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A viscoelastic-micropolar solid with voids and microtemperatures under the effect of the gravity field

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Abstract. The model of two-dimensional plane waves is analyzed in a micropolar-thermoelastic solid with microtemperatures in the context of the three-phase-lag model, dual-phase-lag model, and the Green-Naghdi theory of type III. Harmonic wave analysis is used to hold the solution to the problem. Numerical results of the physical fields are visualized to show the effects of the gravity field, magnetic field, and viscosity. The expression for the field variables is obtained generally and represented graphically for a particular medium.

Keywords: a micropolar; initial stress; microtemperatures; viscoelastic-thermoelastic

1. Introduction

Wave propagations in micropolar materials have many applications in different fields of science and technology. Eringen (1966) discussed the linear theory of micropolar thermoelasticity which is known as micropolar coupled thermoelasticity to include thermal effects. Goodman and Cowin (1972) introduced a continuum theory for granular materials, whose matrix material is elastic and interstices are voids and they explained the concept of distributed body, which represents a continuum model for granular materials and porous materials. Cowin and Nunziato (1973) discussed the linear theory of elastic materials with voids. Marin (1997) discussed the uniqueness and domain of influence that results in thermoelastic solids with voids. Kumar and Rani (2006) introduced the deformation due to moving loads in the thermoelastic medium with voids. Sarkar and Tomar (2019) concerned with the propagation of time-harmonic plane waves in an infinite nonlocal thermoelastic medium with void pores. Hobiny and Abbas (2020) discussed thermoelastic wave assessment in a two-dimension medium with voids.

The discussion of materials with microtemperatures has great importance in the field of continuum mechanics due to the increasing scope and interest in nanotechnology research. The research of nanoparticles with microtemperatures has a great scope in the future of technologies. Ieşan and Quintanilla (2000) constructed the linear theory of thermoelastic materials with an inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. The theory of elastic materials with microstructures goes back to the book of Cosserat and Cosserat (1909). Aouadi (2008) introduced some lemmas in the isotropic theory of microstretch thermoelasticity mediums with microtemperatures. Kumar *et al.* (2017) showed the wave propagation at an interface of elastic and microstretch thermoelastic mediums with microtemperatures. Saci and Djebabla (2020) introduced a new stability number and proved that the unique dissipation due to the microtemperatures is strong enough to drive the system to the equilibrium state in an exponential manner. Chirilă and Marin (2019) discussed a microstretch diffusion thermoelastic solid with microtemperatures and microconcentrations. Various vital approaches for determining the solutions to the governing equations of problems in thermodynamics, and thermoelasticity have been introduced by Hobiny *et al.* (2020), Bhatti *et al.* (2020), and Abbas *et al*. (2009, 2011a, b, 2014a, b, c, 2015, 2018).

In the present work, we discussed the 2D problem of a micropolar viscoelastic-thermoelastic medium with voids and microtemperatures in the context of the three-phase-lag model (3PHL), dual-phase-lag model (Dual), and the Green-Naghdi theory of type III. To analyze and get the numerical solutions for the physical quantities of the problem, harmonic wave analysis is used. The physical quantities are graphically discussed in the absence and presence of the magnetic field as well as the viscosity. Comparisons are made with results for different values of the gravity field. The most significant points are highlighted.

2. Formulation of the problem

A linear, homogeneous, isotropic micropolar magneto-thermoelastic media with microtemperatures vector $\underline{w}=(w_1,w_2,0)$ in xy plane and it has micro-rotation vector $\underline{\varphi}=(0,0,\varphi_3)$. The field equations and constitutive relations can be written by Iesan and Quintanilla (2000), Othman *et al.* (2020), Montanaro (1999), and Said *et al.* (2020) as follows:

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The constitutive relations are

$$\sigma_{ij}=\lambda_I e_{kk}\delta_{ij}+2\mu_I e_{ij}+\beta_I\psi\delta_{ij}+\left(u_{j,i}-\varepsilon_{ijr}\varphi_r\right)k_I^*-\gamma_I\theta\delta_{ij}-P(w_{ij}+\delta_{ij}),\tag{1}$$

$$m_{ij}=\alpha_1\Phi_{r,r}\delta_{ij}+\alpha_5\Phi_{i,j}+\gamma_1\Phi_{j,i}\tag{2}$$

$$q_{ij}=-k_4 w_{l,l}\delta_{ij}-k_5 w_{i,j}-k_6 w_{j,i},\tag{3}$$

$$\mu_I=\mu(1+\alpha'\frac{\partial}{\partial t}),\quad \gamma_I=\gamma(1+\gamma_0\frac{\partial}{\partial t}),\quad \lambda_I=\lambda(1+\lambda_0\frac{\partial}{\partial t}),\quad \beta_I=\beta(1+\beta_0\frac{\partial}{\partial t}),\quad k_I^*=k^*(1+k_0^*\frac{\partial}{\partial t}).\tag{4}$$

The equations of motion

$$\rho\ddot{u}_i=\sigma_{ji,j}+\mu_0(\underline{J}\wedge\underline{H})_i+F_i\tag{5}$$

$$\rho J\frac{\partial^2\varphi}{\partial t^2}=(\alpha_I+\beta_I+\gamma_I')\nabla(\nabla.\underline{\varphi})-\gamma_I'\nabla\wedge(\nabla\wedge\underline{\varphi})+k_I^*(\nabla\wedge\underline{u})-\mu_I(\nabla\wedge\underline{w})-2k_I^*\underline{\varphi},\tag{6}$$

$$\alpha_2\nabla^2\psi-\alpha_3\psi-\alpha_4\frac{\partial\psi}{\partial t}-\alpha_5(\nabla.\underline{u})-\mu_I(\nabla.\underline{w})+m\theta=\rho\alpha_6\frac{\partial^2\psi}{\partial t^2},\tag{7}$$

$$k_6\nabla^2\underline{w}+(k_4+k_5)\underline{\nabla}(\underline{\nabla}.\underline{w})-k_3\underline{\nabla}\theta-k_2\underline{w}-b\frac{\partial\underline{w}}{\partial t}+\mu_1\frac{\partial}{\partial t}\underline{\nabla}\wedge\underline{\varphi}-\mu_2\frac{\partial}{\partial t}\underline{\nabla}\psi=0.\tag{8}$$

