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# Pareto optimality and game theory for pile design having conflicting objectives

Shantanu Hati[a] and Sarat K. Panda*

*Department of Civil Engineering, Indian Institute of Technology (ISM) Dhanbad, India*

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**Abstract.** Based on concept of Pareto-optimal solution and game theory associated with Nash non-cooperative and cooperative solution, a mathematical procedure is presented for optimum design of axially loaded pile structure. The decision making situation is formulated as a constrained optimization problem with two objectives of contradictory in nature. The factor of safety is taken as the design variable. Geometric constraints are considered by imposing a lower and upper bound on the design variable. Two objectives considered are: maximization of ultimate load carrying capacity of pile and minimization of associated cost. The generation of Pareto-optimal solution and methodology based on game theory concept is described. The design problem is mathematically formulated as two-person game. To obtain the starting point of game, Nash non-cooperative solution or Nash equilibrium solution is evaluated for an irrational play. For cooperative game, a negotiation model is developed for overall benefit of all players. Game is terminated when the optimal trade-off between two objectives is reached with maximization of supercriterion. Two numerical examples of practical interest are solved to demonstrate the methodology.

**Keywords:** game theory; Nash cooperative game; Nash non-cooperative solution; pareto-optimal; pile structure; supercriterion

## 1. Introduction

A typical single objective optimization problem can be stated as:

Find an optimum design vector ( $\vec{X}^*$ ) that will minimize an objective function $f(\vec{X})$ satisfying a set of system constraints represented by

$$g_j(\vec{X}) \le 0, \quad \text{where } j=1,2,..m \tag{1}$$

However, in engineering system design there often exist many situations where single criterion is not enough to represent the system performance. In that case system involves multiple objectives. Natures of these objectives are often conflicting. Maximization of one objective leads in a decrease in the value of at least one objective. Mathematically the multi-objective optimization problem (MOOP) can be represented as:

Evaluate the optimum design vector ( $\vec{X}^*$ ) that minimizes $f(\vec{X})$ subject to

$$g_j(\vec{X}) \le 0, \quad \text{where } j=1,2,..m \tag{2}$$

where, $f(\vec{X})$ indicating vector of objective functions and expressed as

$$f(\vec{X}) = \begin{Bmatrix} f_1(\vec{X}) \\ f_2(\vec{X}) \\ . \\ . \\ f_N(\vec{X}) \end{Bmatrix} \tag{3}$$

Generally, in a single design it is impossible to obtain minima of all objectives present in the system. Therefore a compromise solution is required between the minima of all objectives. The optimum solution has a sound physical meaning for single objective optimization problem while it is lost in case of a MOOP.

In presence of several objectives having conflicting nature, designer's task is to find out an optimal trade-off. As a result the optimal solution is evaluated through different solution concepts. A variety of techniques with applications of multi-objective optimization have been introduced in the past several years. The earliest work involves the consideration of multiple objectives in mathematical programming was tackled by Kuhn and Tucker (1951). The progress in the field of multi-criteria optimization was discussed by Hwang and Masud (1979) and Stadler (1986). Another approach through conversion of a single objective optimization problem into a multi-objective problem by treating the constraint functions as additional objectives was mentioned (Duckstein 1981). The weighing method, the constraint method and the minimax approaches for generating Pareto optima has been used by Koski (1981) along with application in engineering problems.

A comparative study of computational efficiencies of the weighting, noninferior set estimation and constraint

*Corresponding author, Associate Professor
E-mail: sarat@iitism.ac.in
[a]Ph.D. Student
E-mail: shantanuhati@gmail.com



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methods has been presented along with capabilities of the methods to produce an approximation Pareto-optimal set (Balachandran and Gero 1984). The multi-objective design has also been solved by Adali (1983) in the form of a trade-off between the objectives and presented in the graphical forms. A numerical method to evaluate Pareto-optimal solution has been proposed to obtain practical result (Koski and Silvennoinen 1982). The application of variety of multi-objective optimization techniques was discussed in the context of solving engineering problem (Rao 1984).

A new hybrid approach based on Taguchi's method and a genetic algorithm is presented by Karen *et al.* (2006) for a better Pareto-optimal set. Ray *et al.* (2001) used evolutionary algorithm for optimal design of two criteria design problem. An algorithm that incorporates a Pareto dominance relation into particle swarm optimization is introduced for solving multi-criteria optimization problem taken from engineering fields (Reddy and Kumar 2007). A formal multi-objective optimization method based on satisfaction metrics is talked by Chen (2001) for designing an engineering system, where three satisfaction-based design models with different trade-off strategies are developed, in which an optimal design solution is taken as the one that maximizes the overall satisfaction of design objectives subjected to necessary design constraints. Another algorithm based on the generalized compound scaling method is used for optimal design of a multi-objective optimization problem (Grandhi *et al.* 1993). The basic idea of scaling technique is to derive a set of scale factors for the design variables. As a result, the design can be brought to the constraint surface and optimum lies on the constraint surface or the intersection of constraints. However, the algorithm generates a partial Pareto-optimal set of solutions. Masutti and Castro (2017) have applied bee-inspired algorithm to solve engineering problem. Another nature-inspired optimization algorithm called bat algorithm is introduced for solving engineering optimization task (Yang and Gandomi 2012). A hybrid particle swarm optimization-cuckoo search algorithm is developed for tackling nonlinear optimization problems taken from engineering field (Ding *et al.* 2019).Brajevic and Ignjatovic (2019) have used upgraded firefly algorithm for constrained engineering optimization problems.

In single objective optimization problem, optimal solution has a clear meaning but in case of multi-objective optimization problem, optimal solution has no explicit meaning. For multi-objective optimization problem, different solution concepts will provide different optimal solutions. The concept of Pareto optimality has been found to be quite effective in this context. Moreover, in case of multi-objective problem, there is no unique solution. To pick up the most compromise result which will provide the optimal trade-off between different conflicting objectives, some solution concepts is necessary that will give an additional criterion.

