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Investigating of free vibration behavior of bidirectional FG beams resting on variable elastic foundation

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Abstract. In the present study, the free vibration of bidirectional functionally graded (FG) beams resting on variable elastic foundation are comprehensively investigated. The beam's behavior is modeled using 2D displacement field that contain undetermined integral terms and involves a reduced unknown functions. The material properties of the FG beam are assumed to be graded in both the thickness and longitudinal directions according to a power law. The beams are considered simply supported and resting on variable elastic foundation. The differential equation system governing the free vibration behavior of bidirectional beams is derived based on the Hamilton principle. The problem is then solved using the Navier solution for a simply supported beam. The accuracy of the used model can be noticed by comparing it with other solutions available in the literature where a good conformance was obtained. A detailed parametric study is conducted to explore the influences of material composition and variable elastic parameters on the vibration characteristics of the beams. The results reveal that the grading indexes in one or both directions as well as the parameters of the elastic foundation strongly impact the fundamental frequencies.

Keywords: BDFG beams; 2D theory; variable elastic foundation; free vibration

1. Introduction

Functionally graded materials (FGM) are advanced composite materials with heterogeneous, fine structures. The mechanical properties of these materials vary smoothly and continuously from one direction to other. This characteristic is obtained by gradually varying the volume fraction of the constituent materials, which are generally ceramics and metals. This advantage, compared to traditional composite materials, avoids problems of delamination, stress concentrations and residual stresses and therefore helps to maintain the integrity of the structure at a desirable level.

Over the past few decades, conventional unidirectional FGMs in which the gradual variation of material exists in only one direction have already been widely reported (Hadji and Avcar 2021, Bachir Bouiadjra *et al.* 2020, Kar and Panda, 2020, Avcar 2019, Nematiand Mahmoodabadi, 2019, Bachiri *et al.* 2018,Selmi 2020, Vinyas 2020, Melaibari *et al.* 2020, Madenci and Gülcü 2020, Hadji 2020, Shahmohammadi *et al.* 2020, Cuong-Le *et al.* 2020, Ahmed *et al.* 2019,Madenci 2019, Ramteke *et al.* 2019, Rahmani *et al.* 2019, Attia 2017, Benferhat *et al.* 2016, Akbas 2015).

Also, for traditional composite structures, many studies have been carried out so far (Madenci *et al.* 2020 a-c, Madenci and Özütok 2020, Özütok and Madenci 2017, Madenci and Özütok 2017).

Since FGMs have been specifically designed to work in severe thermal environments, such as advanced aerospace crafts and shuttles, the temperature or stress distribution could be in two or even three directions. Therefore, conventional FGM may not be useful in the design of such structures as the conventional FGM has a variation of material properties in only one direction (Lu and Chen 2020). Accordingly, understanding the behavior of bi-directional FG beams becomes an important task.

Şimşek (2015) studied the free and forced vibration of bi-directional functionally graded (BDFG) beam subjected to a moving load. He used the Timoshenko beam theory as well as Euler beam theory to derive the equations of motion. Different boundary conditions were studied. In

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another work, Şimşek (2016) investigated the buckling of a two-dimensional functionally graded materials (2D-FGMs) beams with different boundary conditions. Using various beams theories, Karamanli (2017a) analyzed the elastostatic behavior of BDFG beam subjected to various sets of boundary conditions. Pydah and Sabale (2017) employed the Euler-Bernoulli theory to study the flexure of BDFG circular beams. Rajasekaran and Khaniki (2018) analyzed the dynamic behavior of cracked BDFG Euler-Bernoulli beam. Bhattacharya and Das (2019) presented an improved mathematical model based on Timoshenko beam theory and modified couple stress theory to investigate the free vibration BDFG tapered rotating micro-beam. Wang et al (2016) presented an analytical solution based on Euler–Bernoulli beam theory for free vibration of BDFG beam. Hao and Wei (2016) presented an exact solution for the dynamic characteristic of a BDFG Timoshenko beam. To obtain this solution, the authors combined the state space method with the dynamic stiffness method. Shafiei *et al.* (2017) analyzed the vibration behavior of 2D-FG porous nano and micro Timoshenko beam using the Eringen's nonlocal elasticity and modified couple stress theory. Several boundary conditions were studied.

Fariborz and Batra (2019) studied the free vibration of circular BDFG beams by the mean of a shear deformation theory that incorporates through-the-thickness logarithmic variation of the circumferential displacement. Tran and Nguyen (2018) employed a new third-order shear deformation theory (TSDT) to analyze the free vibration of BDFG beams in thermal environment.

By employing the general nonlocal theory (GNT) in conjunction with Reddy's third order shear deformation beam theory, Faroughi *et al.* (2020) studies the wave propagation of BDFG porous rotating nano-beam. Rahmani *et al.* (2020) used the general nonlocal theory to analyze the vibrational behavior of BDFG rotating imperfect nonobeam.

Karamanli (2017b) studied the static behavior of BDFG sandwich beams subjected to various sets of boundary conditions by the mean of a quasi 3D shear deformation theory and Symmetric Smoothed Particle Hydrodynamics (SSPH) method. Tang and Ding (2019) have modeled the nonlinear hygro-thermal dynamics of a BDFG beam with coupled transverse and longitudinal displacements. Li *et al.* (2018) implemented meshless Total Lagrangian (TL) method to investigate the bending behavior of the BDFG beam structures.

In many important applications, the elastic foundation is the soil. This foundation effect greatly the behavior of the beam by changing its load bearing capacity and its natural frequencies.

During the last years, studies of structures resting on elastic foundation have attracted much attention from researchers and several hypotheses models have been introduced (Chami *et al.* 2020, Boulal *et al.* 2020, Jalaei and Civalek 2019). But most of this studies deal with structures on elastic foundations with constant moduli. However, those who treat FG structures resting on variable elastic foundation are really limited. We cite as an example the works of (Eisenberger and Clastornik 1987, Zhou 1993, Pradhan and Murmu 2009, Sobhy 2015, Ali Rachedi *et al.* 2020, Merzoug *et al.* 2020).

Based on the extensive review of the previous studies, and as far as the authors are aware, there is no study on free vibration behavior of BDFG beam resting on variable elastic foundation. The present work attempts to bridge this research gap.

For this purpose, the novelty of this work is to investigate the effect of the inclusion of a variable elastic foundation on the dynamic characteristics of a BDFG beam by using a 2D shear deformation theory that contain undetermined integral terms to reduce the number of unknown's functions. The material properties of FG beam vary along both thickness and axial directions according to a power law. The elastic foundation is modeled as a variable two-parameter Winkler–Pasternak foundation. Equations of motion are derived from the principle of virtual displacements. Analytical solutions for the free vibration behavior are obtained based on Fourier series that satisfy the Navier method. A parametric analysis is led to assess the effect of the variable elastic foundation, FG power index, span-to-depth and others parameters on the natural frequencies of BDFG beam on variable elastic foundation.