The heat conduction equation as Choudhuri (2007) and Said (2020)

$$K^*(1+\tau_v\frac{\partial}{\partial t})\nabla^2\theta+K(1+\tau_\theta\frac{\partial}{\partial t})\nabla^2\dot{\theta}+k_1\underline{\nabla}.\underline{w}=(1+\tau_q\frac{\partial}{\partial t}+\frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2})\left(\rho C_E(n_0\ddot{\theta}+n_1\dot{\theta})+\gamma_I T_0(n_0\ddot{e}+n_1\dot{e})+mn_0T_0\ddot{\psi}++mn_1T_0\dot{\psi}\right)\tag{9}$$

where $e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ are the components of strain, $w_{ij}=\frac{1}{2}(u_{j,i}-u_{i,j})$ are the rotation of displacement vector, σ_{ij} are the components of stress, λ,μ are the elastic constants, α_t is the thermal expansion coefficient, ε_{ijr} is the alternate tensor, δ_{ij} is the Kronecker delta, ψ is the change in the volume fraction field, $\theta=T-T_0$, where T is the temperature above the reference temperature T_0. P is the initial stress, K^*, K are the additional material constant and the coefficient of thermal conductivity respectively, C_E is the specific heat at constant strain, n_0,n_1 are integers, $\tau_\theta,\tau_q,\tau_v$ are the phase-lag of temperature gradient, phase-lag of heat flux and the phase-lag of thermal displacement gradient respectively, ρ is the mass density, $\alpha_1,\beta_1,\gamma_1,k^*$ are the solid constants, $\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,m$ are material constants due to the presence of voids, m_{il} is the couple stress tensor, and J is microinertia. F_i is the gravity force as in Kakar *et al*. (2014), $\alpha_0,\alpha',\gamma_0,\beta_0,k_0^*$ are the viscoelastic parameters, b,μ_1,μ_2, $k_i(i=1,2,..,6)$ are the constitutive coefficients, q_i is the heat flux moment, q_{ij} the first heat flux moment.

Eqs. (5)-(9) are the field equations of the generalized viscoelastic-micropolar solid, applicable to the three-phase-lag (3PHL) model, the Lord-Shulman (L-S) theory, the classical coupled theory (CD), as well as the dual-phase-lag model (Dual), as follows:

a) Equations of the three-phase-lag (3PHL) model when, $n_0=1,\ n_1=0,\ K,\tau_\theta,\tau_q,\tau_v>0$ and the solutions are always (exponentially) stable if $\frac{2K\tau_\theta}{\tau_q}>K+K^*\tau_v>K^*\tau_q$ as in Quintanilla and Racke (2008).

b) Equations of the Green–Naghdi (GN-II) theory without energy dissipation when, $n_0=1,\ n_1=0,\ K=\tau_\theta=\tau_q=\tau_v=0.$

c) Equations of the Green–Naghdi (GN-III) theory with energy dissipation when, $n_0=1,\ n_1=0,\ \tau_\theta=\tau_q=\tau_v=0.$

d) Equations of the Lord-Shulman (L-S) theory when, $n_0=1,\ n_1=0,\ \tau_\theta=\tau_v=0,\quad K,\ \tau_q\neq 0,$

e) Equations of the classical coupled (CD) theory when, $n_0=0,\ n_1=1,\ K=\tau_T=\tau_q=\tau_v=0.$

f) Equations of the dual-phase-lag model (Dual) model when, $n_1=1,\ ,\ \tau_v,>0,\ n_0,K,\tau_\theta=0,\ \tau_q\neq 0,\ \tau_q^2=0$

We introduce the non-dimension quantities:

$$(x',y',u',v')=c\eta(x,y,u,v),\quad \theta'=\frac{\theta}{T_0},\quad \sigma'_{ij}=\frac{\sigma_{ij}}{\rho c^2},\quad w'_i=\frac{w_i}{c\eta},\quad h'=\frac{h}{H_0},\quad (t',\tau'_q,\tau'_v,\tau'_\theta)=c^2\eta(t,\tau_q,\tau_v,\tau_\theta),$$
$$(\varphi'_3,\psi')=(\varphi_3,\psi),\quad g'=\frac{g}{\eta c^3},\quad q'_{ij}=\frac{q_{ij}}{(k_5+k_6)c^2\eta^2},\quad m'_{ij}=\frac{\eta m_{ij}}{\rho c},\quad \eta=\frac{\rho C_E}{K^*},\quad c^2=\frac{\lambda+2\mu}{\rho}.\tag{10}$$

Suppose the following potential functions as Othman *et al.* (2020)

$$u=\frac{\partial\pi_1}{\partial x}+\frac{\partial\pi_2}{\partial y},\quad v=\frac{\partial\pi_1}{\partial y}-\frac{\partial\pi_2}{\partial x},\quad w_1=\frac{\partial q_1}{\partial x}+\frac{\partial q_2}{\partial y},\quad w_2=\frac{\partial q_1}{\partial y}-\frac{\partial q_2}{\partial x}.\tag{11}$$

From Eqs. (10) into Eqs. (5)-(9), and then use Eqs. (11), we get

$$A\frac{\partial^2\pi_1}{\partial t^2}=(\frac{\lambda_I+2\mu_I+k_I^*+\mu_0H_0^3}{\rho c^2})\nabla^2\pi_1+\frac{\beta_I}{\rho c^2}\psi-\frac{\gamma_I T_0}{\rho c^2}\theta-g\frac{\partial\pi_2}{\partial x},\tag{12}$$

$$A\frac{\partial^2\pi_2}{\partial t^2}=\left(\frac{\mu_I+k_I^*-\frac{P}{2}}{\rho c^2}\right)\nabla^2\pi_2+\frac{k_I^*}{\rho c^2}\varphi_3+g\frac{\partial\pi_1}{\partial x},\tag{13}$$

$$\left(1+\frac{k_4+k_5}{k_6}\right)\nabla^2 q_1-\frac{k_3T_0}{k_6c^2\eta^2}\theta+\frac{k_2}{k_6c^2\eta^2}q_1-\frac{b}{k_6\eta}\frac{\partial q_1}{\partial t}-\frac{\mu_2}{k_6\eta}\frac{\partial\psi}{\partial t}=0,\tag{14}$$

$$\nabla^2 q_2 - \frac{k_2}{k_6 c^2 \eta^2} q_2 - \frac{b}{k_6 \eta}\frac{\partial q_2}{\partial t} + \frac{\mu_1}{k_6 \eta}\frac{\partial \varphi_3}{\partial t} = 0, \quad (15)$$

