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An optimized method based on fractal theory to calculate particle size distribution

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Abstract. Particle size distribution has a great influence on the physical properties of granular soils. As an important packing material for engineering, the particle size distribution of granular soils needs to be optimized to yield optimal physical and mechanical performance. There are unknown parameters in existing calculation methods for particle size distribution of granular soils. In order to calculate the optimal particle size distribution curve to reach the densest state under existing conditions without unknown parameters, an optimized method has been proposed based on fractal theory in which all parameters can be obtained by particle screening. With this method, particle size distributions of granular soils can be easily quantified. Compared with experimental data obtained by other researchers, the physical characteristics of soils with a better PSD are better, suggesting the superiority of the proposed method. The fractal dimension of good PSDs calculated in this study ranges from 2.21 to 2.63. Further, laboratory consolidation tests show that the deformation of the prepared specimens calculated by the new method is smaller than that of other specimens with different particle size distributions, which further validates the proposed method.

Keywords: compaction; granular soil; fractal theory; optimized method; particle size distribution

1. Introduction

Granular soils composed of soil particles with various sizes are widely used as engineering materials for road bases, levees and embankment dams. The physical and mechanical behaviors of granular soils are highly affected by the particle size distribution (PSD) (Bayat *et al*. 2015, Minh and Cheng. 2013), which is typically presented as the percentage of the total mass of soil occupied by a given size fraction (Ghasemy *et al*. 2019). It is a fundamental and important soil attribute used by civil engineers to quantify the properties of granular materials (Krzysztof *et al*. 2014, Wang and Li. 2015). In the fields of soil physical and chemical characteristics, the PSD of soils needs to be optimized to yield optimal performance, especially in terms of compactness. For this purpose, many laboratory tests are conducted to obtain the optimal PSD.

Many attempts have been made to characterize PSD curves using different mathematical models, such as power-law models (Lassabatère *et al*. 2006), logarithm models (Zhuang *et al*. 2001), hyperbolic models (Vipulanandan and Ozgurel. 2009), statistical distribution models (Zobeck *et al*. 2015) and fractal models(Bird *et al*. 2010). Among them, the fractal model has several advantages, such as a simpler form and fewer parameters. Many granular geomaterials resulting from weathering or fragmentation follow a power law frequency distribution of sizes, creating fractal sets (Turcotte and L. 1986), which implies that the probability of any size range to break is the same (following the concept of scale invariance) (Mandelbrot. B. B. 1983). This concept has been increasingly used in soil mechanics in models for characterizing particle size distributions (Altuhafi *et al*. 2010, Altuhafi. 2011a, Altuhafi. 2011b, Altuhafi and Coop.2011). However, the current calculation methods contain some unknown parameters, and it is therefore difficult to obtain an accurate grading curve. An effective method to determine the optimal PSD is needed.

Fractal theory is an effective method for delineating the PSD curve of soils and many efforts have been made to study the relationship between fractal dimension and PSD. Bird pointed out that the relationship between the mass and particle size distribution follows a power law distribution (Bird *et al*. 2010), which is described as

$$M(d \le d_i) = cd_i^{3-D} \tag{1}$$

where $M(d<d_i)$ is the mass of particles smaller than d_i, D is the fractal dimension of the PSD, and c is a composite scaling constant. Perrier modified Eq. (1) using a position parameter k, to agree with the actual situation (Perrier *et al*. 1999), and the gradation curve satisfies

$$M(d \le d_i) = k + cd_i^{3-D} \tag{2}$$

When k=0, Eq. (2) is equivalent to Eq. (1). Millán used a fractal model to divide the soil particle size distribution into two scaling regions, and the fractal dimensions of the two regions are different (Millán *et al*. 2003). The overall mass-particle relationship is the superposition of the two regional relationships, as shown as follows:

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$$M(d \le d_i) = c_1 d_i^{3-D_1} (d_i \le d_c) + c_2 d_i^{3-D_2} (d_i \ge d_c) \quad (3)$$

where d_c is the demarcation of the two regions of soil, c_1 is the composite scaling constant, D_1 is the fractal dimension of the first region, c_2 is the composite scaling constant, and D_2 is the fractal dimension of the first region.