Literature reveals that optimization methods have been used in studying various aspects of geotechnical problems e.g. Yamagami and Ueta (1987) have presented nonlinear programming technique for slope stability analysis using limiting equilibrium method. Liu *et al.* (2020) have applied a hybrid model of artificial neural network based on genetic algorithm to predict the unconfined compressive strength of the rock mass ahead of a tunnel face. A robust design optimization procedure is presented for piled-raft foundation to support tall wind turbines in clayey and sandy soil (Ravichandran *et al*. 2018).NSGA-II coupled with Monte Carlo simulation is used considering the total cost of foundation and the standard deviation of differential settlement as the objectives. Chan *et al.* (2009) applied hybrid genetic algorithm for pile group foundation design. Objective is to minimize the material volume of foundation where configuration, number, cross-sectional dimension of the piles as well as thickness of the pile cap are considered as design variables. Arai *et al.* (1983) have described a numerical procedure based on optimization methods for estimating soil properties. Guido *et al.* (1987) derived a method from plate loading test that assumes the load spreads through the fill layer at an angle of $45^{\circ}$ for three-dimensional condition and geosynthetic reinforcement is required to support the weight of a soil pyramid which is not supported by piles. An attempt is made for designing a corrugated-core sandwich structure through probabilistic multi-objective optimization approach (Khalkhali *et al.* 2016). Yu *et al.* (2020) have used the concept of robust geotechnical design to optimize the stone columns reinforced with soft clay foundation. The problem considering multiple objectives including safety, cost and design robustness. The optimum reliability based design of tunnel structure during earthquake extreme events is presented by Azadi *et al.* (2020). Applied loads on tunnel as well as the resistance of structural members are assumed as probabilistic in nature. Hewlett and Randolph (1988) carried out model tests on a granular embankment fill material overlying a rectangular grid of pile caps to evaluate the amount of load transferred to the piles and the foundation soil due to soil arching. The calculations are based on the semi-spherical arches formed in the fill material. Bhattacharya (1990) has analyzed the problem of slope stability using nonlinear programming method. Mirzaei *et al.* (2015) have used particle swarm optimization method for tackling the problem related with optimal design of homogeneous earth dam structure. Kalemci and Ikizler (2020) have considered Rao-3 algorithm for optimal design of reinforced concrete cantilever retaining wall. Total weight of the steel and concrete required for construction are chosen as the objective function. Abusharar *et al.* (2009) developed a new theoretical analysis for embankments on soft ground supported by a rectangular grid of piles and geosynthetic. The main considerations are inclusion of a uniform surcharge load on the embankment, the use of individual square pile caps and taking into account the skin friction mechanism at the soil geosynthetic interface. Raju (1999) has estimated soil parameters, namely Young modulus and Poisson's ratio of soil using Powell's method of optimization. A novel approach was suggested for optimum design of hydraulic section of irrigation canals due to frost heave in cold region (Wang *et al.* 2019). Dey and Basudhar (2012) have used optimization technique to estimate Burger model parameters.

The literature review indicates that concept of game theory and Nash non-cooperative and cooperative solution associated with it, have not been used so far in solving multi-criteria optimization problem in the field of geotechnical engineering. This aspect has been addressed in this study. In conventional pile design problem, generally ultimate load carrying capacity of pile is evaluated for a desired factor of safety. In this study pile design problem is treated as multi objective optimization problem where two objectives are present conflicting with each other. Specifically, in this work, an approach based on Pareto-optimal and game theory with Nash solution concept shall be presented that will form the basis of optimal solution of MOOP. Thereafter, the methodology with computational procedure is demonstrated to obtain numerical results by solving two illustrative examples related to optimum design of axially loaded pile structure.

## 2. Description of the problem

The ultimate load carrying capacity of an axially loaded pile can be expressed as the sum of the loads carried by the pile base and by the pile shaft. The ultimate load (equal to product of working load & factor of safety) depends on the working load as well as factor of safety. For a fixed value of working load, higher factor of safety results in increasing the ultimate load carrying capacity of the pile. The increase of ultimate load affects the volume of the pile.

From a designer point of view, it is desirable to increase the ultimate load carrying capacity of the pile with minimum volume. As the cost is directly related with the volume, designer would be interested on maximizing the load carrying capacity with minimum cost. So the pile design problem has two important conflicting objectives and it becomes a two-objective optimization problem.

In present study, the drilled shaft and driven precast concrete pile have been considered. A simple schematic diagram of an axially loaded pile system is shown in Fig. 1.

It is assumed that the piles are installed in homogeneous profile of sandy soil. However, the ultimate load (Qult) at the head of the pile is balanced by the loads taken by the pile base (Qb,ult) and pile shaft (Qs,ult) respectively. It may be expressed as

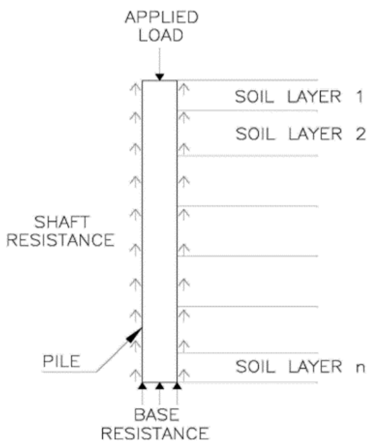


Fig. 1 Axially loaded pile structure

$$Q_{ult} = Q_{b,ult} + Q_{s,ult} \tag{4}$$

The ultimate load is generally taken as the load that causes 10% relative settlement of the pile head (i.e. pile head settlement/pile diameter = 0.1) (Misra 2010) and for this amount of settlement, the shaft resistance reaches its limit capacity (Salgado 2006). The limit shaft resistance $Q_{s,ult}$ is represented as

$$Q_{s,ult} = \sum_{i=1}^{n} q_{sLi} A_{si} \tag{5}$$

where, $q_{sLi}$ is the limit unit shaft resistance within any soil layer $i$ that the pile penetrates, $A_{si}$ is the pile shaft area interfacing with the layer $i$, and $n$ is the total number of soil layers interfacing with the pile shaft. The ultimate base resistance $Q_{b,ult}$ is given by

$$Q_{b,ult} = q_{b,10\%} A_b \tag{6}$$

where, $A_b$ represents the cross sectional area of the pile base and $q_{b,10\%}$ is the ultimate unit base resistance corresponding to 10% relative settlement of the pile head. Finally, the ultimate pile capacity is given by

$$Q_{ult} = q_{b,10\%} A_b + \sum_{i=1}^{n} q_{sLi} A_{si} \tag{7}$$

The equations used for designing different types of pile in sandy soil are provided below.

### *2.1 Equations of ultimate unit base resistance and limit unit shaft resistance for drilled shaft in sandy soil*

For drilled shaft in sand, the limit unit shaft resistance is written as (Salgado and Prezzi 2007)

$$q_{sL} = K\sigma_v^{'} \tan\phi_c \tag{8}$$

where,

$$K = 0.7K_0 \exp\left[\left\{0.0114 - 0.0022\ln\left(\frac{\sigma_v^{'}}{p_A}\right)\right\}D_R\right] \tag{9}$$

in which $\sigma_v^{'}$ is the vertical effective stress at the depth at which the limit capacity is evaluated, $\phi_c$ is the critical state friction angle of the sand, $D_R$ is the relative density of sand expressed as a percentage, $K_0$ is the coefficient of earth pressure at rest and $p_A$ is a reference stress. The ultimate unit base resistance of drilled shaft in sand is represented as (Salgado 2006)

$$q_{b,10\%} = [0.23\exp(-0.0066D_R)]q_{bL} \tag{10}$$

where,

$$\frac{q_{bL}}{p_A} = 1.64\exp[0.1041\phi_c + (0.0264 - 0.0002\phi_c)D_R]\left(\frac{\sigma_h^{'}}{p_A}\right)^{(0.841-0.0047D_R)} \tag{11}$$

in which $\sigma_h^{'}$ is the horizontal effective stress at the depth of the pile base.