$$\gamma_I' c^2 \eta^2 \nabla^2 \varphi_3 - k_I^* \nabla^2 \pi_2 + \mu_I c^2 \eta^2 \nabla^2 q_2 - 2k_I^* \varphi_3 = \rho J c^4 \eta^2 \frac{\partial^2 \varphi_3}{\partial t^2}, \quad (16)$$

$$\alpha_2 \nabla^2 \psi - \frac{\alpha_3}{c^2\eta^2}\psi - \frac{\alpha_4}{\eta}\frac{\partial \psi}{\partial t} - \frac{\alpha_5}{c^2\eta^2}\nabla^2 \pi_1 - \mu_I \nabla^2 q_1 + \frac{mT_0}{c^2\eta^2}\theta = \rho\alpha_6 c^2 \frac{\partial^2 \psi}{\partial t^2}, \quad (17)$$

$$d_3(1+\tau_v \frac{\partial}{\partial t})\nabla^2\theta + d_4(1+\tau_\theta \frac{\partial}{\partial t})\nabla^2\dot{\theta} + d_5 \nabla^2 q_1 =$$
$$\left(1+\tau_q \frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\Big(\rho C_E T_0 (d_1 \ddot{\theta} + d_2 \dot{\theta}) + \quad (18)$$
$$\gamma_I T_0 \nabla^2 (d_1 \ddot{\pi}_1 + d_2 \dot{\pi}_1) + mT_0 (d_1 \ddot{\psi} + + d_2 \dot{\psi})\Big),$$

Where $A = 1 + \frac{\varepsilon_0 \mu_0^2 H_0^2}{\rho}$, $d_1 = n_0 c^4 \eta^2$, $d_2 = n_1 c^2 \eta$, $d_3 = K^* T_0 c^2 \eta^2$,

$d_4 = K T_0 c^4 \eta^3$, $d_5 = k_I^* c^2 \eta^2$.

3. The analytical analysis

To hold the solution with harmonic wave analysis we consider

$$[\pi_1, \pi_2, q_1, q_2, \psi, \varphi_3, \theta](x,y,t) = [\overline{\pi_1}, \overline{\pi_2}, \overline{q_1}, \overline{q_2}, \overline{\psi}, \overline{\varphi_3}, \overline{\theta}](y) \exp\left(\mathrm{i}\,(a x - \zeta t)\right). \quad (19)$$

where $\overline{\pi_1}(y)$, etc. is the amplitude of the function $\pi_1(x,y,t)$ etc., i is the imaginary unit, ζ is the complex time constant and a is the wave number in the $x-$direction.

Introducing Eqs. (19) in Eqs. (12)-(18), thus we get

$$(A_1 \mathrm{D}^2 - N_1)\overline{\pi_1} - \mathrm{i}ag\overline{\pi_2} + A_2\overline{\psi} - A_3\overline{\theta} = 0, \quad (20)$$

$$\sigma_2 = \frac{R}{S} + \frac{M_x T y_S}{I_x} - \frac{M_y T a}{2I_y} \quad (21)$$

$$-(A_6 \mathrm{D}^2 - N_3)\overline{\pi_1} + (\alpha_2 \mathrm{D}^2 - N_4)\overline{\psi} - \mu_1(\mathrm{D}^2 - a^2)\overline{q_1} + N_5\overline{\theta} = 0, \quad (22)$$

$$(A_7 \mathrm{D}^2 - N_6)\overline{\varphi_3} - (A_8 \mathrm{D}^2 - N_7)\overline{\pi_2} + (A_9 \mathrm{D}^2 - N_8)\overline{q_2} = 0, \quad (23)$$

$$(A_{10} \mathrm{D}^2 - N_9)\overline{q_1} - N_{10}\overline{\theta} + N_{11}\overline{\psi} = 0, \quad (24)$$

$$(\mathrm{D}^2 - N_{12})\overline{q_2} - N_{13}\overline{\varphi_3} = 0, \quad (25)$$

$$(A_{14}\mathrm{D}^2 - N_{15})\overline{q_1} + (A_{15}\mathrm{D}^2 - N_{16})\overline{\pi_1} + (A_{13}\mathrm{D}^2 - N_{14})\overline{\theta} + N_{17}\overline{\psi} = 0 \quad (26)$$

where A_i, N_i are given in the appendix and $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}y}$.

The system of Eqs. (20)-(26) are solved to obtain

$$(\mathrm{D}^{14} - E_1 \mathrm{D}^{12} + E_2 \mathrm{D}^{10} - E_3 \mathrm{D}^8 + E_4 \mathrm{D}^6 - E_5 \mathrm{D}^4 + E_6 \mathrm{D}^2 - E_7)\overline{\pi_1}(y) = 0, \quad (27)$$

where E_i's are given in the appendix.

The bounded solution of Eq. (27) is given as

$$\overline{\pi_1}(y) = \sum_{n=1}^{7} I_n \exp^{(-t_n y)}. \quad (28)$$

In a similar manner, we get

$$\overline{\pi_2} = \sum_{n=1}^{7} I_n J_{1n} \exp^{(-t_n y)}, \quad (29)$$

$$\overline{\varphi_3} = \sum_{n=1}^{7} I_n J_{2n} \exp^{(-t_n y)}, \quad (30)$$

$$\overline{q_1} = \sum_{n=1}^{7} I_n J_{4n} \exp^{(-t_n y)}, \quad (31)$$

$$\overline{q_2} = \sum_{n=1}^{7} I_n J_{3n} \exp^{(-t_n y)}, \quad (32)$$

$$\overline{\psi} = \sum_{n=1}^{7} I_n J_{5n} \exp^{(-t_n y)}, \quad (33)$$

$$\overline{\theta} = \sum_{n=1}^{7} I_n J_{6n} \exp^{(-t_n y)} \quad (34)$$

Using Eqs. (28)-(34) in Eqs. (1)-(3) and (11), thus we have

$$\overline{u} = \sum_{n=1}^{7} I_n J_{7n} \exp^{(-t_n y)}, \quad (35)$$

$$\overline{v} = \sum_{n=1}^{7} I_n J_{8n} \exp^{(-t_n y)} \quad (36)$$

$$\overline{w_1} = \sum_{n=1}^{7} I_n J_{9n} \exp^{(-t_n y)}, \quad (37)$$

$$\overline{w_2} = \sum_{n=1}^{7} I_n J_{10n} \exp^{(-t_n y)}, \quad (38)$$

$$\overline{\sigma_{xy}} = \sum_{n=1}^{7} I_n J_{12n} \exp^{(-t_n y)}, \quad (39)$$