Zhang (2017) studied the ultimate particle size distribution of uniform and gap-graded soils using the fractal theory. The number and average size of soil particles can be obtained by the following:

$$N(d_1,d_2) = \frac{3M(d_1,d_2)(D-3)}{4AR\rho\pi D} \frac{d_1^{-D} - d_2^{-D}}{d_1^{3-D} - d_2^{3-D}} \quad (4)$$

$$d_{mean} = \left(\frac{3M(d_1,d_2)}{4ARN(d_1,d_2)\pi\rho} \right)^{1/3} \quad (5)$$

where $N(d_1,d_2)$ is the number of soil particles with sizes ranging from d_1 to d_2 ($d_1 \le d_2$), $M(d_1,d_2)$ is the mass of the soil particles whose sizes are between d_1 and d_2, d_{mean} is the average diameter of soil particles, AR is the ratio between the short axis and the long axis (representing the property of shape factor), and ρ is the density of soil particles. Eq. (4) and Eq. (5) agree well with the real situation in theory, but it is hard to apply this framework in engineering because the number and average diameter of soil particles cannot be measured. Zhu optimized the particle size distribution of rockfill materials, according to the relationship between the fractal dimension, the average particle size, and the maximum dry density (Zhu *et al*. 2012). Bayat summarized the relationship between particle mass and size, and found that the fractal model proposed by Millán has the highest coincidence with the experimental particle distribution curve (Bayat *et al.* 2015, Millán *et al*. 2003).

Many investigations have been conducted to study the grading method of particles, but most of the current methods for soil gradation contain unknown parameters. Further, most formulae for calculating soil gradation are difficult to apply and are impractical for engineering. In practice, it is generally considered that the gradation is adopted when the inhomogeneity coefficient Cu is larger than 10 and curvature coefficient Cc is from 1 to 3. However, the parameters for determining Cu and Cc cannot fully represent the entire gradation of soil particles. Here, a new method for determining the optimum gradation of packing materials is proposed based on the fractal theory.

2. The fractal expression of PSD

The PSDs of packing materials in nature usually exhibit fractal properties (Yasrebi *et al*. 2014). Mandelbrot (1983) pointed this out in two-dimensional PSDs in the 1980s. Tyler and Wheatcraft (1992) extended the two-dimensional expression to three-dimensions based on Mandelbrot's work, and the fractal relationship between particle size and volume is calculated as follows.

$$V(d > d_i) = C_v[1-(d_i / d_{max})^{3-D}] \quad (6)$$

where $V(d>d_i)$ is the volume of particles whose sizes are larger than d_i, C_v is the volume factor, D is the fractal dimension of the PSD, and d_{max} is the maximum particle size. Although the three-dimensional expression of the PSD is closer to the actual state than the two-dimensional one, it is difficult to be used in engineering. The mass-size distribution, not the volume-size distribution, is used in practice, and the mass-size expression of particles can be obtained by

$$\rho V(d > d_i) = M(d > d_i) = \rho C_v[1-(d_i / d_{max})^{3-D}] \quad (7)$$

When the lower limit of the particle size d is set to 0, the total mass of the packing materials M_{max} is:

$$M_{max} = M(d > 0) = \rho C_v[1-(d / d_{max})^{3-D}] = \rho C_v \quad (8)$$

The ratio of the mass over the screen size d_i to the total mass can be obtained from the following:

$$M(d > d_i) / M_{max} = 1-(d_i / d_{max})^{3-D} \quad (9)$$

When the gradation curve is expressed as the relationship between the ratio of the mass of particles passing through a sieve to the total mass of all particles, Eq. (9) can be modified to another expression:

$$M(d \le d_i) / M_{max} = (d_i / d_{max})^{3-D} \quad (10)$$

Eq. (10) is the expression of particle mass-particle size relationship based on the fractal theory. The mass under a sieve and its corresponding percentage content can be plotted as a straight line on a log-log scale, as follows:

$$lg\left(M(d \le d_i) / M_{max}\right) = (3-D)lg(d_i / d_{max}) \quad (11)$$

3. Determination of the optimum PSD based on the fractal theory

According to the traditional fractal expression of a PSD (Eq.11), when the particle size d_i is the minimum value d_{min}, the mass of the particles smaller than d_{min} is zero. This obviously does not agree with reality. Actually, the minimum value of particle diameter depends on the sieve pores (sieving method), so the mass of the particles smaller than d_{min} would not be zero. These are classified as fine-grained soils. It is worth noting that the true minimum particle size in practical engineering is difficult to measure except by water analysis method. Further, due to the uncertainty of fractal dimension D, the traditional gradation curve based on the fractal theory is usually difficult to apply in engineering. Therefore, Eq. (11) needs to be improved.

