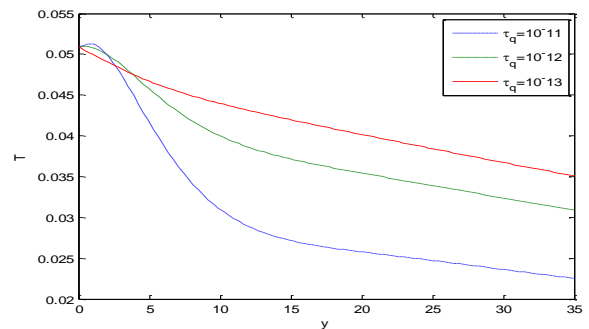


Fig. 8 Distribution of the temperature  $T$

L–S theory. It is clear that the component  $u$  decreases when the time increases through negative values until attaining a certain value and then vanishing. However, in Fig. 2 the displacement  $v$  decreases at  $t = 0.5, t = 1.5$  to the minimum value at  $y = 5$  and  $10$  then takes, the form of the wave until it decays to zero. Fig. 3 illustrate the comparison of the displacement  $T$  in the presence of double porosity at  $t = 0.5, t = 1.5$ . We find in Fig. 3 that the values of  $T$  increase gradually through positive values until vanishing at  $t = 1.5$ . Here also, at  $t = 0.5$  the temperature decreases until it decay to zero. In Fig. 4 the displacement  $\sigma_{yy}$  increases on the same path at  $t = 0.5, t = 1.5$  until it decays to zero. Figs. 5 explains the comparison of  $\sigma_{xy}$  in the presence of double porosity at  $t = 0.5, t = 1.5$  We find in Fig. 5 that  $\sigma_{xy}$  at  $t = 1.5$  decreases to a minimum value at  $y = 7$  and decay gradually until it decays to zero. Here also, at  $t = 1.5$  we

find that  $\sigma_{xy}$  decreases at  $y = 2$  and take the form of wave until it decay to zero. Figs. 6 and 7 show the comparison of the displacement  $u, v$  in the presence of double porosity at  $g = 9.8$  and  $(\tau_q = 10^{-11}, 10^{-12}, 10^{-13})$ . We find that in Fig. 6 describes the variation of the horizontal displacement  $u$  against  $y$  in the frames of the L–S theory. It is clear that the component  $u$  increases when the relaxation time decreases through negative values then vanishing. However, in Fig. 7 the displacement  $v$  decreases at  $\tau_q = 10^{-11}, 10^{-12}, 10^{-13}$  to the minimum value at  $\tau_q = 10^{-11}$  and begin to increase in the curves at  $\tau_q = 10^{-12}, \tau_q = 10^{-13}$  then decays to zero. Fig. 8