$$\overline{m_{yz}} = \sum_{n=1}^{7} I_n J_{13n} \exp^{(-t_n y)}, \quad (40)$$

$$\overline{\sigma_{yy}}(y) = \sum_{n=1}^{7} I_n J_{11n} \exp(-t_n y) - \frac{P}{\rho c^2}\exp\left(\mathrm{i}\,(\zeta t - a x)\right), \quad (41)$$

$$\overline{q_{xy}}(y) = \sum_{n=1}^{3} I_n J_{14n} \exp(-t_n y), \quad (42)$$

$$\overline{q_{yy}}(y) = \sum_{n=1}^{3} I_n J_{15n} \exp(-t_n y). \quad (43)$$

Where J_{in} are given in the appendix.

4. The boundary conditions

In the physical problem, we should suppress the positive exponential that is unbounded at infinity. The constants $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ have been chosen such that the boundary conditions on the surface at $y = 0$ taking the form

$$\theta=\theta_0\,\Gamma_1(x,t),\qquad \sigma_{xy}=0,\quad \sigma_{yy}=-R_p\,\Gamma_2(x,t)-P,$$

$$m_{yz}=0,\quad \frac{\partial\psi}{\partial y}=\psi_0\,\Gamma_3(x,t),\qquad q_{xy}=q_{yy}=0. \tag{44}$$

Where $\Gamma_1(x,t),\Gamma_2(x,t),\Gamma_3(x,t)$ are arbitrary functions, θ_0,ψ_0 are constants, and R_p is the magnitude of initial stress. Using the above boundary conditions, thus we have

$$\sum_{n=1}^{7}J_{5n}\,t_n\,I_n=-\psi_0,\qquad \sum_{n=1}^{7}J_{6n}\,I_n=\theta_0,\qquad \sum_{n=1}^{7}J_{11n}\,I_n=-R_p,$$
$$\sum_{n=1}^{7}J_{12n}\,I_n=0,\qquad \sum_{n=1}^{7}J_{13n}\,I_n=0,\qquad \sum_{n=1}^{7}J_{14n}\,I_n=0, \tag{45}$$
$$\sum_{n=1}^{7}J_{15n}\,I_n=0\cdot$$

After applying the inverse of the matrix method for Eqs. (45), we have

$$\begin{pmatrix}I_1\\I_2\\I_3\\I_4\\I_5\\I_6\\I_7\end{pmatrix}=\begin{pmatrix}t_1J_{51}&t_2J_{52}&t_3J_{53}&t_4J_{54}&t_5J_{55}&t_6J_{56}&t_7J_{57}\\J_{61}&J_{62}&J_{63}&J_{64}&J_{65}&J_{66}&J_{67}\\J_{111}&J_{112}&J_{113}&J_{114}&J_{115}&J_{116}&J_{117}\\J_{121}&J_{122}&J_{123}&J_{124}&J_{125}&J_{126}&J_{127}\\J_{131}&J_{132}&J_{133}&J_{134}&J_{135}&J_{136}&J_{137}\\J_{141}&J_{142}&J_{143}&J_{144}&J_{145}&J_{146}&J_{147}\\J_{151}&J_{152}&J_{153}&J_{154}&J_{155}&J_{156}&J_{157}\end{pmatrix}^{-1}\begin{pmatrix}-\psi_0\\\theta_0\\-R_p\\0\\0\\0\\0\end{pmatrix}. \tag{46}$$

5. Numerical calculations and discussion for the problem

For numerical computations, we consider an example where Magnesium Crystal-like material is modeled as an isotropic generalized magneto-micropolar thermoelastic solid with voids. All the units of the used parameters are given in SI units as Said (2020)

$$T_0=293\,\text{K},\ \alpha_6=1.753\text{x}10^{-20}\,\text{N.m}^{-2},\ \ J=2\text{x}10^{-20}\,\text{m}^2.\text{s}^{-2},$$
$$\mu=7.86\text{x}\,10^{11}\,\text{N.m}^{-2},\quad a=0.4,\quad \alpha_1=1.78\text{x}10^{-5}\,\text{N.m}^{-2},$$
$$\lambda=7.76\text{ x }10^{11}\,\text{N.m}^{-2},\ \ \lambda_0=\ 0.75\,\text{s}^{-1},\quad \beta=1.1386\ \text{x }10^{11}\,\text{N.m}^{-2},$$
$$\alpha_t=1.78\text{x}10^{-5}\,\text{N.m}^{-2},\ \ \alpha_4=7.87\text{x}10^{-9}\,\text{N.m}^{-2},\ \ \tau_q=9\text{x}10^{-7}\text{s},$$
$$\alpha_2=3.668\text{x}10^{-5}\,\text{N.m}^{-2},\ \ \alpha_3=1.478\text{x}10^{10}\,\text{N.m}^{-2},\ \ \mu_0=1.9,\ \ \xi_0=0.3,$$
$$\alpha_5=1.1386\text{x}10^{12}\,\text{N.m}^{-2},\quad \tau_\nu=6\text{x}10^{-7}\,\text{s},\quad \tau_T=7\text{x}10^{-7}\,\text{s},$$
$$\varepsilon_0=0.7,\ \ k_1=0.0035\,\text{Ns}^{-1},\ \ k_2=0.0045\,\text{Ns}^{-1},\ \ k_3=0.0055\,\text{NK}^{-1}\text{s}^{-1},$$
$$C_E=183.3\,\text{J.kg}^{-1}.\text{K}^{-1},\ \rho=8954\text{kg.m}^{-3},\ \ \mu_1=0.0085\text{N},$$
$$k_4=0.0065\,\text{Ns}^{-1}\text{m}^2,\ \ K^*=186\,\text{w.m}^{-1}.\text{K}^{-1}.\text{s}^{-1},\ \ \xi=\xi_0+i\,\xi_1,$$
$$\xi_1=0.3,\quad \mu_2=0.0354\ \text{N},\quad b=1.5\text{x}10^{-18}\text{N},\qquad \text{k}^*=10^{11}\text{Nm}^{-2},\ \beta_0=0.95\,\text{s}^{-1},$$
$$k_6=0.0096\,\text{Ns}^{-1}\text{m}^2,\ \ k_5=0.0076\,\text{Ns}^{-1}\text{m}^2,\quad K=150\,\text{w.m}^{-1}.\text{K}^{-1},$$
$$\text{t}=0.002\text{s},\quad \text{P}=5\text{ x}10^8\text{ Nm}^{-2},\ \ \theta_0=0.5,\ \ \psi_0=0.5,\ \ R_p=0.5,\ \ \text{m}=3,$$
$$x=0.5\text{m},\ \gamma_1=0.0009\,\text{N.m}^{-2},\ \ k_0^*=0.85\text{s}^{-1},\ \gamma_0=0.95\text{s}^{-1}.$$