Based on the fractal theory, we propose a formula for calculating the optimal particle size distribution of packing materials. The PSD curve fitting the fractal theory constantly crosses point A and B:

$$A = \left(lg(100\frac{d_{min}}{d_{max}}), lg(100\frac{M_{min}}{M_{max}}) \right) \quad (12)$$

$$B=\left(lg(100\frac{d_{max}}{d_{max}}),lg(100\frac{M_{max}}{M_{max}})\right) \tag{13}$$

Note that the point B is equal to (2,2), and thus the slope of the gradation curve crossing point A and B in double logarithmic form is expressed as follows:

$$\frac{[2-lg(100M_{min}/M_{max})]}{[2-lg(100d_{min}/d_{max})]}=\frac{lg(M_{min}/M_{max})}{lg(d_{min}/d_{max})} \tag{14}$$

From Eq. (14), the logarithmic formula of the PSD curve shown as follows can be obtained by Eq. (12), Eq. (13) and Eq. (14).

$$lg(100M/M_{max})=\frac{[2-\lg(100M_{min}/M_{max})]}{[2-\lg(100d_{min}/d_{max})]}[(lg(100d/d_{max})-2)]+2 \tag{15}$$

Eq. (15) can be simplified as follows:

$$lg(M/M_{max})=\frac{lg(M_{min}/M_{max})}{lg(d_{min}/d_{max})}lg(d/d_{max}) \tag{16}$$

Converting formula (16) into a power-law form gives:

$$\frac{M}{M_{max}}=\left(\frac{d}{d_{max}}\right)^{\frac{lg(M_{min}/M_{max})}{lg(d_{min}/d_{max})}} \tag{17}$$

According to Eq. (17), the fractal dimension D can be computed as follows:

$$D=3-\frac{lg(M_{min}/M_{max})}{lg(d_{min}/d_{max})} \tag{18}$$

The fractal dimension D ranges from 2 to 3 in nature, thus the slope of the gradation curve should vary between 0 and 1. Therefore, the relationship between d_{min} , d_{max} , d_{min} and d_{max} can be described as:

$$\frac{d_{min}}{d_{max}}<\frac{M_{min}}{M_{max}} \tag{19}$$

The least squares method is used to obtain the regression objective function (Xiaoming *et al*. 2018):

$$F_{obj}=\sum_{j=1}^{k}\left\{lg\left[\sum_{i=1}^{n}M_i\left(d<d_j\right)\right]-lg[M_{max}\cdot P(d<d_j)]\right\}^2 \tag{20}$$

The fitting degree between the theoretical formula and actual data can be calculated by the multiple determination coefficient R^2 (Xiaoming *et al*. 2018):

$$R^2=1-\frac{\sum_{j=1}^{k}\left[M_{max}\cdot P\left(d<d_j\right)-\sum_{i}^{n}M_i(d<d_j)\right]^2}{M_T\cdot\sum_{j=1}^{k}\left[P\left(d<d_j\right)-\frac{1}{k}\sum_{j=1}^{k}P(d<d_j)\right]^2} \tag{21}$$

where k is the number of different apertures of the screen, $P(d<d_j)$ is the particle mass content through the number i pore size d_j of the sieve, n is the number of the fractal dimension D, and M_i $(d<d_j)$ is the particle mass through the number i pore size d_j of the sieve.

The method for calculating the optimal PSD in this paper is based on one-dimension fractal theory, thus n, F_{obj} and R^2 are equal to 1, 0 and 1, respectively. It is deemed that the calculated PSD reaches an ideal state when the porosity is at its the minimum value, and the PSD is the optimal gradation.

According to Eq. (17), the effective particle size d_{10}, the median particle size d_{30}, and the restricted particle size d_{60} can be calculated by

$$d_{60}=d_{max}\times\left(0.6\right)^{\frac{lg(d_{min}/d_{max})}{lg(M_{min}/M_{max})}} \tag{22}$$

$$d_{30}=d_{max}\times\left(0.3\right)^{\frac{lg(d_{min}/d_{max})}{lg(M_{min}/M_{max})}} \tag{23}$$

$$d_{10}=d_{max}\times\left(0.1\right)^{\frac{lg(d_{min}/d_{max})}{lg(M_{min}/M_{max})}} \tag{24}$$

The inhomogeneity coefficient Cu is as follows:

$$Cu=\frac{d_{60}}{d_{10}}=6^{\frac{lg(d_{min}/d_{max})}{lg(M_{min}/M_{max})}}=6^{\frac{1}{3-D}}>10 \tag{25}$$

and the curvature coefficient Cc can be obtained by

$$Cc=\frac{d_{30}\cdot d_{30}}{d_{60}\cdot d_{10}}=1.5^{\frac{lg(d_{min}/d_{max})}{lg(M_{min}/M_{max})}}=1.5^{\frac{1}{3-D}}=1\sim3 \tag{26}$$

According to the relationship between Cu and Cc, the slope of the gradation curve can be calculated by

$$0.37<k=3-D=\frac{lg(M_{min}/M_{max})}{lg(d_{min}/d_{max})}<0.79 \tag{27}$$

Therefore, the fractal dimension D is from 2.21 to 2.63.

The ratio of the mass of the minimum particle size to the mass of the maximum particle size M_{min} / M_{max} can be calculated by

$$\left(\frac{d_{min}}{d_{max}}\right)^{0.79}<\frac{M_{min}}{M_{max}}<\left(\frac{d_{min}}{d_{max}}\right)^{0.37} \tag{28}$$

The upper and lower envelopes of the curves of proper gradation for packing can be obtained according to Eq. (27), as shown in Fig. 1.