### 2.2 Equations of ultimate unit base resistance and limit unit shaft resistance for precast concrete driven pile in sandy soil

The limit unit shaft resistance of precast concrete driven pile in sand is written as (Salgado and Prezzi 2007)

$$q_{sL} = 0.02\tan(0.95\phi_c)[1.02 - 0.0051D_R]q_{bL} \quad (12)$$

where, $q_{bL}$ is the limit unit base resistance at the depth at which $q_{sL}$ is required. The limit unit base resistance is given by Eq. (11) and ultimate unit base resistance of concrete driven pile in sand is represented as

$$q_{b,10\%} = [1.02 - 0.0051D_R]q_{bL} \quad (13)$$

in which the limit unit base resistance $q_{bL}$, given by Eq. (11), is evaluated at the pile base.

*Statement of the problem*

Evaluate the optimal solution which gives best compromise among two different conflicting objectives present in the problem. Two different conflicting objectives considered in the problem are ultimate load carrying capacity of the pile and associated cost. The design variable in this optimization problem may be taken as factor of safety which affects both the objectives. The constraint set would be formed by imposing a lower bound and an upper bound on the values of design variable. The task of the designer would be to evaluate an optimal value of design variable from its feasible space that would give best trade-off between the two objectives.

Mathematically this problem can be stated in an optimization format as given below:

Find the optimum solution (optimum value of design variable $X=x$) that would satisfy the constraint set $x_l \le x \le x_u$ where, suffix '$l$' and '$u$' represent lower and upper values and shall find the best trade-off between the following two objectives.

Maximize $f_1(X)$= ultimate load carrying capacity of pile and minimize $f_2(X)$ = cost of concrete volume.

To solve the above problem, the concept of Pareto-optimal solution and Nash concept of game theory has been proposed. These two methods are described in the following two sections.

## 3. Pareto-optimal solution

The vector $X^* \in S$ is called Pareto-optimal if there exists no feasible vector $X \in S$ which would decrease some objective function without causing a simultaneous increase in at least one objective function (Rao 1987), where $S$ represents the feasible domain for the problem. Generally several Pareto optima exist for a vector optimization problem and additional information is required to order the Pareto-optimal set. So it becomes possible to make the multi-objective approach is a flexible technique for most design problems although consideration of additional information is not included in the optimization model. Several numerical methods have been proposed for solving a vector optimization problem. In general, each method produces a different Pareto-optimal solution.

Some common approaches to solve the vector optimization problem are briefly mentioned, e.g. optimization problem can be solved by weighting method. In this method a new objective function, which is formed by linear combination of actual objectives as

$$F(\vec{X}) = w_1 f_1(\vec{X}) + w_2 f_2(\vec{X}) + ..... + w_N f_N(\vec{X})$$

where, $w_i$ represent nonnegative weighting factors indicating relative importance of various objective functions present in the problem and not all zero but sum is equal to unity. Then MOOP is solved as minimize $F(\vec{X})$ satisfying a set of constraints represented by

$$g_j(\vec{X}) \le 0, \text{ where } j=1,2,...m \quad (14)$$

When weight vector ($w_i$) is known, the optimal solution can be evaluated by minimizing the new objective function. The resulting solution is Pareto-optimal solution (POS). Therefore different optimal solutions can be obtained by assigning different values of weight vector. This means nature of POS is non-unique. Disadvantage of weighting method is incapable of whole Pareto-optimal set in case of certain nonconvex problem. In ϵ-constraint method, vector optimization problem can be described as minimize $f_i(\vec{X})$ subject to following set of constraints

$$f_j(\vec{X}) \le \varepsilon_j,\ j = 1,2,....,k \qquad j \ne i \quad (15)$$

where, $f_i^*$ represent ideal feasible solution for objective function $f_i$ and $\varepsilon_j = f_j^* + \Delta_j$, $\Delta_j$ indicating positive increments. This means one objective function is considered as scalar objective and other objective functions are treated as additional constraints by suitably choosing the values of $\varepsilon_j$. Even in nonconvex problems the entire set of POS can be generated by gradually varying $\varepsilon_j$. Solutions of ϵ-constraint method are non-unique in nature.

Thus it is evident from above discussion of POS method, some important features are:

• The optimal solution set is non-unique.

• One objective shall improve at the cost of at least another one.

• The method provides enough flexibility for the system designer.

## 4. Game theory approach

Based on the concept of game theory, attempts have been made for developing a methodology that was first introduced by Von Neumann and advanced by Von Neumann with Morgenstern (1947) later. To evaluate the solution for game theory problems, Nash (1951,1953) suggested two solution concepts. Domaszewski *et al.* (1999) used game theory technique for tackling a MOOP related with bending of 2-D structure. Rao (1987) used

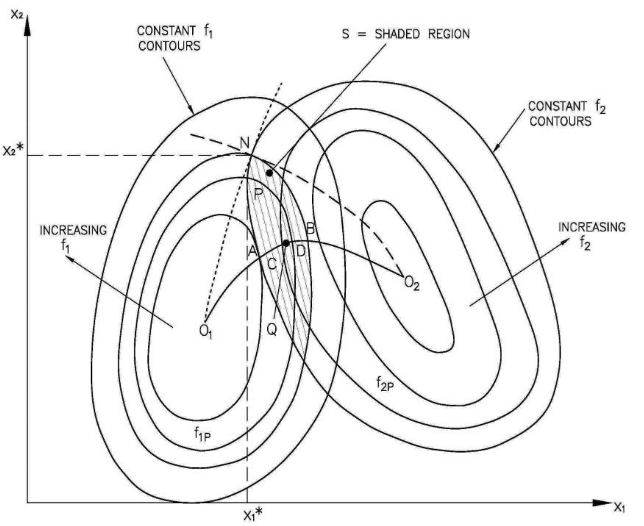


Fig. 2 Graphical representations of non-cooperative and cooperative solutions

Nash supercriterion model for optimum design of truss structure. The concept of game theory is applied to solve multi objective optimization problems related with optimum design of four bar joist rack structure (Meng *et al.* 2010) and earthquake resistant structure (Cheng 1999). Hati and Panda (2021) applied Nash supercriterion model in optimum maintenance problem of an aged structure. Additional mass imposed on the structure and extra residual life of structure due to modification are taken as objectives. From the literature it is found that concept of game theory has been used in solving MOOP related to structural design problem, but not yet attempted to solve problems in geotechnical field.