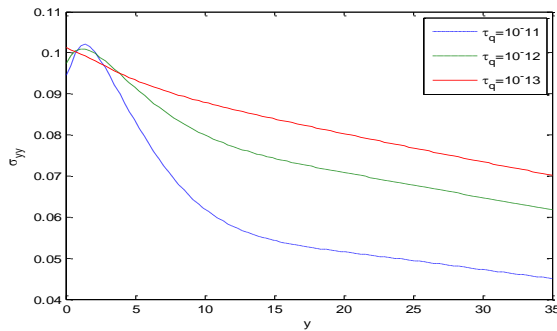


Fig. 9 Distribution of the stress  $\sigma_{yy}$

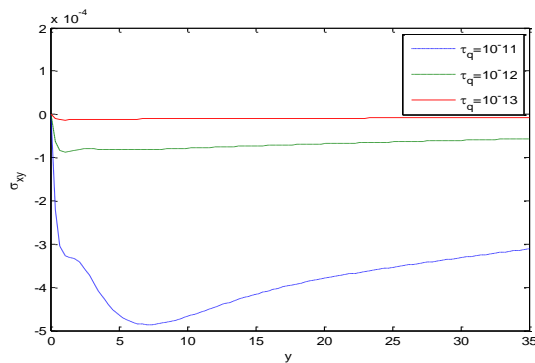


Fig. 10 Distribution of the stress  $\sigma_{xy}$

illustrates the comparison of the displacement  $T$  in the presence of double porosity at ( $\tau_q = 10^{-11}$ ,  $10^{-13}$ ).  $10^{-12}$ , We find in Fig. 8 that the values of  $T$  increase gradually through positive values until it reach to maximum value in three curves then slowly begin to decrease until it vanishing to zero. In Fig. 9 the displacement  $\sigma_{yy}$  increases on the same path at  $y = 2$  in three curves at ( $\tau_q = 10^{-11}$ ,  $10^{-12}$ ,  $10^{-13}$ ) then decreases through negative values then vanishing. Fig. 10 explains the comparison of  $\sigma_{xy}$  in the presence of double porosity and gravity at ( $\tau_q = 10^{-11}$ ,  $10^{-12}$ ,  $10^{-13}$ ). We find in Fig. 10 that  $\sigma_{xy}$  at  $\tau_q = 10^{-13}$  take a minimum value around zero. Here also, at  $\tau_q = 10^{-12}$  we find that  $\sigma_{xy}$  decreases at  $y = 1$  to minimum value until it decay to zero. However, at  $\tau_q = 10^{-11}$  we find that  $\sigma_{xy}$  decreases at  $y = 7$  to minimum value until it reach to zero.

## 6. Conclusions

In the figures obtained by comparing the functions at two different values of the time and three different values of the relaxation times, important phenomena are observed:

Analytic solutions based upon normal mode analysis of the thermoelastic problem in solids have been developed, which used in the present article is applicable to a wide range of problems in hydrodynamics and thermoelasticity.

There are significant differences in the physical component under two different times.

All the physical quantities satisfy the boundary conditions.

The relaxation time  $\tau_q$  in the presence of gravity plays a significant role in all physical quantities.

The value of all the physical quantities converges to zero, and all the functions are continuous. The problem though theoretical can provide useful information for experimental researchers working in the field of geophysics, and earthquake engineering, along with seismologists working in the field of mining tremors and drilling into the crust of the earth. The numerical treatment of the general system of equations and conditions governing the phenomenon may be useful in getting rid of the limitations of the method of normal modes' technique and this task is in progress.

## Declaration of conflicting interests

The authors have declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## Author contributions

The authors contributed equally to this work and approved it for publication.