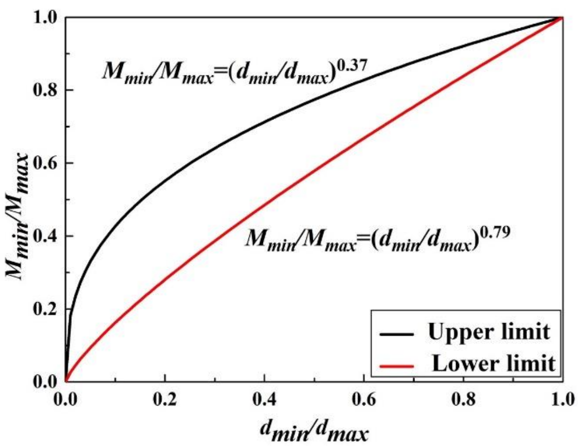
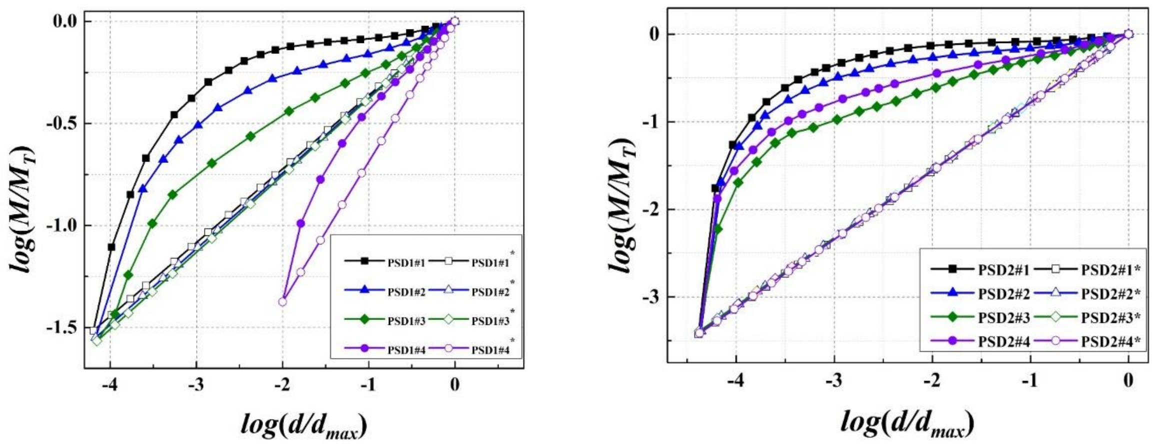


Fig. 1 Envelope curve of proper



(a)The first group experimental data obtained from Li (2013)

(b)The second group experimental data obtained from Li (2013)

Fig. 2 Comparison of different samples with different PSDs and their linear computation

Table 1 The deformation of different samples obtained from Li (2013).

Sample	PSD 1#1	PSD 1#2	PSD 1#3	PSD 1#4	PSD 2#1	PSD 2#2	PSD 2#3	PSD 2#4
Deformation(mm)	6.6	6.12	8.72	4.72	8.04	6.47	6.29	7.13

In the application of traditional fractal theory to the PSD curve, the target fractal dimension cannot be calculated, thus the optimal gradation curve of particles cannot be obtained. The advantage of formula (18) is that it can calculate the target fractal dimension according to the maximum and minimum particle size and the corresponding mass percentage content. If the curve of the double logarithm is closer to a straight line, the gradation is better.

4. Model validation

A total of 4 groups of data deemed to have an optimal PSD were obtained from previous experiment conducted by Li (2013) and Ovalle (2014) and used for benchmark testing. PSD is the experimental data obtained by double logarithm processing and PSD* is the data obtained using formula (16). The results in Fig.2 and Fig.3 show that the distance between the better grading curve PSD data and its linear grading curve PSD* obtained by formula (16) is small, supporting the usefulness of the formula proposed in this paper.

The data in Fig.2 (a) and Fig.2 (b) are obtained from direct shear tests for different PSDs of mixtures up to 6.0 mm by Li (2013). The content of coarse particles and fine particles has an important influence on the mechanical properties of soils. The deformation of different samples is shown in Table1. Compared with PSD1#1, PSD1#2 and PSD1#3, the deformation of the sample with PSD1#4 is the smallest under the same external load of 400kPa. In Fig.2 (a), the distance between the PSD1#4 and PSD1#4* is also the smallest. Compared with the PSD2#1, PSD2#2 and PSD2#4, the deformation of the sample with PSD2#3 is the smallest under the same external load of 400kPa. In Fig.2 (b), the distance between the PSD1#4 and PSD1#4* is also the smallest.