The game theory approach can be seen with reference to a two criteria, two controller optimization problem whose graphical representation is shown in Fig. 2. Let, $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ represent two scalar criteria and $x_1$ and $x_2$ two scalar design variables. It is assumed that one player is associated with each criterion. The first player wants to select a design variable $x_1$ which will minimize his/her pay-off $f_1$ and similarly the second player seeks a variable $x_2$ which will minimize his/her own pay-off $f_2$. If $f_1$ and $f_2$ are continuous, then the contours of constant values of $f_1$ and $f_2$ appear as represented in Fig. 2. The dotted lines passing through $O_1$ and $O_2$ indicate the loci of rational (minimizing) choices for the first and the second player for a fixed value of $x_2$ and $x_1$, respectively. The intersection of these two lines, if it exists, is a candidate for the two criteria minimization problem assuming that the players do not cooperate with each other (non-cooperative game). In Fig. 2, the point $N(x_1^*, x_2^*)$ indicates such a point. This point (N), known as a Nash equilibrium solution, represents a stable equilibrium condition in the sense that no player can deviate unilaterally from this point for further improvement of his/her own criterion (Nash 1950).

This point has the characteristic that

$$f_1\left(x_1^*, x_2^*\right) \leq f_1\left(x_1, x_2^*\right) \tag{16}$$

and

$$f_2\left(x_1^*, x_2^*\right) \leq f_2\left(x_1^*, x_2\right) \tag{17}$$

where, $x_1$ may lie left or right of $x_1^*$ in Eq. (16) and $x_2$ may be above or below of $x_2^*$ in Eq. (17).Combining Eqs. (16)-(17), and extending the idea to a $N$-player non-cooperative game gives the mathematical definition of a Nash equilibrium solution as

$$\left\{\begin{array}{c} f_1\left(x_1^*, x_2^*, \ldots, x_N^*\right) \leq f_1\left(x_1, x_2^*, \ldots, x_N^*\right) \\ f_2\left(x_1^*, x_2^*, \ldots, x_N^*\right) \leq f_2\left(x_1^*, x_2, \ldots, x_N^*\right) \\ . \\ . \\ f_N\left(x_1^*, x_2^*, \ldots, x_N^*\right) \leq f_N\left(x_1^*, x_2^*, \ldots, x_N\right) \end{array}\right\} \tag{18}$$

So far it was assumed that there exists only one Nash equilibrium point, i.e. the dotted lines in Fig. 2 intersect only at one point. An interesting situation happens when the two lines intersect at more than one point. In this case, since the values of $f_1$ and $f_2$ are different at Nash equilibrium points, any player can have the advantage of declaring his/her move first thereby forcing the other players to play at the equilibrium point of his/her own choice.

In a cooperative game, the two players agree to cooperate with each other and hence any point in the shaded region S of Fig. 2 will give both of them with a better solution than their respective Nash equilibrium solutions (Nash 1953). Since the region S does not provide a unique solution, the concept of Pareto-optimal (non-inferior) solutions can be developed to eliminate many solutions from the region S. It can be seen that all points in the region S can be eliminated except those lying on the continuous line $O_1ACQDBO_2$which represents the loci of tangent points between the contours of $f_1$ and $f_2$. Every point lying on this line has the property that it is not dominated by any other point in its neighbourhood, i.e.,

$$f_1(Q) \leq f_1(P) \tag{19}$$

and

$$f_2(Q) \leq f_2(P) \tag{20}$$

where, Q is a point lying on the line $O_1O_2$ and P is a neighbouring point. So all points of S that do not lie on the line $O_1O_2$ need not be considered during cooperative play. The set of all points lying on the line AB is known as a Pareto-optimal set and it may be denoted by $S_p$ (Ho 1970).

*Nash non-cooperative game*

A game is defined as the actions of a group of players who desire to maximize their own gains according to their individual strategies. The game is called non-cooperative when the players are acting independently without cooperation with each other and resulting solution is known as Nash non-cooperative solution. Often it is called Nash Equilibrium Solution (NES).

*Nash cooperative game*

The players desire to form a coalition with expectation that cooperating with each other, everyone can obtain an outcome which is better than Nash solution (Nash 1950, 1953). Measure of success of cooperative game is embodied in pareto-optimality concept where a single objective function is formed by convex combination of various objective functions. Thus various combinations would result different pareto optimal solutions and to overcome this issue, Nash formulated a supercriterion. Often it is called Nash supercriterion which give ultimate basis of the optimal solution of cooperative game. A procedure for finding the Pareto-optimal set and Nash solution based on a supercriterion or bargaining model is presented in the following section.

## 5. Computational procedure

### *5.1 Computational steps for obtaining Pareto-optimal solution*

Let the multi-criteria optimization problem involved minimization of the objective functions $f_i(X), i=1,2,...,N$ subject to $g_j(\vec{X}) \le 0,$ where $j=1,2,...m$. The algorithm for evaluating the Pareto-optimal solution can be stated as follows:

*Step 1:*

Define a new objective function $F_i(X)$ as

$$F_i(X) = m_i f_i(X) \tag{21}$$

by choosing the values of the scaling constants $m_i$ as

$$m_i = A / f_i(X_S) \tag{22}$$

where, $A$ is an arbitrary positive number and $X_s$ is any starting feasible design vector.

*Step 2:*

Construct a combined objective function( $F_C(X)$ )such that

$$F_C(X) = \sum_{i=1}^{N} w_i F_i(X) \tag{23}$$

where,$w_i$ is a weight attached with the individual objective $F_i(X)$, $w_i \ge 0, i=1,2,...,N$ ,and

$$\sum_{i=1}^{N} w_i = 1 \tag{24}$$

*Step 3:*

The Pareto-optimal set $S_p$ can be obtained by considering the various combinations of $w_i$ (where $i$=1,2,...,$N$) and minimizing $F_C(X)$ with respect to design variables in feasible domain. Since various combinations of $w_i$ would result various Pareto-optimal solution, thus it is non-unique in nature.

### *5.2 Computational steps for evaluating Nash solution*

In order to apply the concept of game theory to evaluate the solution of a MOOP, different objectives are considered to correspond different players in the game, each player for one objective. All players in the game are equally intelligent and also rational. Play start with the initial point known as non-cooperative solution. During non-cooperative game, trend of each player is to maximize his/her own gain compared to their respective starting point. It is considered that each player in the game has analysed their individual objective to know best and worst possible outcomes respectively. Also assumed that each player knows two extreme values of all players. For the negotiation process, a reasonable bargaining model also called supercriterion is developed on the basis of Nash solution. All players put their joint effort for maximization of supercriterion to achieve cooperative solution. When supercriterion is maximized, the game is terminated.