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## Appendix

$$\begin{aligned}
a_1 &= \frac{\mu}{\rho c_1^2}, \quad a_2 = \frac{(\mu + \lambda)}{\rho c_1^2}, \quad a_3 = \frac{b \alpha_1}{\rho c_1^2 K_1 w_1^2}, \quad a_4 = \frac{d \alpha_1}{\rho c_1^2 K_1 w_1^2}, \\
a_5 &= \frac{\beta T_0}{\rho c_1^2}, \quad a_6 = \frac{\gamma^*}{c_1^2}, \quad a_7 = g, \quad a_8 = \frac{a}{c_e}, \quad a_9 = \frac{\beta}{\rho c_e}, \\
a_{10} &= \frac{\gamma_1 \alpha_1}{\rho c_e K_1 w_1^2}, \quad a_{11} = \frac{\gamma_2 \alpha_1}{\rho c_e K_1 w_1^2}, \quad a_{12} = \frac{\alpha}{c_1^2 K_1}, \quad a_{13} = \frac{b}{c_1^2 K_1}, \\
a_{14} &= \frac{\alpha_1}{K_1 w_1^2}, \quad a_{18} = \frac{b_1}{c_1^2 K_2}, \quad a_{17} = \frac{\varrho \rho}{\alpha_1}, \quad a_{16} = \frac{\gamma_1 T_0}{\alpha_1}, \quad a_{15} = \frac{\alpha_3}{K_1 w_1^2}, \\
a_{22} &= \frac{\alpha_2}{w_1^2 K_2}, \quad a_{21} = \frac{\alpha_3}{w_1^2 K_2}, \quad a_{20} = \frac{d K_1}{\alpha_1 K_2}, \quad a_{19} = \frac{\gamma}{c_1^2 K_2}, \\
a_{27} &= \frac{\varrho \alpha_1}{\rho \beta_2 K_1 w_1^2}, \quad a_{26} = \frac{d T_0}{\beta_2 \rho}, \quad a_{25} = \frac{\gamma^*}{\beta_2 \rho}, \quad a_{24} = \frac{m \rho K_1}{\alpha_1 K_2}, \quad a_{23} = \frac{\gamma_2 T_0 K_1}{\alpha_1 K_2}, \\
a_{31} &= \frac{\rho c_1^2}{w_1}, \quad a_{30} = \beta_1 d T_0, \quad a_{29} = \beta_1 \gamma^*, \quad a_{28} = \frac{m \alpha_1}{\rho \beta_2 K_1 w_1^2}, \\
a_{33} &= \beta_1 \beta_2 \rho, \quad a_{32} = \tau^0 \rho c_1^2, \\
a_{34} &= a_1 + a_2, \quad a_{35} = i a a_7, \quad a_{36} = \tau_q \omega, \quad a_{37} = a_{14} + \omega^2, \\
a_{38} &= a_{22} + \omega^2, \quad a_{39} = a_{31} \omega + a_{32} \omega^2, \quad a_{40} = \frac{\lambda}{\beta T_0}, \quad a_{41} = \frac{2\mu + \lambda}{\beta T_0}, \\
a_{42} &= \frac{b \alpha_1}{K_1 w_1^2 \beta T_0}, \quad a_{43} = \frac{d \alpha_1}{K_1 w_1^2 \beta T_0}, \quad a_{44} = \frac{\gamma^* \rho}{\beta T_0}, \quad s_1 = (a_{34} a^2 + \omega^2), \\
s_2 &= (a_1 a^2 + \omega^2), \quad s_3 = a_{36} a_9 a^2, \quad s_4 = (a_{36} + a^2), \quad s_5 = \frac{b}{\alpha_1} a^2, \\
s_6 &= (a_{12} a^2 + a_{37}), \quad s_7 = (a_{13} a^2 + a_{15}), \quad s_8 = a_{20} a^2, \quad s_9 = (a_{18} a^2 + a_{21}), \\
s_{10} &= (a_{19} a^2 + a_{38}), \quad s_{11} = \mu a^2, \quad s_{12} = \nu a^2, \quad s_{13} = a_{30} a^2, \\
s_{14} &= a_{39} + a_{33} a^2, \quad B_1 = a_{36} a_9, \quad B_2 = a_{36} a_{10}, \quad B_3 = a_{36} a_{11}, \\
B_4 &= a_{36} a_8, \quad B_5 = \frac{-b}{\alpha_1}, \quad B_6 = 2a_{29} a^2, \quad B_7 = a_{29} a^4, \\
\mu &= \frac{\beta_1 \alpha_1 \varrho}{K_1 w_1^2}, \quad \nu = \frac{\beta_1 \alpha_1 m}{K_1 w_1^2}, \quad \eta_1 = \frac{\alpha_1}{k_1 w_1^2}, \quad \eta_2 = \frac{b_1 \alpha_1}{\alpha k_1 w_1^2}, \quad \eta_3 = \frac{\gamma \alpha_1}{\alpha k_1 w_1^2}, \\
H_{1n} &= \frac{a_{35}}{(a_1 k_n^2 - s_2)}, \\
H_{2n} &= - \{ [B_3 a_{17} - B_4 (a_{13} k_n^2 - s_7)] [a_6 a_{16} - a_5 a_{17}] \\
&\quad + [a_4 a_{17} + a_6 (a_{13} k_n^2 - s_7)] [(k_n^2 + s_4) a_{17} - B_4 a_{16}] \} \\
&\quad \{ [a_{23} (-a_{33} k_n^2 + s_{14}) - a_{24} (a_{30} k_n^2 - s_{13})] [B_4 (a_{34} k_n^2 \\
&\quad - s_1 + a_{35} H_{1n}) + a_6 (B_1 k_n^2 - s_3)] + [a_5 B_4 + a_6 (k_n^2 + s_4)] \\
&\quad [(-a_{33} k_n^2 + s_{14}) (-a_{20} k_n^2 + s_8) \\
&\quad - a_{24} (a_{29} k_n^4 - B_6 k_n^2 + B_7)] \} \\
&\quad + \{ [a_{17} (B_1 k_n^2 - s_3) - B_4 (B_5 k_n^2 + s_5)] \\
&\quad [a_6 a_{16} - a_5 a_{17}] [a_{17} (k_n^2 + s_4) - B_4 a_{16}] \} \\
&\quad \{ [a_4 B_4 + B_3 a_6] [a_{23} (-a_{33} k_n^2 + s_{14} \\
&\quad - a_{24} (a_{30} k_n^2 - s_{13}))] + [a_5 B_4 + a_6 (k_n^2 + s_4)] \\
&\quad [(-a_{33} k_n^2 + s_{14}) (a_{19} k_n^2 - s_{10}) - a_{24} (a_{35} k_n^2 - s_{12})] \} / \\
&\quad \{ [B_4 a_3 + B_4 B_2] [a_{23} (-a_{33} k_n^2 + s_{14}) - a_{24} (a_{30} k_n^2 - s_{13})] \}
\end{aligned}$$