The data in Fig. 3 (a) and Fig. 3 (b) are obtained from particle crushing tests for different PSDs of mixtures by Ovalle (2014). It is believed that the strength of soil changes with particle size. The experimental results show that the specimens with PSD1#1 in Fig. 3 (a) and PSD2#1 are more difficult to be crushed compared to those with PSD1#2 and PSD2#2. The results in Fig.3(a) show that the distance between PSD1#1 and PSD1#1* is smaller than that between PSD1#2 and PSD1#2*. The results in Fig. 3 (b) show that the distance between PSD2#1 and PSD2#1* is smaller than that between PSD2#2 and PSD2#2*.

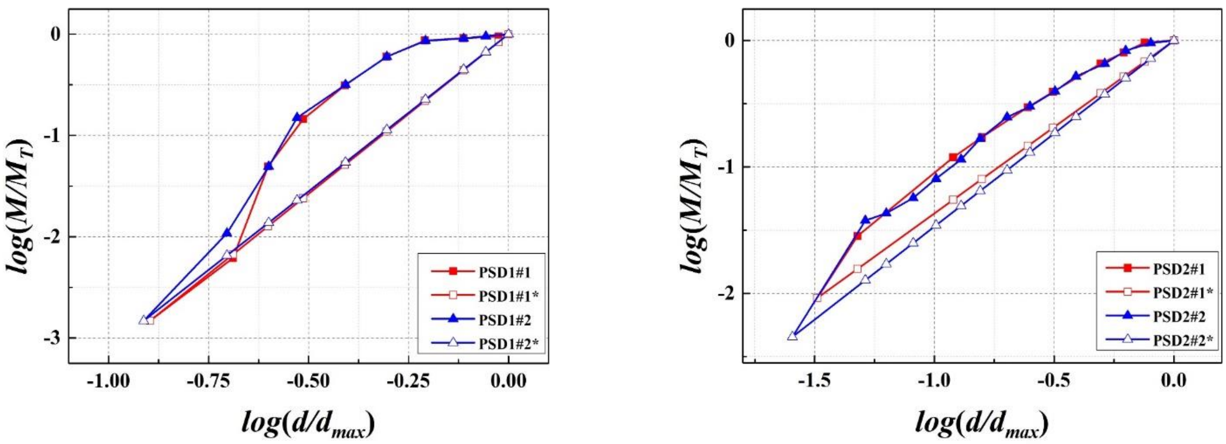
At present, traditional methods for calculating the optimal PSD contain parameters that are difficult to obtain or unrealistic. Additionally, the gradation range used in design specifications is relatively wide. In practice, it is difficult to determine the optimum gradation of packing materials. Compared with traditional methods, the method proposed in this paper contains no unknown parameters, and the calculated PSD can better meet the requirements for compactness.

5. Laboratory consolidation experiment

5.1 Low-pressure consolidation experiment

The test material used was rounded sand with low inclination. The consolidation instrument was selected as the test instrument, where the inner diameter of the ring cutter is 61.8 mm and the height is 20 mm. A displacement sensor with an accuracy of 0.2% of the total range for deformation was used as the measuring equipment.

A total of 10 samples with different PSDs were prepared for the low-pressure consolidation test, and they were divided into two groups (group 1 and group 2), as listed in Tables 2 and 3. The weight of each sample was 100 g. In



(a) The first group experimental data obtained from Ovalle (2014)

(b) The second group experimental data obtained from Ovalle (2014)

Fig. 3 Comparison of different samples with different PSDs and their linear computation

(a) Particle sieving

(b)Sample preparation

(c) Laboratory consolidation

Fig. 4 Sketch of the consolidation progress

Table 2 The PSDs of samples in group 1

Sample	Particle mass smaller than the particle size (g)							
	4.75mm	2.36mm	2.0mm	1.18mm	0.6mm	0.3mm	0.15mm	0.075mm
S1#1	100	60.26	53.70	36.56	22.39	13.60	8.32	5
S1#2	100	60	50	40	30	20	10	5
S1#3	100	60	55	45	30	25	10	5
S1#4	100	90	60	50	30	20	10	5
S1#5	100	80	60	40	30	15	10	5

Table 3 The PSDs of samples in group 2

Sample	Particle mass smaller than the particle size (g)						
	4.75mm	2.36mm	2.0mm	1.18mm	0.6mm	0.3mm	0.15mm
S2#1	100	63.1	56.2	38.9	25.1	15.7	10
S2#2	100	60	50	40	30	20	10
S2#3	100	60	55	45	30	25	10
S2#4	100	90	60	50	30	20	10
S2#5	100	80	60	40	30	15	10

group 1, the minimum sieving aperture was 0.075 mm and the maximum sieving aperture was 4.75 mm. According to Eq. (28), the mass percent of the particle size less than 0.075 mm was 3.7%~21.5%, and a value of 5% was selected here. Therefore, the fractal dimension of the best grading sample is 2.28 according to Eq. (18). The moisture content of each sample was set as 8% because that the dry density of cohesionless soil with a moisture content of 8%