Figs. 1-4 display the graphs for w_2, u, m_{yz} and q_{xy} against the vertical distance y in the absence and presence of the magnetic field. In Fig. 1, it is observed that the presence of the magnetic field decrease the values of microtemperatures component w_2 then increase and again decrease. It takes oscillatory behavior in the range of the distance y. Fig. 2 depicts that the presence of the magnetic field increase the values of the horizontal displacement u then decrease and again increase. Fig. 3 shows that the couple stress m_{yz} begins from a zero value and obeys the boundary condition at y=0 (see Eq. (42)). In the presence of the magnetic field, m_{yz} starts by decreasing to reach its minimum value then increases to reach its maximum value and moves with wave behavior. However, in the absence of the magnetic field, m_{yz} starts by increasing to approach its maximum value then decreases to reach its minimum value, and moves with wave behavior. Fig. 4 displays that the heat flux moment component q_{xy} begins from a zero value and satisfies the boundary condition at y =0 (see Eq. (42)). The values of q_{xy} start by increasing to approach their maximum value then decrease to reach their minimum value, and move with wave behavior.

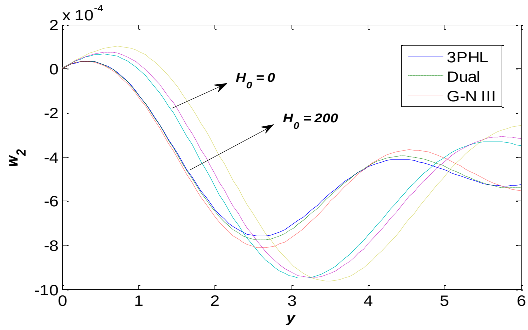

Fig. 1 Variation of the microtemperatures component w_2 with and without the magnetic field effect

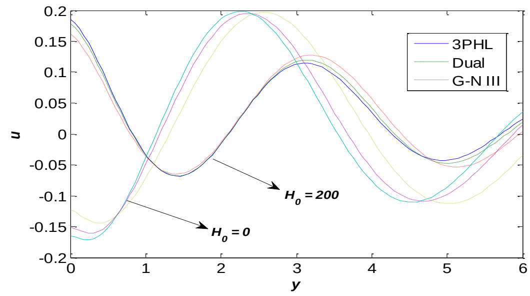

Fig. 2 Variation of the horizontal displacement u with and without the magnetic field effect

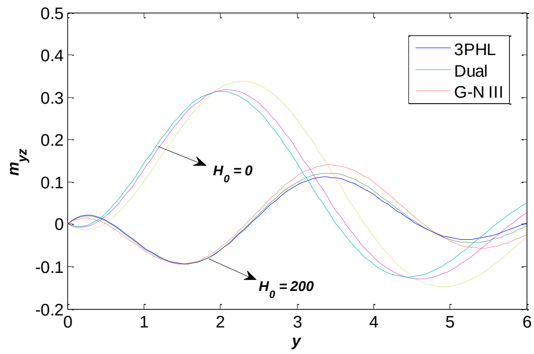

Fig. 3 Variation of the couple stress m_{yz} with and without the magnetic field effect

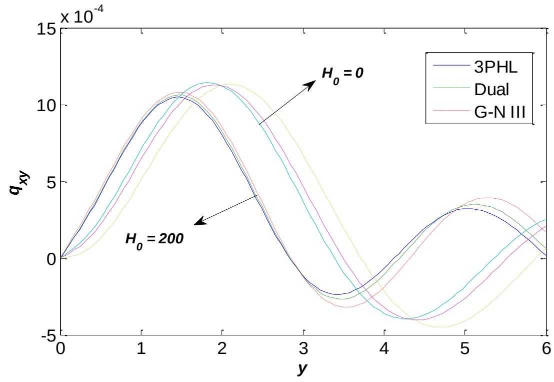

Fig. 4 Variation of the heat flux moment component q_{xy} with and without the magnetic field effect

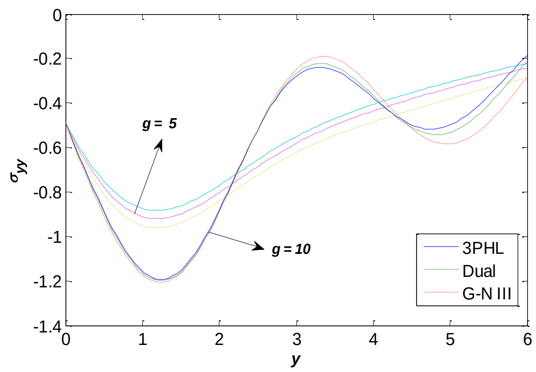

Fig. 5 Variation of the stress component σ_{yy} for different values of the gravity field

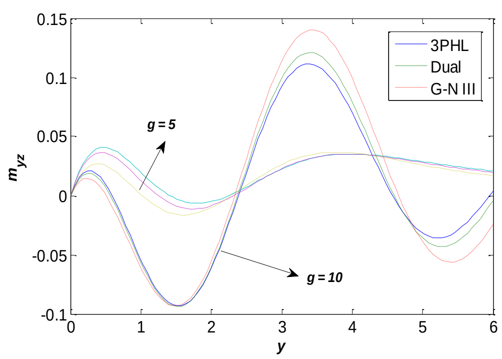

Fig. 6 Variation of the couple stress m_{yz} for different values of the gravity field