2. Theoretical developments

As shown in Fig.1, we consider a bi-directional FG beam model with length L, thickness hand width b. The beam is assumed to rest on a variable elastic foundation.

The Cartesian coordinate system (x, z) in the figure is introduced such that the x axis is on the mid-plane, and the z axis is perpendicular to the mid-plane, and it directs upward.

The beam material is assumed to consist of two ceramics (referred to as ceramic1 and ceramic2) and two metals (referred to as metal1 and metal2) whose volume fraction varies in both the thickness and longitudinal directions as (Tran and Nguyen 2018)

$$\begin{aligned} V_{c1} &= \left(\frac{z}{h}+\frac{1}{2}\right)^{n_z}\left[1-\left(\frac{x}{L}\right)^{n_x}\right] \\ V_{c2} &= \left(\frac{z}{h}+\frac{1}{2}\right)^{n_z}\left(\frac{x}{L}\right)^{n_x} \\ V_{m1} &= \left[1-\left(\frac{z}{h}+\frac{1}{2}\right)^{n_z}\right]\left[1-\left(\frac{x}{L}\right)^{n_x}\right] \\ V_{m2} &= \left[1-\left(\frac{z}{h}+\frac{1}{2}\right)^{n_z}\right]\left(\frac{x}{L}\right)^{n_x} \end{aligned} \tag{1}$$

where n_z and n_x are the grading indexes, which dictate the variation of the constituent materials in the thickness and longitudinal directions, respectively.

In Fig. 2, the variation of the volume fraction of ceramic 1 and ceramic 2 in both directions is represented for two values of grading indexes n_z and n_x.

P is the effective material properties (such as the elastic

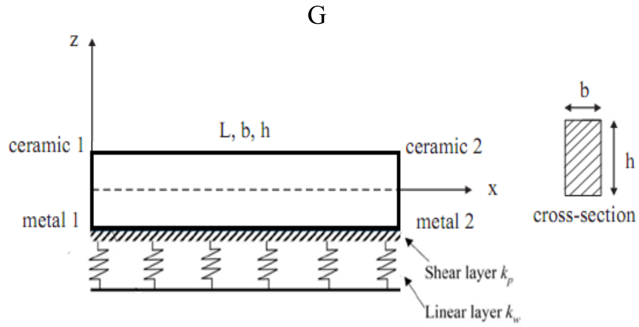


Fig. 1 Geometry and coordinates of BDFG beam resting on variable elastic foundation

modulus and mass density, etc.) which is determined using the Voigt model as follows (Tran and Nguyen 2018)

$$P = V_{c1}P_{c1} + V_{c2}P_{c2} + V_{m1}P_{m1} + V_{m2}P_{m2} \tag{2}$$

where P_{c1}, P_{c1}, P_{m1}, and P_{m2} denote respectively the properties of the ceramic1, ceramic2, metal 1 and metal2. Substituting Eq. (1) into Eq. (2), one gets (Tran and Nguyen 2018)

$$P(x,z) = \left[(P_{C1} - P_{m1})\left(\frac{z}{h}+\frac{1}{2}\right)^{n_z} + P_{m1}\right]\left[1-\left(\frac{x}{L}\right)^{n_x}\right] + \left[(P_{C2} - P_{m2})\left(\frac{z}{h}+\frac{1}{2}\right)^{n_z} + P_{m2}\right]\left(\frac{x}{L}\right)^{n_x} \tag{3}$$

We note that, In Eq. (3) if n_x=0 , we will have a unidirectional FG beam with ceramic2 and metal2. Moreover, if n_z=0 , we will have an axially unidirectional FG beams made from ceramic1 and ceramic2.

2.1 Kinematics

The displacement field of the conventional HSDT is given by:

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + f(z)\phi_x(x,y,t) \tag{4a}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{4b}$$

where u_0; w_0, φ_x are three unknown displacements of the mid-plane of the beam, $f(z)$ denotes shape function representing the variation of the transverse shear strains and stresses within the thickness. By considering that $\varphi_x = \int \theta(x,y)dx$, the displacement field of the present model can be expressed in a simpler form as:

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + k_1 f(z)\int \theta(x,y,t)dx \tag{5a}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{5b}$$

In this work, the present higher-order shear deformation plate theory is obtained by setting

$$f(z) = 5z\left(\frac{1}{4} - \frac{z^2}{3h^2}\right),\ g(z) = \frac{df(z)}{dz} \tag{6}$$

The kinematic relations can be obtained as follows:

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s\ ,\ \{\gamma_{xz}\} = g(z)\{\gamma_{xz}^0\} \tag{7}$$

where

$$\{\varepsilon_x^0\} = \left\{\frac{\partial u_0}{\partial x}\right\},\ \begin{Bmatrix} k_x^b \\ k_x^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ k_1\theta \end{Bmatrix}, \tag{8a}$$

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s\ ,\ \{\gamma_{xz}^0\} = \left\{k_1 A'\frac{\partial\theta}{\partial x}\right\} \tag{8b}$$

The integrals used in the above equations shall be resolved by a Navier type method and can be given as follows:

$$\int \theta\ dx = A'\frac{\partial\theta}{\partial x}.$$

where the coefficient A' is expressed according to the type of solution used, in this case via Navier. Therefore, A' and k_1 are expressed as follows:

$$A' = -\frac{1}{\alpha^2}\ ,\ k_1 = \alpha^2 \tag{9}$$

where α defined in expression (25).

For elastic and isotropic FGMs, the constitutive relations can be expressed as:

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \end{Bmatrix}, \tag{10}$$

where (σ_x, τ_{xz}) and (ε_x, γ_{xz}) are the stress and strain components, respectively. Using the material properties defined in Eq. (1), stiffness coefficients, C_{ij}, can be given as

$$C_{11} = E(x,z),\ C_{55} = \frac{E(x,z)}{2(1+\nu)} \tag{11}$$

The beam is assumed to rest on two-parameter elastic foundation model, which consists of closely spaced springs interconnected through a shear layer made of incompressible vertical elements, which deform only by transverse shear. The response equation of this foundation is given:

$$R(x,y) = \overline{K}(x)\,w(x,y) + \overline{G}\,\nabla^2 w(x,y) \tag{12}$$

where R is the density of the reaction force of elastic foundation, $\overline{K}$ is Winkler parameter depended on x only. It is assumed to be linear, parabolic or sinusoidal (Sobhy 2015, Ali Rachedi *et al.* 2020, Merzoug *et al.* 2020, Pradhan and Murmu 2009):

$$\overline{K}(x)=\frac{J_1h^3}{L^4}\begin{cases}1+\xi\dfrac{x}{L} & Linear\\ 1+\xi\left(\dfrac{x}{L}\right)^2 & Parabolic\\ 1+\xi\sin\left(\pi\dfrac{x}{L}\right) & Sinusoidal\end{cases} \tag{13}$$

in which J_1 is a constant and ξ is a varied parameter. G is the shear layer foundation stiffness ∇^2 is the Laplace operator in x, and w is the deflection of the beam. Note that, if ξ=0, the elastic foundation becomes Pasternak foundation and if the shear layer foundation stiffness is neglected, the Pasternak foundation becomes the Winkler foundation.