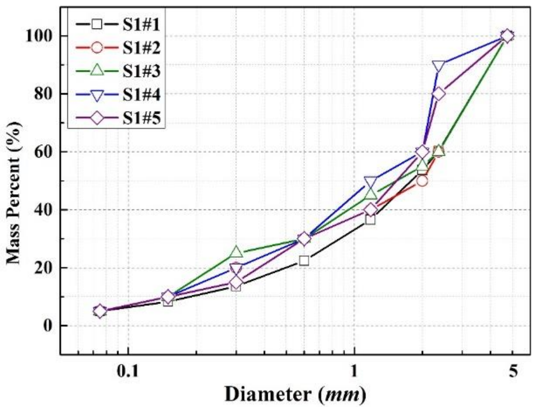


Fig. 5 PSDs of samples in group 1

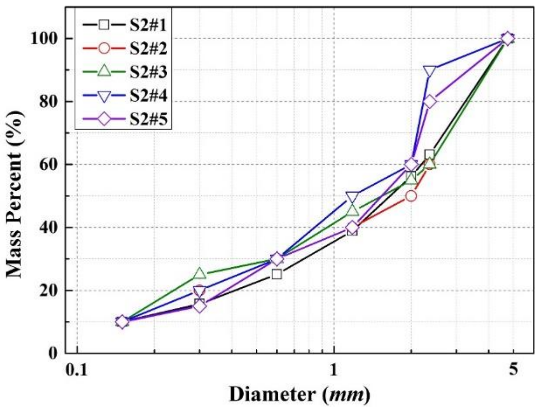


Fig. 6 PSDs of samples in group 2

Table 4 *Cu* and *Cc* of different samples in group 1 and group 2

Sample	1#1	1#2	1#3	1#4	1#5	2#1	2#2	2#3	2#4	2#5
Cu	11.65	15.7	15.7	13.3	13.3	14.7	15.7	15.7	13.3	13.3
Cc	1.74	1.02	1.02	1.20	1.20	1.84	1.02	1.02	1.20	1.20

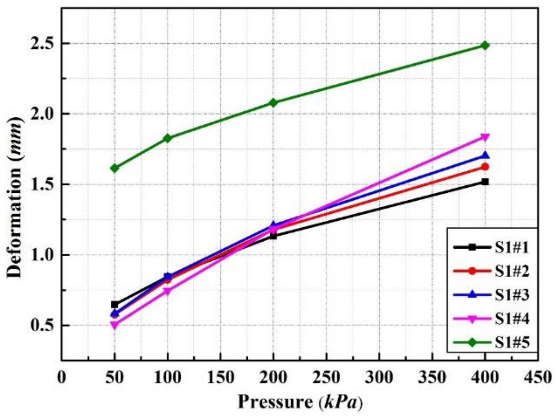


Fig. 7 Deformation of samples in group 1 under different pressures

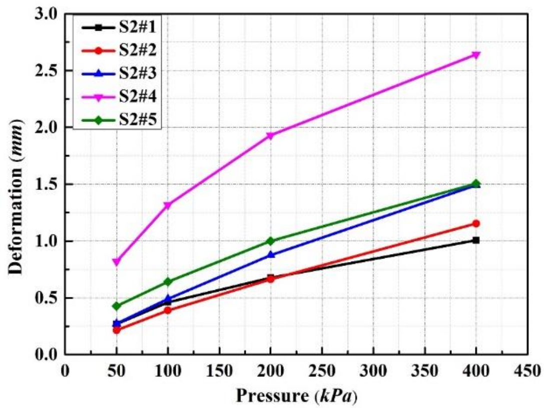


Fig. 8 Deformation of samples in group 2 under different pressures

is close to the maximum dry density. The PSD of S1#1 was computed by the method proposed in this paper and the other samples were well graded. In group 2, the minimum and maximum sieving aperture were 0.15 mm and 4.75 mm, respectively. The mass percent of particle size less than 0.15 mm was 6.5%~27.5% based on Eq. (28), and a value of 10% was selected in this paper. The fractal dimension of the best grading sample was 2.33. PSD of S1#1 was computed by the proposed method and the other samples were well graded. The sample with the optimal PSD had the minimum deformation. The inhomogeneity coefficient influences the physical characteristics of granular aggregates (Yifei Sun *et al, 2019*). The *Cu* and *Cc* of each specimen calculated are listed in Table 4, and the inhomogeneity coefficients of S1#2, S1#4, S2#2, S2#4 are equal to S1#3, S1#5, S2#3, S2#5, respectively.