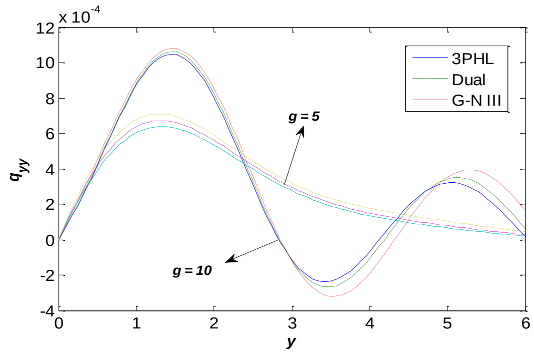

Fig. 7 Variation of the heat flux moment component q_{yy} for different values of the gravity field

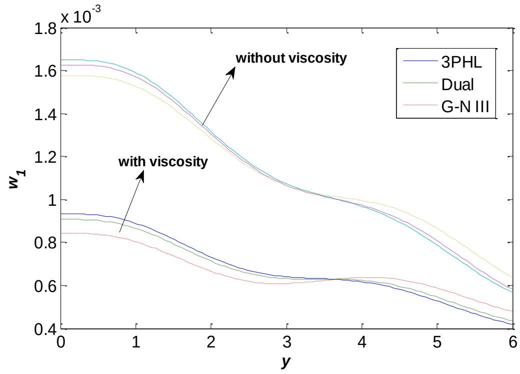

Fig. 8 Variation of the microtemperatures component w_1 with and without the viscosity

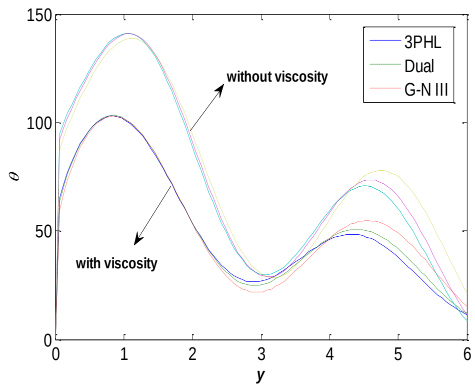

Fig. 9 Variation of the temperature θ with and without the viscosity

Figs. 5-7 introduce the graphs for σ_{yy}, m_{yz}, and q_{yy} against the vertical distance y for different values of the gravity field. Fig. 5 shows that the variation of the stress component σ_{yy} starts from a negative value and satisfies the boundary condition at $y = 0$ (see Eq. (41)). It is observed that the increase of the gravity field decrease the values of σ_{yy} then increase and again decrease. It is observed from Fig. 6 that the increasing of the gravity field decreases the values of the couple stress m_{yz} then increase and again decrease. Fig. 7 depicts that the heat flux moment component q_{yy} begins from a zero value and obeys the boundary condition at $y = 0$ (see Eq. (42)). The increase of the gravity field increase the values of q_{yy} then decrease and again increase. Figs. 8 and 9 introduce the graphs for w_1, θ against the vertical distance y with and without viscosity. It is observed that the presence of the viscosity decrease values of w_1, θ.

Figs. 10-13 are giving 3D surface curves for the couple stress m_{yz}, the heat flux moment component q_{yy}, the thermal temperature θ, and the micro-rotation component φ_3 to study plane waves in a micropolar-thermoelastic solid with microtemperatures under influence of gravity field

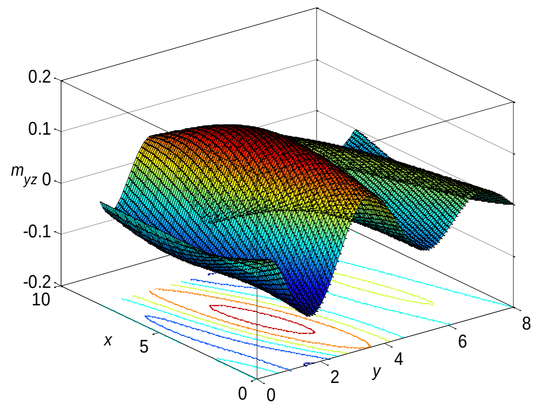

Fig. 10 Variation of the couple stress m_{yz} in the context of three-phase-lag model

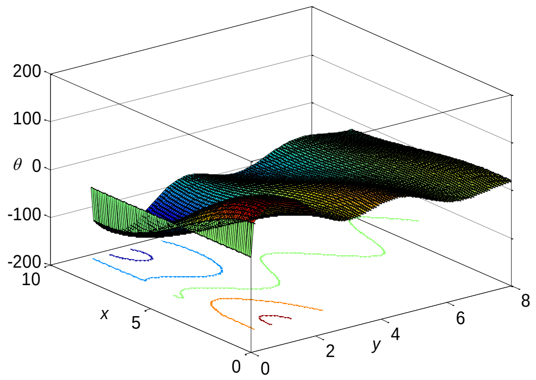

Fig. 12 Variation of the thermal temperature θ in the context of three-phase-lag model

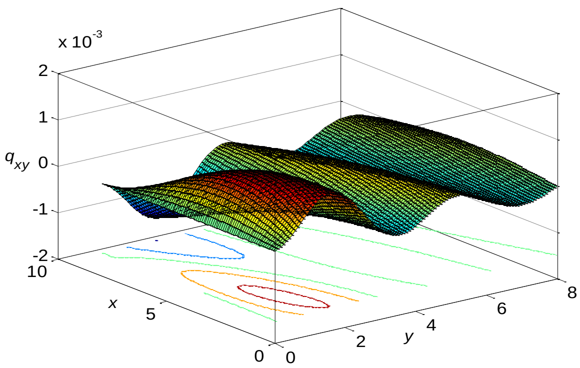

Fig. 11 Variation of the heat flux moment component q_{xy} in the context of three-phase-lag model

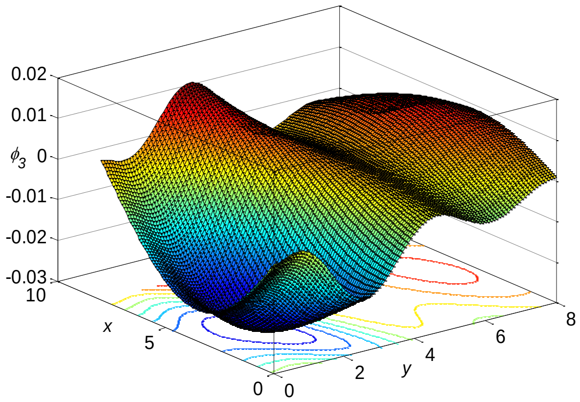

Fig. 13 Variation of the micro-rotation component φ_3 in the context of three-phase-lag model

using the three-phase-lag model. These figures are very important to study the dependence of these physical quantities on the vertical component of distance. All the physical quantities are moving in wave propagation.