2.2 Equations of motion

Hamilton's principle is herein utilized to determine the equations of motion:

$$0=\int_0^t(\delta U+\delta V-\delta K)\,dt \tag{14}$$

where δU is the variation of strain energy; δV is the variation of potential energy of the foundation; and δK is the variation of kinetic energy.

The variation of strain energy of the plate is given by

$$\begin{aligned}\delta U&=\int_x\int_z\left[\sigma_x\delta\varepsilon_x+\tau_{xz}\delta\gamma_{xz}\right]dxdz\\&=\int_x\left[N_x\delta\varepsilon_x^0+M_x^b\delta k_x^b+M_x^s\delta k_x^s+Q_{xz}\delta\gamma_{xz}^0\right]dx\end{aligned} \tag{15}$$

where the stress resultants N, M and Q are defined by

$$\left(N_x,M_x^b,M_x^s\right)=\int_{-h/2}^{h/2}(1,z,f)\,\sigma_x dz,$$

$$\text{and}\quad \left(Q_{xz}\right)=\int_{-h/2}^{h/2}g\left(\tau_{xz}\right)dz \tag{16}$$

The variation of potential energy of the foundation

$$\delta V=\int_x R\delta w dx \tag{17}$$

The variation of kinetic energy of the plate can be expressed as

$$\begin{aligned}\delta K&=\int_x\int_z\left[\dot{u}\delta\dot{u}+\dot{w}\delta\dot{w}\right]\rho(z)\,dxdz\\&=\int_x\Big\{I_0\left[\dot{u}_0\delta\dot{u}_0+\dot{w}_0\delta\dot{w}_0\right]-I_1\left[\dot{u}_0\frac{\partial\delta\dot{w}_0}{\partial x}+\frac{\partial\dot{w}_0}{\partial x}\delta\dot{u}_0\right]\\&+J_1k_1A'\left[\dot{u}_0\frac{\partial\delta\dot{\theta}}{\partial x}+\frac{\partial\dot{\theta}}{\partial x}\delta\dot{u}_0\right]+I_2\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x}\\&+K_2\left(k_1A'\right)^2\frac{\partial\dot{\theta}}{\partial x}\frac{\partial\delta\dot{\theta}}{\partial x}-J_2k_1A'\left[\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{\theta}}{\partial x}+\frac{\partial\dot{\theta}}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x}\right]\Big\}dx\end{aligned} \tag{18}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density given by Eq. (3); and (I_i, J_i, K_i) are mass inertias expressed by

$$0=\int_0^t(\delta U+\delta V-\delta K)\,dt \tag{19a}$$

$$\left(J_1,J_2,K_2\right)=\int_{-h/2}^{h/2}\left(f,zf,f^2\right)\rho(z)dz \tag{19b}$$

By substituting Eqs. (15), (17) and (18) into Eq. (14), the following can be derived:

$$\int_x\left(\frac{\partial N_x}{\partial x}-I_0\ddot{u}_0+I_1\frac{\partial\ddot{w}_0}{\partial x}-k_1A'J_1\frac{\partial\ddot{\theta}}{\partial x}\right)\delta u_0dx=0 \tag{20a}$$

$$\int_x\begin{pmatrix}\dfrac{\partial^2M_x^b}{\partial x^2}-I_0\ddot{w}_0-I_1\dfrac{\partial\ddot{u}_0}{\partial x}+I_2\dfrac{\partial\ddot{w}_0}{\partial x}\\-J_2k_1A'\dfrac{\partial^2\ddot{\theta}}{\partial x^2}+R\end{pmatrix}\delta w_0dx=0 \tag{20b}$$

$$\int_x\begin{pmatrix}-k_1A'\dfrac{\partial^2M_x^s}{\partial x^2}+k_1A'\dfrac{\partial Q_{xz}^s}{\partial x}\\+J_1k_1A'\dfrac{\partial\ddot{u}_0}{\partial x}+K_2\left(k_1A'\right)^2\dfrac{\partial^2\ddot{\theta}}{\partial x^2}\\-J_2k_1A'\dfrac{\partial^2\ddot{w}_0}{\partial x^2}\end{pmatrix}\delta\theta dx=0 \tag{20c}$$

Substituting Eq. (8) into Eq. (10) and the subsequent results into Eqs. (16), the stress resultants are obtained in terms of strains as following compact form:

$$\begin{Bmatrix}N\\M_x^b\\M_x^s\end{Bmatrix}=\begin{bmatrix}A&B&B^s\\B&D&D^s\\B^s&D^s&H^s\end{bmatrix}\begin{Bmatrix}\varepsilon_x^0\\k_x^b\\k_x^s\end{Bmatrix} \tag{21a}$$

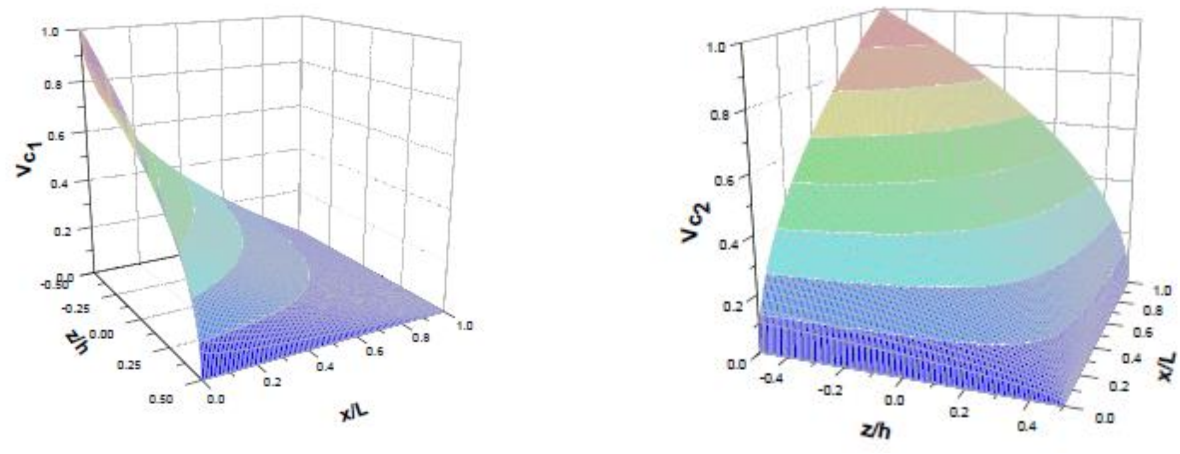
where

$$\left\{A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\right\}=\int_{-h/2}^{h/2}C_{11}\left(1,z,z^2,f(z),z\,f(z),f^2(z)\right)dz \tag{21b}$$