The computational procedure is presented below:

*Step 1: Non-cooperative or individual solution*

Consider a MOOP where *'N'* objective functions $f_1(X)$, $f_2(X)$, …. $F_N(X)$are present and *'m'* constraints represented by $g_j(\vec{X}) \le 0,$ with $j = 1, 2,...m$. $\vec{X}$ is a vector of design variables with dimension ($n$×1). It is assumed out of $N$ objectives, there are *'p'* objectives where designer's target is to minimize them (such as weight, cost, loss, time etc) and other hand remaining *'q'* objectives ($p$+$q$=$N$) for which designer attempts to maximize them (e.g. reliability, profit, residual life of old structure etc). In general, it is obvious that there never exist any design vector $\vec{X}$ which minimizes all *'p'* objective functions and simultaneously maximizes all remaining *'q'* objective functions. Therefore object is to find out best trade off between different objectives present in the problem (best with respect to a subjective criterion). It is considered there are *'N'* players in the game corresponding to *'N'* different objectives. The aim of each player is to find out a design vector which satisfies the constraint set and also minimizes/maximizes his/her objective function only. If first player is given a chance (by remaining other players) to select a design vector which will be most suitable to him/her, immediately he/she will choose a feasible vector $X^*_1$ such as $f_1(X^*_1)$ = min $f_1(X)$ satisfying the constraints $g_j(\vec{X}) \le 0$ , $j$ = 1, 2,….$m$. Generally for $i \le p$, such a move is given to player *'i'* by remaining other players in the game, then he/she will select a feasible vector $X^*_1$ that will minimize $f_i(X)$. On other hand when for $p+1 \le i \le N$, such a chance is given to player *'i'* by remaining other players in the game, then he/she will select the design vector $X^*_i$ that will maximize $f_i(X)$.

*Step 2: Bounds of each player and Pay-off matrix*

It is assumed that such a move is allowed to all players in the game, a matrix can be formed as below

$$[P] = \begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) & \dots & f_p(X_1^*) & f_{p+1}(X_1^*) & \dots & f_N(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) & \dots & f_p(X_2^*) & f_{p+1}(X_2^*) & \dots & f_N(X_2^*) \\ . & . & \dots & . & . & \dots & . \\ f_1(X_p^*) & f_2(X_p^*) & \dots & f_p(X_p^*) & f_{p+1}(X_p^*) & \dots & f_N(X_p^*) \\ . & . & \dots & . & . & \dots & . \\ f_1(X_N^*) & f_2(X_N^*) & \dots & f_p(X_N^*) & f_{p+1}(X_N^*) & \dots & f_N(X_N^*) \end{bmatrix} \tag{25}$$

The matrix [$P$] is called Pay-off matrix. From above [$P$] matrix it is obvious that diagonal elements indicating best

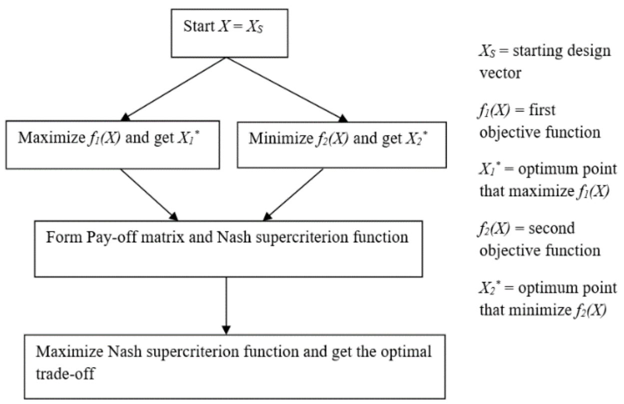


Fig. 3 Flow diagram to obtain the cooperative solution

values in corresponding column (i.e. corresponding player). The word best represents the lowest value of objective function for minimization problem, on other hand it corresponds to highest value for maximization problem. It should also be pointed out that the elements in any column of [*P*] matrix indicate all possible outcome for action of himself/herself as well as all other players in the game. The elements in any row (Say *'j'*) of [*P*] matrix represent state of all players in the game when player *'j'* acts as irrational way.

*Step 3: Starting point of the cooperative game*

In general for $i \le p$, the worst value of any player (Say *'j'*) can be represented as $max f_j(X^*_k)$, where, $'j' = 1,2,3,.....,p$ and $'k'=1,2,3,....,N$. When $p+1 \le i \le N$, worst value of any player (e.g. *'j'*) can be expressed as $min f_j(X^*_k)$, where, $'j' = p+1, p+2, ....., p+q=N$ and $'k' = 1,2,3,....N$ .

*Step 4: Nash supercriterion and the cooperative game solution*

It is assumed that players start bargaining from their respective worst position, a supercriterion can be developed as

$$S = \prod_{j=1}^{p} G_j . \prod_{j=p+1}^{p+q} G_j = \prod_{j=1}^{p} \{f_{jw} - f_j(X)\} . \prod_{j=p+1}^{p+q} \{f_j(X) - f_{jw}\} \quad (26)$$

where, $G_j$ indicate gain of $j^{th}$ player compared with his/her own initial point and $f_{jw}$ represent the worst value of $j^{th}$ player. Now all players in the game have formed the coalition, all of them should give their joint effort for maximization of the supercriterion function *S(X)* in feasible design space, to evaluate cooperative game solution of MOOP.

*Computational steps for specific problem*

In specific problem first objective function $f_1(X)$ represents ultimate load carrying capacity (factor of safety) and second objective function $f_2(X)$ indicates cost (volume of the pile). So it is essential to maximize $f_1(X)$ and minimize $f_2(X)$. Computational steps involved for this problem are given below.

*Step 1:* Maximize $f_1(X)$ within the feasible region ($2.0 \le x \le 5.0$). Assume when $X = X^*_1$, $f_1(X)$reaches it's maximum value. Next minimize $f_2(X)$ within the feasible region. Say $f_2(X)$ gets its minimum value when $X = X^*_2$.

*Step 2:* Pay-off matrix is formed as below

$$[P] = \begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) \end{bmatrix} = \begin{bmatrix} f_1(X_1^*) & f_{2w} \\ f_{1w} & f_2(X_2^*) \end{bmatrix} \quad (27)$$

*Step 3:* Best and worst values for first objectives function are $f_1(X^*_1)$ and $f_1(X^*_2)$ (it may be denoted as $f_{1w}$) while those values for second objective function are $f_2(X^*_2)$ and $f_2(X^*_1)$ (it may be denoted as $f_{2w}$), respectively.

*Step 4:* Nash supercriterion is developed as,

$$S(X) = \{f_1(X) - f_{1w}\}\{f_{2w} - f_2(X)\} \quad (28)$$

Task is now to maximize Nash supercriterion within the feasible zone. Where supercriterion reaches it's maximum value, corresponding values of objective functions indicate the optimum solution.

The flow chart for obtaining cooperative solution based on Nash supercriterion is presented in Fig. 3.