6. Conclusions

With the view of theoretical analysis and numerical computation introduce above, we can conclude the following important phenomena:

- The microtemperatures theory plays a great role in the field of geophysics and earthquake engineering and for seismologists working in the field of mining tremors and drilling into the earth's crust.
- The viscosity is an important property and has a great effect on the distribution of the considered physical fields which are pretty clear from Figs. 8 and 9.
- The magnetic field is an effective physical factor in the distributions of the physical fields which are pretty clear from Figs 1-4.
- The gravity field plays an important role in the variations of all the physical fields which are pretty clear from Figs 5-7.
- The harmonic wave analysis is used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.
- Vertical displacement acts an important role in the distributions of the physical fields which are pretty clear from Figs 10-13.

Declaration of conflicting interests

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Appendix

$$A_1=\frac{\lambda(1-\mathrm{i}\zeta\lambda_0)+2\mu(1-\mathrm{i}\zeta\mu_1)+k^*(1-\mathrm{i}\zeta k_0^*)+\mu_0H_0^3}{\rho c^2},\quad A_2=\frac{\beta(1-\mathrm{i}\zeta\beta_0)}{\rho c^2},$$

$$A_3=\frac{\gamma T_0(1-\mathrm{i}\zeta\gamma_0)}{\rho c^2},\quad A_4=\frac{\mu(1-\mathrm{i}\zeta\mu_1)+k^*(1-\mathrm{i}\zeta k_0^*)-\frac{P}{2}}{\rho c^2},\quad A_5=\frac{k^*(1-\mathrm{i}\zeta k_0^*)}{\rho c^2},$$

$$A_6=\frac{\alpha_5}{c^2\eta^2},\quad A_7=\gamma c^2\eta^2(1-\mathrm{i}\zeta\gamma_0),\quad A_8=k^*(1-\mathrm{i}\zeta k_0^*),$$

$$A_9=\mu c^2\eta^2(1-\mathrm{i}\zeta\mu_1),\quad A_{10}=1+\frac{k_4+k_5}{k_6},\quad A_{14}=k_1c^2\eta^2,$$

$$A_{13}=d_3(1-\mathrm{i}\zeta\tau_\upsilon)-\mathrm{i}\zeta d_4(1-\mathrm{i}\zeta\tau_T),$$

$$A_{15}=\gamma T_0(1-\mathrm{i}\zeta\gamma_0)\left(1-\mathrm{i}\zeta\tau_q-\frac{1}{2}\tau_q^2\zeta^2\right)\left(d_1\eta^2+\mathrm{i}\zeta d_2\right),\quad N_1=A_1a^2-A\zeta^2,$$

$$N_2=A_4a^2-A\zeta^2,\quad N_3=A_6a^2,\quad N_4=\alpha_2a^2+\frac{\alpha_3}{c^2\eta^2}-\frac{\mathrm{i}\zeta\alpha_4}{\eta}-\rho\,\alpha_6c^2\zeta^2,$$

$$N_5=\frac{mT_0}{c^2\eta^2},\quad N_6=A_7a^2+2k^*(1-\mathrm{i}\zeta k_0^*)-J\rho c^4\eta^2\zeta^2,\quad N_7=A_8a^2,$$

$$N_8=A_9a^2,\quad N_9=A_{10}a^2+\frac{k_2}{k_6c^2\eta^2}-\frac{\mathrm{i}\zeta b}{k_6\eta},\quad N_{10}=\frac{T_0k_3}{k_6c^2\eta^2},$$

$$N_{11}=\frac{\mathrm{i}\zeta\mu_2}{k_6\eta},\quad N_{12}=a^2+\frac{k_2}{k_6c^2\eta^2}-\frac{\mathrm{i}\zeta b}{k_6\eta},\quad N_{13}=\frac{\mathrm{i}\zeta\mu_1}{k_6\eta},$$

$$N_{14}=A_{13}a^2-\rho C_ET_0\left(1-\mathrm{i}\zeta\tau_q-\frac{1}{2}\tau_q^2\zeta^2\right)\left(d_1\zeta^2+\mathrm{i}\zeta d_2\right),\quad N_{15}=A_{14}a^2,\quad N_{16}=A_{15}a^2,$$

$$N_{17}=mT_0\left(d_1\zeta^2+\mathrm{i}\zeta d_2\right),\quad N_{18}=N_5A_1-A_6A_3,\quad N_{19}=N_5N_1-N_3A_3,$$

$$N_{20}=N_4A_3-A_2N_5,\quad N_{21}=N_{10}A_2-A_3N_{11},\quad N_{22}=A_{15}A_3-A_{13}N_1-A_1N_{14},$$

$$N_{23}=N_1N_{14}-A_3N_{16},\quad N_{24}=N_{17}A_3-A_2N_{14},\quad N_{25}=A_8A_5-A_7N_2-A_4N_6,$$

$$N_{26}=A_5N_7-N_2N_6,\quad N_{27}=N_{25}+A_4A_9N_{13}-A_4A_7N_{12},$$

$$N_{28}=N_{25}N_{12}+N_{26}+A_4N_8N_{13}+A_9N_2N_{13},\quad N_{29}=N_{12}N_{26}+N_{13}N_2N_8,$$

$$N_{30}=N_{13}A_9-A_7N_{12}-N_6,\quad N_{31}=N_{12}N_6-N_{13}N_8,$$

$$N_{32}=N_{18}N_{21}+N_{10}N_{20}A_1+A_3\alpha_2N_1N_{10},\quad N_{33}=N_{19}N_{21}+N_{10}N_1N_{20},$$

$$N_{34}=N_5N_{21}+N_{10}N_{20},\quad N_{35}=\mu_1N_{21}+N_{20}A_{10}+A_3\alpha_2N_9,$$

$$N_{36}=\mu_1N_{21}a^2+N_{20}N_9,\quad N_{37}=N_{10}A_1A_2A_{13}-A_1A_{13}N_{21},$$