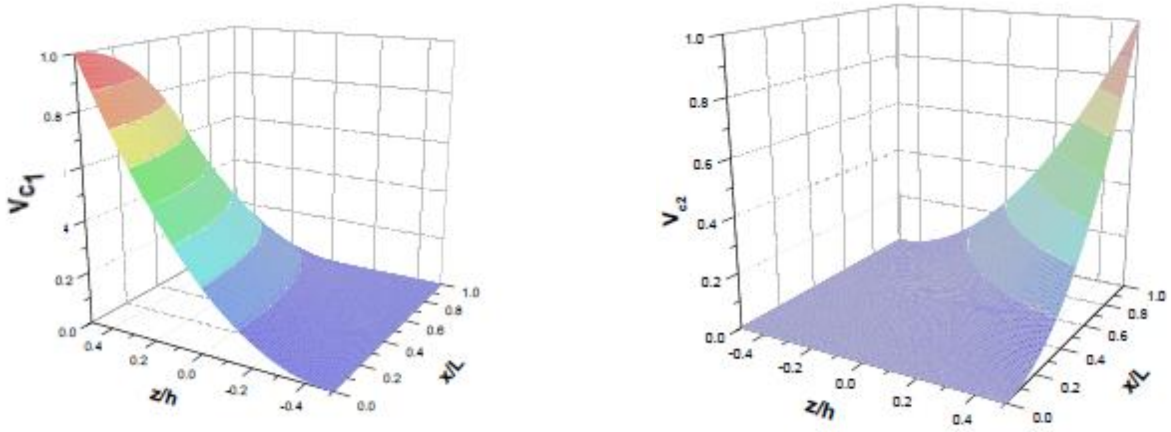
$$A_{44}^s=\int_{-h/2}^{h/2}C_{55}\,g^2(z)\,dz \tag{21c}$$

Introducing Eq. (21) into Eq. (20), the equations of motion can be expressed in terms of displacements (u_0, w_0, θ) and the appropriate equations take the form:

$$\int_x\begin{bmatrix}A_{11}d_{11}u_0+d_1A_{11}d_1u_0-B_{11}d_{111}w_0\\-d_1B_{11}d_{11}w_0+B_{11}^sk_1A'd_{111}\theta\\+d_1B_{11}^sk_1A'd_{11}\theta-I_0\ddot{u}_0+I_1d_1\ddot{w}_0\\-J_1A'k_1d_1\ddot{\theta}\end{bmatrix}\delta u_0dx=0 \tag{22a}$$



(a) $n_x = n_z = 1/2$



(b) $n_x = n_z = 2$

Fig. 2 Variation of volume fractions of ceramics in both direction (a) $n_x = n_z = 1/2$ and (b) $n_x = n_z = 2$

$$\int_x \begin{bmatrix} B_{11}d_{111}u_0 + d_{11}B_{11}d_1u_0 - D_{11}d_{1111}w_0 \\ -d_{11}D_{11}d_{11}w_0 + k_1A'D_{11}^s d_{1111}\theta \\ +k_1A'd_{11}D_{11}^s d_{11}\theta - I_0\ddot{w}_0 - I_1d_1\ddot{u}_0 \\ +I_2d_{11}\ddot{w}_0 - k_1A'J_2d_{11}\ddot{\theta} \end{bmatrix} \delta w_0 dx = 0 \qquad (22b)$$

$$\int_x \begin{bmatrix} -k_1A'\left(B_{11}^s d_{111}u_0 + d_{11}B_{11}^s d_1u_0\right) \\ +k_1A'\left(D_{11}^s d_{1111}w_0 + d_{11}D_{11}^s d_{11}w_0\right) \\ -\left(k_1A'\right)^2\left(H_{11}^s d_{1111}\theta + d_{11}H_{11}^s d_{11}\theta\right) \\ +A_{44}^s d_{11}\theta + d_1A_{44}^s d_1\theta + k_1A'J_1d_1\ddot{u}_0 \\ -k_1A'J_2d_{11}\ddot{w}_0 + \left(k_1A'\right)^2 K_2d_{11}\ddot{\theta} \end{bmatrix} \delta\theta dx = 0 \qquad (22c)$$

2.3 *Analytical solution for simply-supported FG plates*

In this paragraph, the Navier solution for simply supported beams will be used to solve the problem. The following boundary conditions are imposed at the edges:

$$u_0 = w_0 = \theta = \frac{\partial\theta}{\partial x} = N_x = M_x^b = M_x^s = 0 \qquad (23)$$

at $x = 0, L$

The following representation for the displacement quantities, that satisfy the above boundary conditions, is appropriate in the case of our problem

$$\begin{Bmatrix} u_0 \\ w_0 \\ \theta \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m e^{i\omega t} \dfrac{\partial X(x)}{\partial x} \\ W_m e^{i\omega t} X(x) \\ X_m e^{i\omega t} X(x) \end{Bmatrix} \qquad (24)$$

For simply supported boundary conditions

$$X(x) = \sin(\alpha x) \qquad (25)$$

where ω is the frequency of free vibration of the plate, $\sqrt{i} = -1$ the imaginary unit.

With

$$\alpha = m\pi / a \qquad (26)$$

Substituting Eq. (24) into Eq. (23), the following problem is obtained:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} U_m \\ W_m \\ X_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{27}$$

where

$$a_{11} = \int_x \begin{bmatrix} A_{11} d_{111} X + d_1 A_{11} d_{11} X \\ + I_0 d_1 X \, \omega^2 \end{bmatrix} d_1 X dx$$

$$a_{12} = \int_x \begin{bmatrix} -B_{11} d_{111} X - d_1 B_{11} d_{11} X \\ - I_1 d_1 X \, \omega^2 \end{bmatrix} d_1 X dx$$

$$a_{13} = \int_x \begin{bmatrix} k_1 A' B_{11}^s d_{111} X + k_1 A' d_1 B_{11}^s d_{11} X \\ + k_1 A' J_1 d_1 X \, \omega^2 \end{bmatrix} d_1 X dx$$

$$a_{21} = \int_x \begin{bmatrix} B_{11} d_{1111} X + d_{11} B_{11} d_{11} X \\ + I_1 d_{11} X \, \omega^2 \end{bmatrix} X dx$$

$$a_{21} = \int_x \begin{bmatrix} -D_{11} d_{1111} X - d_{11} D_{11} d_{11} X + I_0 X \, \omega^2 \\ - I_2 d_{11} X \, \omega^2 - \overline{K}(x) X + \overline{G} d_2 X \end{bmatrix} X dx \tag{28}$$

$$a_{32} = \int_x \begin{bmatrix} k_1 A' D_{11}^s d_{1111} X + k_1 A' d_{11} D_{11}^s d_{11} X \\ + k_1 A' J_2 d_{11} X \, \omega^2 + J_1^{st} X \, \omega^2 \end{bmatrix} X dx$$

$$a_{31} = \int_x \begin{bmatrix} -k_1 A' B_{11}^s d_{1111} X - k_1 A' d_{11} B_{11}^s d_{11} X \\ - k_1 A' J_1 d_{11} X \, \omega^2 \end{bmatrix} X dx$$

$$a_{32} = \int_x \begin{bmatrix} k_1 A' D_{11}^s d_{1111} X + k_1 A' d_{11} D_{11}^s d_{11} X \\ + k_1 A' J_2 d_{11} X \, \omega^2 + J_1^{st} X \, \omega^2 \end{bmatrix} X dx$$

$$a_{33} = \int_x \begin{bmatrix} -(k_1 A')^2 H_{11}^s d_{1111} X - (k_1 A')^2 d_{11} H_{11}^s d_{11} X \\ -2 k_1 A' R d_{11} X + A_{44}^s d_{11} X + d_1 A_{44}^s d_1 X \\ -(k_1 A')^2 K_2 d_{11} X \, \omega^2 + K_2^{st} X \, \omega^2 \end{bmatrix} X dx$$