A fast consolidation test was conducted for the 10 specimens in group 1 and group 2. The maximum consolidation pressure was 400 *kPa*. The deformation results of each sample (Fig. 6 and Fig. 7) show that the deformation increases with increasing consolidation pressure. Among the samples in group 1 under the final consolidation pressure of 400 *kPa*, the deformation of S1#1 was the smallest (1.517 mm), whereas the deformation of S1#5 was the largest (2.485 mm). For the samples in group 2, the final deformation of S2#1 was the smallest (1.007 mm), whereas the final deformation of S2#4 was the largest (2.642 mm). The results show that the PSD has a significant influence on the compactness of packing materials, and the sample with the optimal PSD presents the highest compression. The specimen prepared according to the proposed method exhibits the lowest compressibility when the pressure exceeds 200 *kPa*, confirming the validity of the proposed method.

In practice, the inhomogeneity coefficient *Cu* and curvature coefficient *Cc* are widely used to evaluate the gradation of soils. However, *Cu* and *Cc* are related to three particle sizes (i.e., d_{10}, d_{30} and d_{60}), which cannot characterize the complete particle size distribution. As shown in Fig. 6 and Fig. 7, even if the packing materials have the same *Cu* and *Cc*, their deformations are different, suggesting that there are shortcomings to judge gradation using *Cu* and *Cc*.

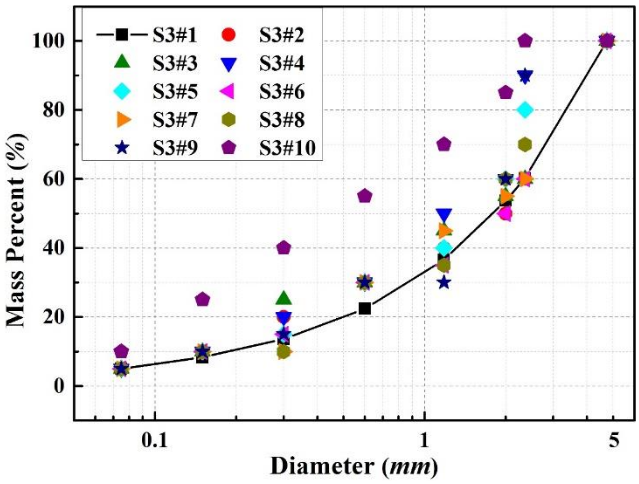


Fig. 9 PSDs of different samples in the high-pressure consolidation test

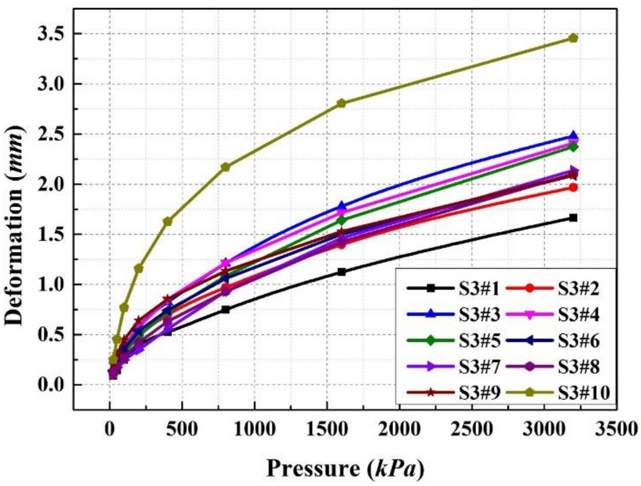


Fig. 10 Deformation of different samples under various pressures in the high-pressure consolidation test

5.2 High-pressure consolidation experiment

A total of 10 samples were prepared for the high-pressure consolidation test. In the test, the maximum pressure was set as 3200 *kPa*. Particle breakage will not occur under this pressure (Shen *et al*. 2018), which ensures that the PSDs of the specimens does not change during the consolidation process. S3#1 is the best graded specimen according to the proposed method, and the fractal dimension of the PSD of S3#1 was 2.28. The other samples were also well graded. The gradation of each sample is shown in Fig. 8.

The deformations of the ten specimens under different pressures are shown in Fig. 9. With increasing external pressure, the deformations became larger. The deformations of specimens with different PSDs were different under the same pressure. Compared with other samples, the final deformation of S3#1 under a pressure of 3200 *kPa* was the smallest (1.665 mm) whereas the deformation of S3#10 was the largest (3.456 mm). Figure 9 also showed that the growth rate of deformation of S3#1 was the smallest among all the samples.