$$N_{38}=N_{10}N_{24}A_1-N_{10}N_1A_2A_{13}-N_{22}N_{21},$$

$$N_{39}=N_{10}N_1N_{24}+N_{23}N_{21},$$

$$N_{40}=A_{13}N_{21}-N_{10}A_2A_{13},\quad N_{41}=N_{10}N_{24}+N_{14}N_{21},$$

$$N_{42}=A_{10}N_{24}-N_9A_2A_{13}+A_{14}N_{21},$$

$$N_{43}=N_9N_{24}+N_{15}N_{21},$$

$$N_{44}=N_{37}A_3A_{10}\alpha_2-N_{10}\alpha_2A_1A_{10}A_3A_{13}A_2,$$

$$N_{45}=N_{32}A_{13}A_{10}A_2-N_{10}N_{42}\alpha_2A_1A_3-N_{35}N_{37}+N_{38}\alpha_2A_3A_{10},$$

$$N_{46}=N_{10}N_{43}\alpha_2A_1A_3+N_{32}N_{42}-N_{33}A_{10}A_{13}A_2+N_{36}N_{37}-N_{35}N_{38}-N_{39}\alpha_2A_3A_{10},$$

$$N_{47}=N_{38}N_{36}+N_{35}N_{39}-N_{32}N_{43}-N_{33}N_{42},\quad N_{48}=N_{33}N_{43}-N_{39}N_{36},$$

$$N_{49}=N_{10}\alpha_2A_{10}A_{13}A_3A_2+N_{40}\alpha_2A_3A_{10},$$

$$N_{50}=N_{10}N_{42}\alpha_2A_3-N_{34}A_{10}A_{13}A_2-N_{35}N_{40}-N_{41}\alpha_2A_3A_{10},$$

$$N_{51}=N_{36}N_{40}+N_{35}N_{41}-N_{10}N_{43}\alpha_2A_3-N_{34}N_{42},$$

$$N_{52}=N_{34}N_{43}-N_{36}N_{41},\quad E_0=N_{44}A_4A_7,$$

$$E_1=\frac{-1}{E_0}\left(N_{44}N_{27}+N_{45}A_4A_7\right),$$

$$E_2=\frac{1}{E_0}\left(N_{45}N_{27}+N_{46}A_4A_7-N_{44}N_{28}-a^2g^2A_7N_{49}\right),$$

$$E_3=\frac{1}{E_0}\left(-N_{44}N_{29}+N_{45}N_{28}-N_{46}N_{27}-N_{47}A_4A_7-a^2g^2A_7N_{50}-a^2g^2N_{30}N_{49}\right),$$

$$E_4=\frac{1}{E_0}\left(N_{44}N_{29}-N_{46}N_{28}+N_{47}N_{27}+N_{48}A_4A_7+a^2g^2A_7N_{51}+a^2g^2N_{30}N_{50}+a^2g^2N_{31}N_{49}\right),$$

$$E_5=\frac{1}{E_0}\left(N_{47}N_{28}-N_{46}N_{29}-N_{48}N_{27}-a^2g^2A_7N_{52}-a^2g^2N_{30}N_{51}-a^2g^2N_{31}N_{50}\right),$$

$$E_7=-\frac{1}{E_0}\left(N_{48}N_{29}+a^2g^2N_{31}N_{52}\right),\quad E_6=\frac{1}{E_0}\left(N_{47}N_{29}-N_{48}N_{28}+a^2g^2N_{30}N_{52}+a^2g^2N_{31}N_{51}\right),$$

$$J_{1n}=\frac{\mathrm{i}ag\left(A_7t_n^4+N_{30}t_n^2+N_{31}\right)}{N_{28}t_n^2-A_7A_4t_n^6-N_{27}t_n^4-N_{29}},\quad J_{2n}=\frac{\left(N_2-A_4t_n^2\right)J_{1n}-\mathrm{i}ag}{A_5},$$

$$J_{3n}=\frac{N_{13}J_{2n}}{t_n^2-N_{12}},\quad J_{4n}=\frac{N_{37}t_n^4+N_{38}t_n^2-N_{39}+\mathrm{i}agJ_{1n}\left(N_{40}t_n^2-N_{41}\right)}{A_3\left(A_{10}A_{13}A_2t_n^4+N_{42}t_n^2-N_{43}\right)},$$

$$J_{8n}=-t_n-\mathrm{i}aJ_{1n},\quad J_{5n}=\frac{N_{19}-N_{18}t_n^2+\mathrm{i}agN_5J_{1n}+\mu_1A_3J_{4n}\left(t_n^2-a^2\right)}{A_3\alpha_2t_n^2-N_{20}},$$

$$J_{7n}=\mathrm{i}a-t_nJ_{1n},\quad J_{6n}=\frac{A_1t_n^2-N_1-\mathrm{i}agJ_{1n}+A_2J_{5n}}{A_3},\quad J_{9n}=\mathrm{i}aJ_{4n}-t_nJ_{3n},$$

$$J_{10n}=-\mathrm{i}aJ_{3n}-t_nJ_{4n},$$

$$J_{11n}=\frac{\mathrm{i}a\lambda J_{7n}(1-\mathrm{i}\zeta\lambda_0)-t_nJ_{8n}\left(\lambda(1-\mathrm{i}\zeta\lambda_0)+2\mu(1-\mathrm{i}\zeta\mu_1)+k^*(1-\mathrm{i}\zeta k_0^*)\right)+\beta J_{5n}(1-\mathrm{i}\zeta\beta_0)-\gamma T_0J_{6n}(1-\mathrm{i}\zeta\gamma_0)}{\rho c^2},$$

$$J_{15n}=\frac{(k_5+k_6)t_nJ_{10n}+k_4\left(t_nJ_{10n}-\mathrm{i}aJ_{9n}\right)}{k_5+k_6}.$$

$$J_{12n}=\frac{\mathrm{i}aJ_{8n}\left(\mu(1-\mathrm{i}\zeta\mu_1)+k^*(1-\mathrm{i}\zeta k_0^*)-\frac{P}{2}\right)-J_{2n}k^*(1-\mathrm{i}\zeta k_0^*)-t_nJ_{7n}\left(\mu(1-\mathrm{i}\zeta\mu_1)+\frac{P}{2}\right)}{\rho c^2},$$

$$J_{13n}=\frac{-\gamma_1\eta^2t_nJ_{2n}}{\rho},\quad J_{14n}=\frac{k_5t_nJ_{9n}-\mathrm{i}ak_6J_{10n}}{k_5+k_6}.$$