3. Numerical results and discussion

In this section, numerical investigations are carried out to check the accuracy of the present model and to study the effects of grading indexes, side-to-thickness ratio and the variable elastic foundation parameter on the natural frequencies of BDFG beams.

As a first example, the non-dimensional frequencies of unidirectional transverse beams (n_x = 0) of the present solution are compared with those obtained by Chaabane *et al.* (2019), the higher order shear deformation theory of Ould Larbi *et al.* (2013) and the Timoshenko beam theory (TBT) of Şimşek (2010). For this example, the FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (Alumina, Al_2O_3): E_c = 380 GPa, υ = 0.3, ρ_c = 3960 kg/m^3 .

Metal (Aluminium, Al): E_m = 70 GPa, υ = 0.3, ρ_m = 2707 kg/m^3

The following dimensionless form is used to compute the frequencies $\bar{\omega}$: $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$

Table 1 Non-dimensional frequencies $\bar{\omega}$ of unidirectional transverse FG beams (n_x=0)

L/h	Sources	Power-law index nz					
		0	0.5	1	2	5	10
5	Present	5.1575	4.4137	3.9926	3.6278	3.4019	3.2820
	Chaabane *et al.* (2019)	5.1633	4.4180	3.9963	3.6303	3.4004	3.2846
	Ould Larbi *et al.* (2013)	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
	TBT (Şimşek 2010)	5.1527	4.4111	3.9904	3.6264	3.4012	3.2816
20	Present	5.4654	4.6543	4.2074	3.8376	3.6492	3.5394
	Chaabane *et al.* (2019)	5.4657	4.6545	4.2076	3.8378	3.6491	3.5395
	Ould Larbi *et al.* (2013)	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	TBT (Şimşek 2010)	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390

From the table 1, it can be seen that the results obtained by the proposed model are in good agreement with those found in the literature for slender and short FG-beams.

Another validation can be seen in Table 2, the non-dimensional frequencies $\hat{\omega}$ of simply supported BDFG beam without elastic foundation in the reference temperature (ΔT = 0) computed with the present solution are compared with those of Nguyen *et al.* (2017) and Tran and Nguyen (2018).

In the following, unless otherwise specified, a BDFG beam with an aspect ratio L/h=20 composed of alumina (Al_2O_3) as ceramic 1, zirconia (ZrO2) as ceramic 2, stainless steel (SUS304) as metal 1 and aluminum (Al) as metal 2 is used. The material properties are the following:

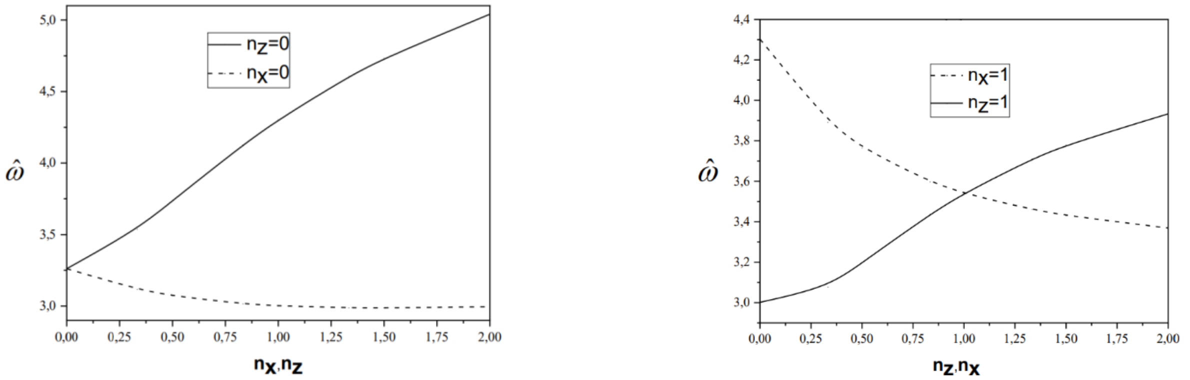
$$E_{c1} = 390 GPa, E_{c2} = 200 GPa,$$
$$\rho_{c1} = 3960 Kg/m^3, \rho_{c2} = 5700 Kg/m^3$$
$$E_{m1} = 210 GPa, E_{m2} = 70 GPa,$$
$$\rho_{m1} = 7800 Kg/m^3, \rho_{m2} = 2702 Kg/m^3$$

For all materials, υ=0.3.The frequencies of the BDFG beam in Table 2 are in the following dimensionless form: $\hat{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_{m2}}{E_{m2}}}$

As expected, a good agreement can be observed between the present results and the available results. As can be seen from this table, the maximum difference between the results of the present solution and those of Tran and Nguyen (2018) not more than 7.31% for a BDFG beams and it is of the order of 0% in the case of an unidirectional

Table 2 Comparison of non-dimensional frequencies $\hat{\omega}$ of simply supported BDFG beam without elastic foundation