*Optimization method:*

Nash cooperative game problem as discussed above is to find out a design vector $\vec{X}$ which would maximize Nash supercriterion *S(X)* and shall satisfy given set of constraint represented by $g_j(\vec{X}) \le 0$ . To solve the optimization problem, grid search method is employed in present investigation.

*Grid search method:*

In grid search method, the feasible domain is discretised into a set of grid points. Objective function (it is Nash supercriterion in present study) is evaluated at all grid points lies in the feasible space. Optimum solution is chosen by compare the values of Nash supercriterion at all grid points.

$S^*$= $\max S(\vec{X})$ satisfying set of constraint $g_j(\vec{X}) \le 0$, $j = 1, 2,....m$; where, $\vec{X}$ belongs to a typical grid point within the mesh constructed by discretization the feasible domain formed by constraint boundaries.

## 6. Illustrative examples

To demonstrate the above methodology, two illustrative examples that have been considered in this study are presented in this section.

Two examples considered are: the drilled shaft and driven precast concrete pile. It is assumed that the design shall be based on the working stress method. It is also assumed that the piles are installed in homogeneous profile of sandy soil and the soil profile is so chosen that the construction of both types of piles is technically feasible. It need to be mentioned that there are some constraints on the availability of raw materials, equipment or technical expertise required for the design and construction of the piles. The water table is assumed to be at the ground surface in this study. Numerical data that are considered in present study are stated below:

### *6.1 Input data*

1. Critical state friction angle of sand is 30°
2. Relative density of sand is 60%

3. Coefficient of earth pressure at rest = 0.4
4. Bulk unit weight of sand $\gamma_{sat}$ = 19.93 kN/m$^3$
5. Unit weight of water $\gamma_2$ = 9.81 kN/m$^3$
6. Reference stress $p_A$ = 100 kPa
7. Working load is taken as 2000 kN
8. Length of pile is 12 m.

*Data related to optimization*

9. Reference value of design variable ($X_S$) =3.0
10. Arbitrary positive number ($A$) =100.0

### 6.2 Mathematical statement of design problem

Find out the Pareto-optimal set and Nash solution for the multi-objective optimum design of axially loaded pile structure. In present study two different conflicting objectives are ultimate load carrying capacity of the pile and associated cost. Here it can be observed that for a fixed value of working load, the ultimate load carrying capacity is directly related with factor of safety (F.O.S.). So, F.O.S. can be treated as first objective function ($f_1(X)$) for a fixed value of working load. Second objective ($f_2(X)$) is taken as volume of pile. In this study F.O.S. is taken as the design variable ($X$) for the problem. Lower and upper bound constraint on the design variable are

$$2.0 \le x \le 5.0 \qquad (29)$$

An interesting feature that is worth to state that in this formulation the first objective and the design variable are identical.

### 6.3 Example 1: (Drilled shaft installed in a sandy soil)

#### 6.3.1 Results for shaft and base resistance

The design calculations (as discussed in section 2.1) of shaft and base resistances for the drilled shaft in sand are shown in Table 1. It may be noted here that the pile length is equally divided by twelve segments.

Using Eq. (7) and results of Table 1 it can be seen that within a feasible domain, different F.O.S. leads to different volume of pile. The different values for F.O.S. versus volume of pile are presented in following section.

Table 1 Design calculations for drilled shaft in sand

| Layer No. | Layer width (m) | Limit unit shaft resistance $q_{sL}$ (kPa) | Limit unit base resistance $q_{bL}$ (kPa) | Ultimate unit base resistance $q_{b,10\%}$ (kPa) |
|---|---|---|---|---|
| 1 | 1 | 2.40 | - | - |
| 2 | 1 | 6.23 | - | - |
| 3 | 1 | 9.71 | - | - |
| 4 | 1 | 13.01 | - | - |
| 5 | 1 | 16.17 | - | - |
| 6 | 1 | 19.28 | - | - |
| 7 | 1 | 22.22 | - | - |
| 8 | 1 | 25.19 | - | - |
| 9 | 1 | 28.01 | - | - |
| 10 | 1 | 30.91 | - | - |
| 11 | 1 | 33.74 | - | - |
| 12 | 1 | 36.48 | - | - |
| Total | | 243.35 | 8454.03 | 1308.61 |

Table 2 Different values of F.O.S. versus volume of pile

| F.O.S. | Volume of pile (m$^3$) |
|---|---|
| 2.0 | 25.04 |
| 2.5 | 32.60 |
| 3.0 | 40.38 |
| 3.5 | 48.31 |
| 4.0 | 56.11 |
| 4.5 | 64.20 |
| 5.0 | 72.28 |

Table 3 Pareto-optimal solution

| Serial No. | $w_1$ | $w_2$ | Minimum value of $F_C(X) = w_1F_1 + w_2F_2$ | Optimum design variable ($X^*$) | $F_1(X^*)$ | $F_2(X^*)$ |
|---|---|---|---|---|---|---|
| 1 | 0.1 | 0.9 | 71.186 | 2.0 | 150.0 | 62.4289 |
| 2 | 0.2 | 0.8 | 79.9431 | 2.0 | 150.0 | 62.4289 |
| 3 | 0.3 | 0.7 | 88.7002 | 2.0 | 150.0 | 62.4289 |
| 4 | 0.4 | 0.6 | 96.342 | 2.3 | 130.4348 | 73.6134 |
| 5 | 0.5 | 0.5 | 99.859 | 2.8 | 107.1429 | 92.5751 |
| 6 | 0.6 | 0.4 | 99.2361 | 3.4 | 88.2353 | 115.7373 |
| 7 | 0.7 | 0.3 | 94.1408 | 4.2 | 71.4286 | 147.1359 |
| 8 | 0.8 | 0.2 | 83.7956 | 5.0 | 60.0 | 178.9783 |
| 9 | 0.9 | 0.1 | 71.8978 | 5.0 | 60.0 | 178.9783 |

#### 6.3.2 F.O.S. versus volume of pile

The variation of concrete volume with F.O.S. is presented in Table 2. As expected, it can be seen from the results that concrete volume constantly increases with increasing F.O.S.

Since there exist many possible feasible combinations (between F.O.S. and concrete volume), the task of designer would be to select a suitable combination that will give best trade-off. This is achieved by Pareto-optimal solution method, the results of which are presented below. A typical Pareto-optimal solution is obtained by attaching weights to the individual objective and forming a convex combination between them and minimizing the combined objective function.

#### 6.3.3 Pareto-optimal solution

The results obtained by minimizing the combined objective function ($F_c(X)$) for various combinations of $w_i$, are presented in Table 3. The procedure to construct the combined objective function through convex combination of two objectives is discussed in section 5.1.