$$\begin{aligned}
&\quad + [a_5 B_4 + a_6 (k_n^2 + s_4)] [(a_{18} k_n^2 - s_9) (-a_{33} k_n^2 + s_{14}) \\
&\quad - a_{24} (a_{34} k_n^2 - s_{11})] \} \{ [B_3 a_{17} - B_4 (a_{13} k_n^2 - s_7)] \\
&\quad [a_6 a_{16} - a_5 a_{17}] + [a_4 a_{17} + a_6 (a_{13} k_n^2 - s_7)] \\
&\quad [a_{17} (k_n^2 + s_4) - B_4 a_{16}] \} - \{ [B_2 a_{17} \\
&\quad - B_4 (a_{12} k_n^2 - s_6)] [a_6 a_{16} - a_5 a_{17}] \\
&\quad + [a_3 a_{17} + a_6 (a_{12} k_n^2 - s_6)] [a_{17} (k_n^2 + s_4) - B_4 a_{16}] \} \\
&\quad \{ [a_4 B_4 + B_3 a_6] [a_{23} (-a_{33} k_n^2 + s_{14}) - a_{24} (a_{30} k_n^2 - s_{13})] \\
&\quad + [a_5 B_4 + a_6 (k_n^2 + s_4)] [(-a_{33} k_n^2 + s_{14}) (a_{19} k_n^2 - s_{10}) \\
&\quad - a_{24} (a_{35} k_n^2 - s_{12})] \} \\
&\quad - \{ [a_4 B_4 + B_3 B_6] [a_{23} (-a_{33} k_n^2 + s_{14}) - a_{24} (a_{30} k_n^2 - s_{13})] \\
&\quad + [B_5 B_4 + a_6 (k_n^2 + s_4)] [(-a_{33} k_n^2 + s_{14}) (a_{19} k_n^2 - s_{10}) \\
&\quad - a_{24} (a_{35} k_n^2 - s_{12})] \} \\
H_{3n} &= \frac{-a_{24} (a_{35} k_n^2 - s_{13}) \{ [a_{23} (-a_{33} k_n^2 + s_{14}) - a_{24} (a_{30} k_n^2 - s_{13})] [B_4 (a_{34} k_n^2 - s_1 + a_{35} H_{1n}) \\
&\quad + a_6 (B_1 k_n^2 - s_3)] + [a_5 B_4 + a_6 (k_n^2 + s_4)] \{ [(-a_{33} k_n^2 + s_{14}) (-a_{20} k_n^2 + s_8) \\
&\quad - a_{24} (a_{29} k_n^2 - B_6 k_n^2 + B_7)] \} + \{ [B_4 a_3 + B_4 B_2] [a_{23} (-a_{33} k_n^2 + s_{14}) \\
&\quad - a_{24} (a_{30} k_n^2 - s_{13})] + [a_5 B_4 + a_6 (k_n^2 + s_4)] [a_{18} k_n^2 - s_9] (-a_{33} k_n^2 + s_{14}) \\
&\quad - a_{24} (a_{34} k_n^2 - s_{11}) \} \\
&\quad + [a_5 B_4 + a_6 (k_n^2 + s_4)] \}, \\
H_{4n} &= \frac{[a_5 B_4 + a_6 (k_n^2 + s_4)] \{ [B_4 (a_{34} k_n^2 - s_1 + a_{35} H_{1n})] + a_6 (B_1 k_n^2 - s_3) \} \\
&\quad + \{ [B_4 a_3 + B_4 B_2] H_{2n} + [a_4 B_4 + B_3 a_6] H_{3n} \}}{a_6}, \\
H_{5n} &= \frac{(a_{34} k_n^2 - s_1 + a_{35} H_{1n} + a_3 H_{2n} + a_4 H_{3n} - a_5 H_{4n})}{a_6}, \\
H_{6n} &= i a + k_n H_{1n}, \quad H_{7n} = -k_n + i a H_{1n}, \\
H_{8n} &= a_{36} (-a^2 + i a k_n) + a_{37} (k_n^2 - k_n H_{1n}) + a_{38} H_{2n} \\
&\quad + a_{39} H_{3n} - H_{4n} - a_{40} H_{5n}, \\
H_{9n} &= -a_{36} k_n H_{7n} + i a_{37} a H_{6n} + a_{38} H_{2n} + a_{39} H_{3n} + H_{4n} - a_{40} H_{5n}, \\
H_{10n} &= \mu [i a H_{7n} - k_n H_{6n}], \quad H_{11n} = -(\eta_1 k_n H_{2n} + \eta_2 k_n H_{3n}), \\
H_{12n} &= -(\eta_3 k_n H_{2n} + \eta_2 k_n H_{3n}).
\end{aligned}$$