$\hat{\omega}$	Sources	c	n_x= 1/3	n_x= 1/2	n_x= 5/6	n_x= 1	n_x= 4/3	n_x= 3/2	n_x= 2
n_z= 0	Nguyen *et al.* (2017)	3.3018	3.7429	3.9148	4.1968	4.3139	4.5118	4.5956	4.8005
	Tran and Nguyen (2018)	3.3018	3.7428	3.9146	4.1966	4.3137	4.5116	4.5954	4.8003
	Present	3.3018	3.5766	3.7793	4.1773	4.3562	4.6653	4.7965	5.1115
	diff %	0.00	-4.44	-3.46	-0.46	0.99	3.41	4.38	6.48
n_z= 1/3	Nguyen *et al.* (2017)	3.1542	3.5050	3.6305	3.8252	3.9022	4.0277	4.0792	4.2009
	Tran and Nguyen (2018)	3.1543	3.5050	3.6305	3.8251	3.9022	4.0276	4.0791	4.2008
	Present	3.1543	3.3285	3.4880	3.8011	3.9389	4.1705	4.2661	4.4881
	diff %	0.00	-5.04	-3.93	-0.63	0.94	3.55	4.58	6.84
n_z= 1/2	Nguyen *et al.* (2017)	3.1068	3.3285	3.5397	3.7087	3.7745	3.8805	3.9236	4.0245
	Tran and Nguyen (2018)	3.1069	3.4285	3.5397	3.7087	3.7745	3.8805	3.9235	4.0244
	Present	3.1070	3.2481	3.3948	3.6837	3.8101	4.0210	4.1073	4.3058
	diff %	0.00	-5.26	-4.09	-0.67	0.94	3.62	4.69	6.99
n_z= 5/6	Nguyen *et al.* (2017)	3.0504	3.3296	3.4206	3.5548	3.6059	3.6569	3.7194	3.7847
	Tran and Nguyen (2018)	3.0505	3.3296	3.4206	3.5547	3.6058	3.6869	3.7193	3.7946
	Present	3.0507	3.1431	3.2721	3.5287	3.6403	3.8247	3.8993	4.0687
	diff %	0.01	-5.60	-4.34	-0.73	0.96	3.74	4.84	7.22
n_z= 1	Nguyen *et al.* (2017)	3.0359	3.2984	3.3819	3.5035	3.5495	3.6219	3.6508	3.7177
	Tran and Nguyen (2018)	3.0359	3.2983	3.3818	3.5034	3.5493	3.6217	3.6507	3.7175
	Present	3.0362	3.1095	3.2319	3.4771	3.5835	3.7588	3.8295	3.9892
	diff %	0.01	-5.72	-4.43	-0.75	0.96	3.79	4.90	7.31



(a) (b)

Fig. 3 Variation of the of non-dimensional frequencies $\hat{\omega}$ versus the grading indexes of FG beam without elastic foundation L/h=10

transverse FG beams (n_x=0).

The slight difference between the results of the present study and those of Nguyen *et al.* (2017) and Tran and Nguyen (2018) for the BDFG beams may be resulted from the different beam theory and the solution used herein to solve the problem with the previously references.

In effect, in the present work we have used a 2D HSDT and the Navier solution for simply supported beams to determine the frequencies. Nguyen *et al.* (2017) employed finite element formulation based on Timoshenko beam theory and Tran and Nguyen (2018) used the third-order shear deformation theory and the finite element formulation based on the shear rotation.

Fig. 3(a) and 2(b) shows the variation of the dimensionless frequency as a function of the volume fractions in the two directions n_x and n_z. Two cases are studied:

-Fig. (a), n_z=0 with variation of n_x (Solid line), n_x=0 with variation of n_z (Dash line)

-Fig. (b), and n_z=1 with variation of n_x and n_x=1 with variation of n_z

Various information can be obtained from these figures:

- For Fig. (a), both variations start from the same point. This is logical because it corresponds to the case of the

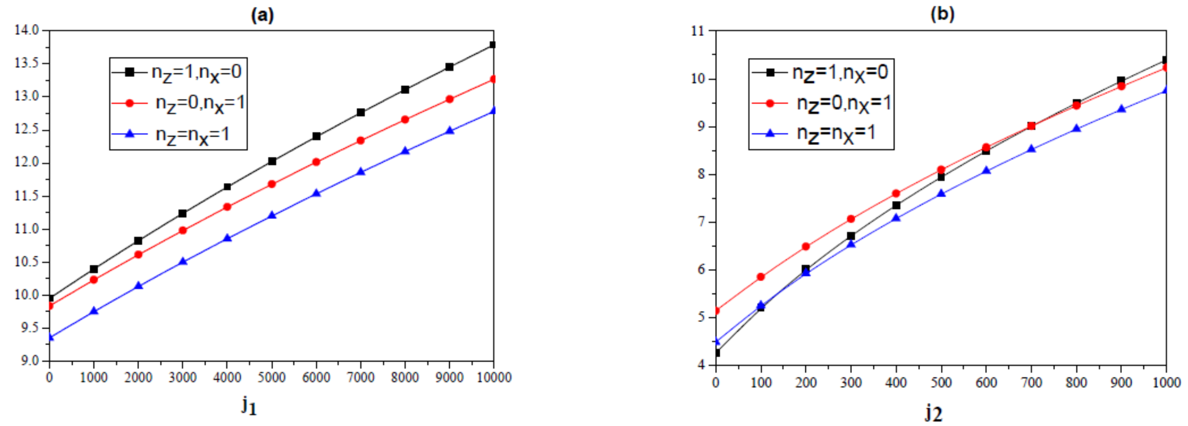


Fig. 4 Variation of the of non-dimensional frequencies $\hat{\omega}$ versus elastic foundation parameters L/h=10, ξ = 0 (a) j_2 =1000 and (b) j_1 =1000

Table 3 Effect of elastic foundation parameter's and aspect ratio L/h on the non-dimensional frequencies $\hat{\omega}$ ($n_x = n_z = 1$, $\xi = 0$)

j_1, j_2	L/h						
	5	10	15	20	25	30	50
0,0	3.3853	3.5404	3.5722	3.5835	3.5888	3.5917	3.5960
1000,0	4.3477	4.4866	4.5155	4.5259	4.5307	4.5334	4.5372
1000,1000	9.6063	9.7514	9.7820	9.7930	9.7982	9.8010	9.8052