From the results of Table 3 it can be seen that

- For different convex combinations, $F_1(X)$) varies from 150.0 to 60.0 and $F_2(X)$ varies from 62.4289 to 178.9783.
- Different convex combination gives different optimal results. Hence the optimal solutions are non-unique.
- The designer has flexibility to choose a particular optimal solution that meets his/her interest but not

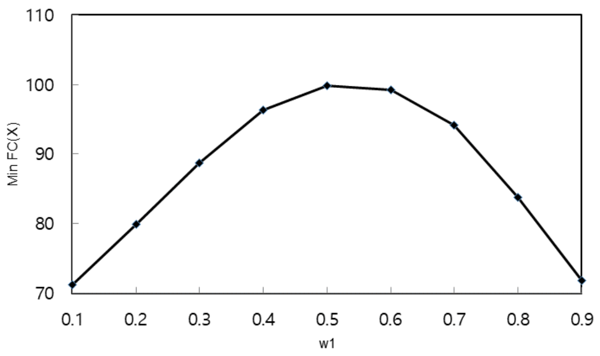


Fig. 4 Variation of min $F_C(X)$ with $w_1$

Table 4 Optimum solution for individual objective functions (Nash equilibrium solution)

| Individual Optimum Solution of two players | |
|---|---|
| Maximization of F.O.S., $f_1(X)$ | Minimization of volume of pile, $f_2(X)$, m$^3$ |
| $X^*_1 = 5.0$<br>$f_1(X^*_1) = 5.0$ | $X^*_2 = 2.0$<br>$F_2(X^*_2) = 25.04$ |

considered in mathematical formulation.

• Some additional consideration need to be introduced in the formulation to make the optimal solution unique.

Variation of minimum value of $F_c(X)$ with respect to $w_1$ is presented in Fig. 4. This Fig. 4 provides the trajectory of minimum $F_c(X)$ for changes in the weighting factor $w_1$.

### *6.3.4 Nash Equilibrium solution, Pay-off matrix, cooperative game and supercriterion*

As mentioned earlier, in this method it is assumed two players are present in the game. One player corresponds to one objective i.e., first player corresponds to maximization of F.O.S. $f_1(X)$ and the second player corresponds to minimization of volume of pile $f_2(X)$. Nash equilibrium solution or optimum solution for individual objective function is given in Table 4. It should be pointed out from numerical results of Table 4 that the design variable takes its upper bound value when F.O.S. is maximized. On other hand, the design variable hits to it lower bound value when volume of pile is minimized. A contradictory situation arises between two optimum solutions correspond to two players participating in the game.

As per Eq. (27),the pay-off matrix $[P]$ is obtained and given below

$$[P] = \begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) \end{bmatrix} = \begin{bmatrix} 5.0 & 72.28 \\ 2.0 & 25.04 \end{bmatrix} \qquad (30)$$

For first objective function (maximization), best and worst values are 5.0 and 2.0 while those are 25.04 and 72.28, respectively in case of second objective function (minimization). It is considered this information is known to both players at the initial stage of cooperative game. Assume both players have an agreement to play the cooperative game and their non-cooperative solution is chosen as starting point of cooperative game. Nash supercriterion function *S(X)* for cooperative game as mentioned in Eq. (28) is formulated as,

$$S(X) = \{f_1(X) - 2.0\}\{72.28 - f_2(X)\} \qquad (31)$$

here, $f_1(X)$ and $f_2(X)$ are objective functions as described in section 6.2. There are two terms within the second bracket at right hand side of Eq. (31) which indicate gain of two players and it is measured from respective starting point in cooperative game. These initial values for both players are 2.0 and 72.28 respectively.

Table 5 Results for supercriterion maximization evaluated by grid search method(gain is measured from their respective non-cooperative values)

| Optimum Solution |
|---|
| $X^* = 3.5$<br>$S_{max} = 35.955$<br><br>with $f_1 = 3.5$ gain= 1.5<br>$f_2 = 48.31$ gain= 23.97 |

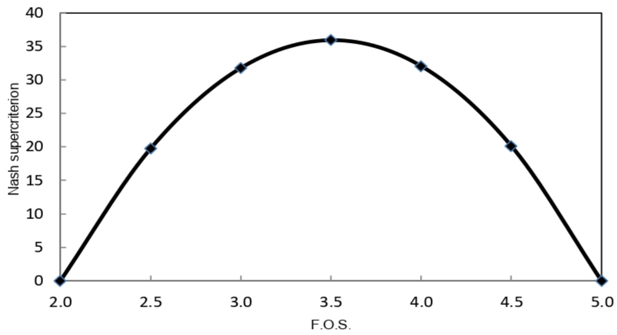


Fig. 5 F.O.S. versus Nash supercriterion

### *6.3.5 Optimization results for supercriterion maximization*

Numerical results for supercriterion maximization have been evaluated using grid search method. The results are presented in Table 5.

From the results of Table 5 it is found that at the end of play, although no player could drive his/her objective near to his/her own best value, both of them have improved their values significantly in comparison to their respective reference values. At optimal point, volume of pile is 48.31 m$^3$ with factor of safety equals to 3.5.

The variation of Nash supercriterion with respect to F.O.S. is plotted and it is shown in Fig. 5.

### *6.4 Example 2: (Driven concrete pile installed in a sandy soil)*

In this example driven concrete pile is considered. All results are evaluated like previous example discussed in above section 6.3.

#### *6.4.1 Results for shaft and base resistance*

The design calculations (as discussed in section 2.2) of shaft and base resistances are shown in Table 6.

#### *6.4.2 F.O.S. versus volume of pile*

The variation of concrete volume with F.O.S. is

Table 6 Design calculations for driven concrete pile in sand

| Layer No. | Layer Width (m) | Limit Unit Shaft Resistance $q_{sL}$ (kPa) | Limit Unit Base Resistance $q_{bL}$ (kPa) | Ultimate Unit Base Resistance $q_{b,10\%}$ (kPa) |
|---|---|---|---|---|
| 1 | 1 | 11.06 | - | - |
| 2 | 1 | 20.47 | - | - |
| 3 | 1 | 27.23 | - | - |
| 4 | 1 | 32.92 | - | - |
| 5 | 1 | 37.81 | - | - |
| 6 | 1 | 42.31 | - | - |
| 7 | 1 | 46.43 | - | - |
| 8 | 1 | 50.34 | - | - |
| 9 | 1 | 53.97 | - | - |
| 10 | 1 | 57.40 | - | - |
| 11 | 1 | 60.73 | - | - |
| 12 | 1 | 63.86 | - | - |
| Total | | 504.53 | 8454.03 | 6036.17 |

Table 7 Different values of F.O.S. versus volume of pile

| F.O.S. | Volume of pile (m$^3$) |
|---|---|
| 2.0 | 5.44 |
| 2.5 | 7.13 |
| 3.0 | 8.86 |
| 3.5 | 10.59 |
| 4.0 | 12.24 |
| 4.5 | 14.02 |
| 5.0 | 15.80 |