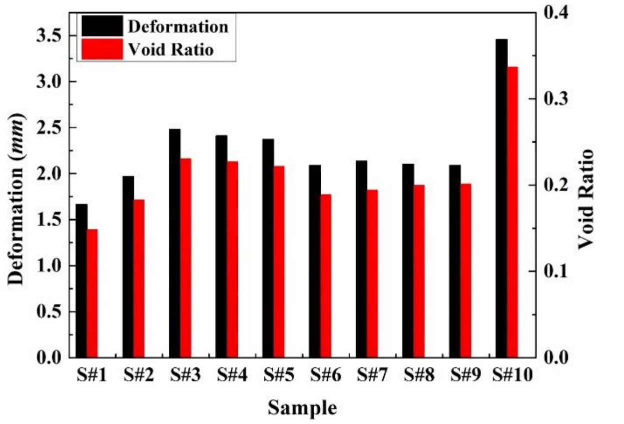


Fig. 11 Deformations and pore ratios of different samples

The initial void ratio of each sample before the consolidation test is shown in Fig.10. It can be concluded that the final deformation of the sample has a positive relationship with the initial void ratio. The sample with a larger porosity tends to have a larger deformation under the consolidation test. As shown in Fig.10, S3#1 had the smallest porosity (0.15), smaller than 0.2, which is equivalent to smallest porosity obtained by Wu W and Li W (2017). The porosity of S3#10 was 0.34, which was the largest one measured. Fig.10 also suggests that the PSD of the sample prepared by the proposed method is the optimal one of all the samples since it is in the densest state.

6. Discussions

The results of the consolidation test show that PSD has a significant influence on the density of granular soil samples. Even if the samples have the same *Cu* and *Cc*, the deformation varies greatly. When the applied external pressure is small, that is, below 400 *kPa*, the difference of the deformation of different samples is not obvious. This may be because the effect of a small load on the sample is to move soil particles until they reach a new stabilized position. However, with increasing external load, the difference between deformations of different samples becomes notable. When the external pressure is 3200 *kPa*, the deformation of S3#1 was almost the half that of S3#10.

Particles are divided into three types according to their sizes: coarse particles (5~1 mm), medium particles (1~0.25 mm) and fine particles (≤0.25 mm). The structure and morphology of each type of particle is different: (1) the aspect ratio of coarse particles is small, (2) medium particles are mostly dendritic or flaky, with a higher aspect ratio, and (3) fine particles are mostly flaky, with a much higher aspect ratio (Shen *et al*. 2018; Tae-Woong Park *et al.* 2018). The sand used for the experimental test in this paper includes coarse particles, medium particles and fine particles, exhibiting multi-sized characteristics. The applied external loads used do not cause breakage of particles, which means they will not lead to any change of gradation.

When the number of coarse particles is much more than that of fine particles, the whole system reflects the properties of coarse particles which act as bearing skeleton

(Matheson and M. 1986). In this case, the number of fine particles is not enough to fill the pores, producing much porous space that could be compressed. Therefore, the compressibility of the whole particle system is large (Hyodo Masayuki *et al.* 2017). When the number of coarse particles is much less than that of fine particles, the whole particle system exhibits the properties of fine particles, and the coarse particles are completely wrapped in the fine particles. In this condition, the skeleton cannot be formed. As a result, the ability of the system to withstand external loads is reduced, and the compressibility of the whole particle system is large. Simulations reveal that in an assembly of coarse and fine particles, neither all of the fine particles nor all of the coarse ones participate in the force chains to carry the external loads (Bei-Bing Dai *et al.* 2019, Gong Jian and Liu Jun, 2017). Only when the number of coarse and fine particles reaches the optimum balance, coarse particles can form a complete skeleton, while fine particles fill the pores. Therefore, coarse and fine particles interact with each other to transmit external force together, and the porosity that can be compressed is the minimized and the soil sample is in a dense state.

The PSD calculated by the proposed method can meet the requirement of maximum compactness and good gradation. In this case, medium particles could fill the pores of coarse particles, and fine particles fill the pores of medium particles. Thus, the relative content of particles with different sizes is optimized and the system is the densest.

7. Conclusions

In this study, an optimized method for the particle size distribution of multi-sized granular soils base on fractal theory has been proposed. The following conclusions can be obtained by the laboratory tests and the model validation.

- When there is only one fractal dimension, the PSD of granular soils can reach the densest condition with the maximum density.
- The formula proposed can meet the requirements of optimal gradation. When the value of the fractal dimension of the PSD ranges from 2.21 to 2.63, the PSD of granular soils is excellent.
- When the external load is small, the compactness advantage of the sample prepared according to the formula proposed is not obvious, but with increasing external load, the compactness advantage can be better appreciated.

Notations

M: the mass of particles smaller than d
d_{min}: the smallest diameter of the particle
d_{max}: the largest diameter of the particle
M_{min}: the mass of the particle size smaller than d_{min}
M_{max}: the total mass of the particles
D: the fractal dimension

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