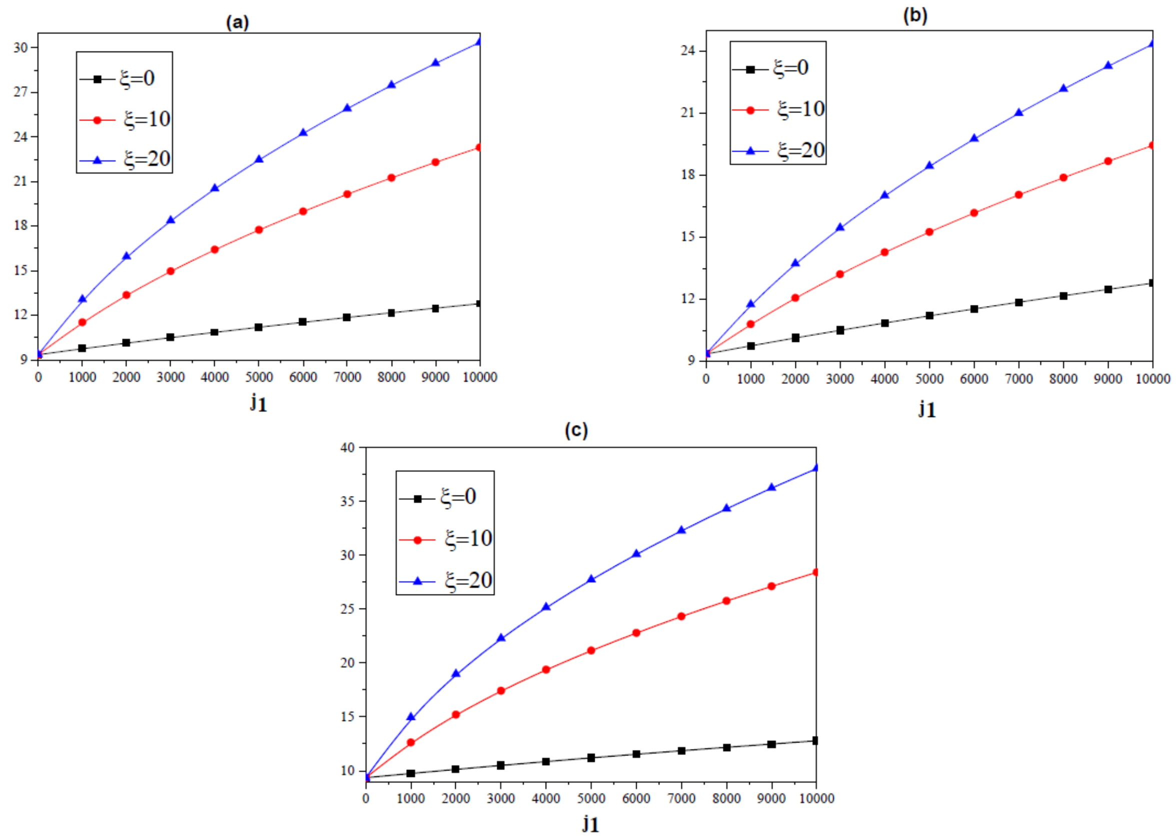


Fig. 5 Variation of the non-dimensional frequencies $\hat{\omega}$ of BDFG beam against elastic foundation parameters j_1 for three values of the parabolic parameter and for three cases of variable elastic foundation (a) Linear, (b) Parabolic, (c) Sinusoidal (L/h =10, $n_x = n_z = 1$, $j_2 = 1000$)

isotropic beam n_x=n_z=0.

- In this figure too, in the case where n_z=0 (Solid Line), the increase of n_x causes the increase of the frequency, therefore a reduction of the period. This case corresponds to

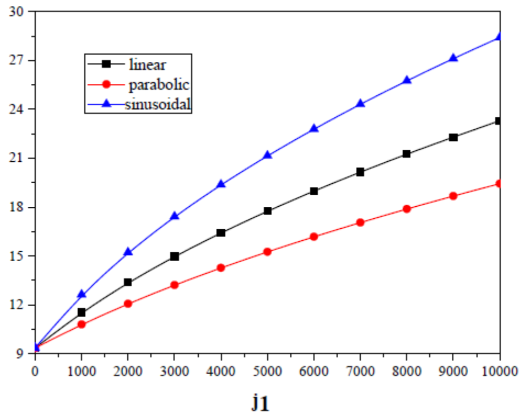


Fig. 6 Effect of the nature of the elastic foundation on the non-dimensional frequencies $\hat{\omega}$ of BDFG beam (L/h =10, $n_x = n_z = 1$, $j_2 = 1000$)

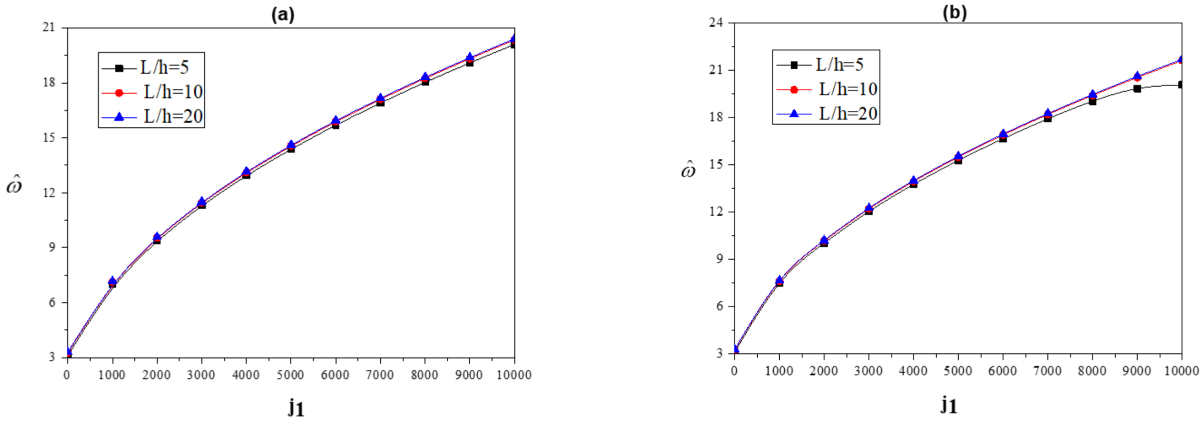


Fig. 7 Effect of aspect ratio L/h on the non-dimensional frequencies $\hat{\omega}$ j_2=0, ξ =10(linear) (a) $n_x = n_z = 0$ and (b) $n_x = n_z = 1$

a rigid axially unidirectional FG beams. In contrast to the second case where n_x=0 with variation of n_z, a decrease of frequency is observed.

- The same remark is observed in figure (b). For n_z=1 with variation of n_x (axially unidirectional FG beams), the frequencies increase. The opposite is observed for the second case.

- In figure (b), the two curves intersect at the point n_x=n_z=1. At this point, we have an FG beam where the variation of the volume fraction is the same in both directions.

Fig. 4 shows the effect of the elastic foundation parameters on the dimensionless frequency for three types of FG beams.

- A transversely unidirectional FG beam (n_z=1 and n_x=0),
- An axially unidirectional FG beam (n_z=0 and n_x=1),
- A BDFG beam (n_z=1 and n_x=1).

In this figure we can see that the increase in the values of the two parameters of the elastic foundation (Winkler and Pasternak) leads to an increase in frequency. This can be explained by the fact that the incorporation of an elastic foundation increases the rigidity of the beam, thus reducing its period and consequently increasing its frequency.

In Fig. 4(a), the transversely unidirectional FG beam gives the highest frequencies, the BDFG beam gives the lowest frequencies and the axially unidirectional beam is in between.

In Fig. 4(b), the curves are quite close and the frequency difference between the three beams is not very large.

Table 3 lists the non-dimensional frequencies $\hat{\omega}$ of BDFG beam for various values of the aspect ratio L/h and three cases of elastic foundation parameters. The table shows a significant influence of the elastic foundation on the fundamental frequency of the beam. Indeed, the inclusion of an elastic foundation contributes to the increase of the stiffness of the beam and consequently increases its frequencies. However, the values of the L/h ratio, which give an indication of the slenderness of the beam, have very little influence on the frequencies.

The variation of non-dimensional frequencies $\hat{\omega}$ of BDFG beam with the elastic foundation parameter J1 is depicted in Fig. 5 for various values of the parabolic parameter ξ and for three cases of elastic support (linear, parabolic and sinusoidal). As can be seen from this figure, increasing the values of the parabolic parameter leads to an increase in frequencies. In addition, between the three supports used, the use of a sinusoidal foundation generates an increase in frequencies.