Table 8 Pareto-optimal solution

| Serial No. | $w_1$ | $w_2$ | Minimum value of $F_C(X) = w_1F_1 + w_2F_2$ | Optimum design variable ($X^*$) | $F_1(X^*)$ | $F_2(X^*)$ |
|---|---|---|---|---|---|---|
| 1 | 0.1 | 0.9 | 71.2403 | 2.0 | 150.0 | 62.4892 |
| 2 | 0.2 | 0.8 | 79.9913 | 2.0 | 150.0 | 62.4892 |
| 3 | 0.3 | 0.7 | 88.7424 | 2.0 | 150.0 | 62.4892 |
| 4 | 0.4 | 0.6 | 96.3474 | 2.3 | 130.4348 | 73.6225 |
| 5 | 0.5 | 0.5 | 99.8139 | 2.8 | 107.1429 | 92.485 |
| 6 | 0.6 | 0.4 | 99.1453 | 3.4 | 88.2353 | 115.5105 |
| 7 | 0.7 | 0.3 | 94.0112 | 4.2 | 71.4286 | 146.704 |
| 8 | 0.8 | 0.2 | 83.6643 | 5.0 | 60.0 | 178.3217 |
| 9 | 0.9 | 0.1 | 71.8322 | 5.0 | 60.0 | 178.3217 |

presented in Table 7.

### *6.4.3 Pareto-optimal solution*

The results of Pareto-optimal solution are presented in Table 8.

It can be noted from the numerical results presented in Table 8, same pattern is observed like the previous example of section 6.3.

Variation of minimum value of $F_C(X)$ with respect to $w_1$ is presented in Fig. 6.

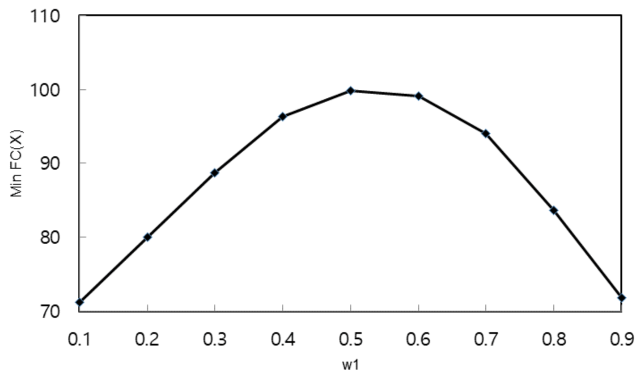


Fig. 6 Variation of min $F_C(X)$ with $w_1$

Table 9 Optimum solution for individual objective functions (Nash equilibrium solution)

| Individual Optimum Solution of two players | |
|---|---|
| Maximization of F.O.S., $f_1(X)$ | Minimization of volume of pile, $f_2(X)$, m$^3$ |
| $X^*_1$= 5.0<br>$f_1(X^*_1)$= 5.0 | $X^*_2$= 2.0<br>$f_2(X^*_2)$= 5.44 |

Table 10 Results for supercriterion maximization evaluated by grid search method(gain is measured from their respective non-cooperative values)

| Optimum Solution |
|---|
| $X^*$= 3.5<br>$S_{\max}$ = 7.815<br>with $f_1$= 3.5 gain= 1.5<br>$f_2$= 10.59 gain= 5.21 |

### *6.4.4 Nash Equilibrium solution, Pay-off matrix, cooperative game and supercriterion*

Nash equilibrium solution or optimum solution for individual objective function is given in Table 9. A conflicting situation arises between two optimum solutions correspond to two players participating in the game.

As per Eq. (27), the pay-off matrix [$P$] is evaluated and given below

$$[P] = \begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) \end{bmatrix} = \begin{bmatrix} 5.0 & 15.80 \\ 2.0 & 5.44 \end{bmatrix} \quad (32)$$

For first objective function (maximization), best and worst values are 5.0 and 2.0 while those are 5.44 and 15.8, respectively in case of second objective function (minimization). It is assumed this information is known to both players at the initial stage of cooperative game. Nash supercriterion function $S(X)$ for cooperative game as described in Eq. (28) is formulated as,

$$S(X) = \{f_1(X) - 2.0\}\{15.8 - f_2(X)\} \quad (33)$$

here, $f_1(X)$ and $f_2(X)$ are objective functions as discussed in section 6.2.

### *6.4.5 Optimization results for supercriterion maximization*

Numerical results for supercriterion maximization have

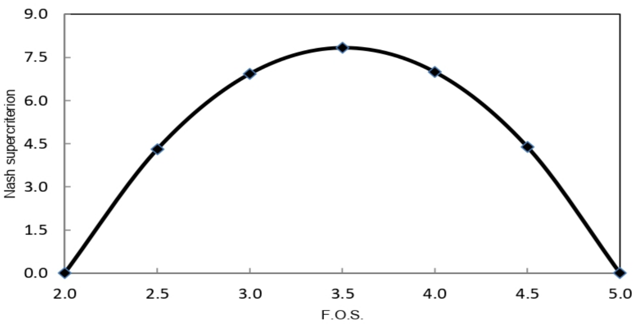


Fig. 7 F.O.S. versus Nash supercriterion

been evaluated using grid search method. The results are presented in Table 10.

Like the previous example, here also one can note that at the termination point of the game, although no player could achieve the maximum possible gain, but each of them have improved their values significantly compared to their respective non-cooperative values. At optimal point, volume of pile is 10.59 $m^3$ with factor of safety equals to 3.5.

The variation of Nash supercriterion with respect to F.O.S. is presented in Fig. 7.

It can be seen from the graph that the value of supercriterion is gradually increases from 0.0 to 7.815 (at corresponding F.O.S. = 3.5) and then gradually decreases to 0.0. At optimal point, the gain of 1st player is 1.5 and gain of 2nd player is 5.21.

## 7. Conclusions

The concepts of Pareto-optimal solution and game theory are presented to evaluate the optimum design of axially loaded pile structure. The design problem is formulated mathematically as constrained optimization problem with two conflicting objectives. The generation of Pareto-optimal solutions and the selection of most compromise solution according to game theory approach are illustrated. The solution concepts of Pareto-optimal, game theory with Nash non-cooperative solution and cooperative solution based on a negotiation model (also called Nash supercriterion) are implemented as basis for evaluating the optimal solution of decision making problem. Two numerical example problems are solved to demonstrate the proposed methodology. The methodology provides:

• Pareto-optimal solutions are non-unique and gain of one player affects that of the other.

• The Pareto-optimal solution set provides enough flexibility to the designer in decision making process.

• Extreme values of two objectives which are important information for decision making process.

• Optimal trade-off between two conflicting objectives that is based on a supercriterion developed from non-cooperative solution.

• Optimum design point corresponding with best trade-off between two conflicting objectives.

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