The effect of the nature of the variable elastic foundation on the non-dimensional frequencies $\hat{\omega}$ of BDFG beam is plotted in figure 6.As has been motioned before, the use of a sinusoidal foundation leads to an increase in frequencies.

Fig. 7 shows the effect of the *L/h* ratio on the non-dimensional frequenciesfor two cases (a) an isotropic beam and (b) a BDFG beam. As can be clearly seen in this figure, the frequencies increase with the increase of the foundation parameter J_1 but remain insensitive to the *L/h* ratio.

4. Conclusions

In this research, the free vibration analysis of BDFG beam resting on variable elastic foundation has been presented. The material properties of the beams are assumed to vary in both the thickness and longitudinal directions. A 2D theory is used to evaluate the natural frequency of BDFG beam on variable elastic foundation. The equations of motion have been derived from Hamilton's principle and they were solved by Navier solution for simply supported beam. The obtained results were compared with the available solutions in literature and show a good agreement. A detailed parametric study is presented to discuss the significant impacts of different parameters on the dynamic behavior of BDFG beam resting on variable elastic foundation.

According to the results of the study, the followings can be drawn:

- For the case of rigid axially unidirectional FG beams ("nz"=0), the increase of "nx" causes the increase of the frequency, therefore a reduction of the period.

- Otherwise ("nx"=0), with variation of nz, a decrease of frequency is observed.

- The increase in the values of the two parameters of the elastic foundation (Winkler and Pasternak) leads to an increase in frequency.

- For a FG beam on elastic foundation, the transversely unidirectional FG beam gives the highest frequencies, the BDFG beam gives the lowest frequencies and the axially unidirectional beam is in between.

- The increasing in the values of the parabolic parameter leads to an increase in frequencies. In addition, between the three supports used (Linear, Parabolic and Sinusoidal), the use of a sinusoidal foundation generates an increase in frequencies.

- The frequencies increase with the increase of the foundation parameter J1 but remain insensitive to the L/h ratio.

Although this document deals with the analysis of free vibration, the extension of this study is also envisaged by considering other types of materials and other models with shear deformation effect (Civalek *et al.* 2021, Si Tayeb *et al.* 2020, Panjehpour *et al.* 2018, Timesli 2020, Al-Basyouni *et al.* 2020, Al-Osta 2019, Kiani 2019, Yaghoobi and Taheri 2020, Demir and Civalek 2017).

References

Al-Basyouni, K. S., Ghandourah, E., Mostafa, H.M. and Algarni, A. (2020), "Effect of the rotation on the thermal stress wave propagation in non-homogeneous viscoelastic body", *Geomech. Eng.*, **21**(1), 1-9. https://doi.org/10.12989/gae.2020.21.1.001.

Ali Rachedi, M., Benyoucef, S., Bouhadra, A., Bachir Bouiadjra, R., Sekkal, M. and Benachour, A. (2020), "Impact of the homogenization models on the thermoelastic response of FG plates on variable elastic foundation", *Geomech. Eng.*, **22**(1), 65-80. http://doi.org/10.12989/gae.2020.22.1.065.

Al-Osta, M.A. (2019), "Shear behaviour of RC beams retrofitted using UHPFRC panels epoxied to the sides", *Comput. Concrete*, **24**(1), 37-49. http://doi.org/10.12989/cac.2019.24.1.037

Ahmed, R.A., Fenjan, R.M. and Faleh, N.M. (2019), "Analyzing post-buckling behavior of continuously graded FG nanobeams with geometrical imperfections", *Geomech. Eng.*, **17**(2), 175-180. https://doi.org/10.12989/gae.2019.17.2.175.

Akbas, S.D. (2015), "Wave propagation of a functionally graded beam in thermal environments", *Steel Compos. Struct.*, **19**(6), 1421-1447. https://doi.org/10.12989/SCS.2015.19.6.1421.

Attia, M.A. (2017), "On the mechanics of functionally graded nanobeams with the account of surface elasticity", *Int. J. Eng. Sci.*, **115**, 73-101. https://doi.org/10.1016/j.ijengsci.2017.03.011.

Avcar, M. (2019), "Free vibration of imperfect sigmoid and power law functionally graded beams", *Steel Compos. Struct.*, **30**(6), 603-615. https://doi.org/10.12989/scs.2019.30.6.603.

Bachir Bouiadjra, R., Bachiri, A., Benyoucef, S., Fahsi, B. and Bernard, F. (2020), "An investigation of the thermodynamic effect of FG beam on elastic foundation", *Struct. Eng. Mech.*, **76**(1), 115-127. https://doi.org/10.12989/sem.2020.76.1.115.

Bachiri, A., Bourada, M., Mahmoudi, A., Benyoucef, S. and Tounsi, A. (2018), "Thermodynamic effect on the bending response of elastic foundation FG plate by using a novel four variable refined plate theory", *J. Therm. Stresses*, **41**(8), 1042-1062. https://doi.org/10.1080/01495739.2018.1452169.

Benferhat, R., Daouadji, T.H. and Adim, B. (2016), "A novel higher order shear deformation theory based on the neutral surface concept of FGM plate under transverse load", *Adv. Mater. Res.*, **5**(2), 107-120. https://doi.org/10.12989/amr.2016.5.2.107.

Bhattacharya, S. and Das, D. (2019), "Free vibration analysis of bidirectional-functionally graded and double-tapered rotating micro-beam in thermal environment using modified couple stress theory", *Compos. Struct.*, **215**, 471-492. https://doi.org/10.1016/j.compstruct.2019.01.080.

Boulal, A., Bensattalah, T., Karas, A., Zidour, M., Heireche, H. and Adda Bedia, E.A. (2020), "Buckling of carbon nanotube reinforced composite plates supported by Kerr foundation using Hamilton's energy principle", *Struct. Eng. Mech.*, **73**(2), 209-223. https://doi.org/10.12989/sem.2020.73.2.209.

Chaabane, L.A., Bourada, F., Sekkal, M., Zerouati, S., Zaoui, F.Z., Tounsi, A., Derras, A., Bousahla, A.A. and Tounsi, A., (2019), "Analytical study of bending and free vibration response of functionally graded beam resting on elastic foundation", *Struct. Eng. Mech.*, **71**(2), 185-196. https://doi.org/10.12989/sem.2019.71.2.185.

Chami., K., Massafer, T. and Hadji, L. (2020), "Analytical modeling of bending and free vibration of thick advanced composite beams resting on Winkler-Pasternak elastic foundation", *Earthq. Struct.*, **19**(2), 91-101. http://doi.org/10.12989/eas.2020.19.2.091.

Civalek, Ö., Dastjerdi, S., Akbaş, Ş. and Akgöz, B. (2021), "Vibration analysis of carbon nanotube-reinforced composite microbeams", *Math. Meth. Appl. Sci.* https://doi.org/10.1002/mma.7069.

Cuong-Le, T., Nguyen, K.D., Nguyen-Trong, N., Khatir, S., Nguyen-Xuan, H. and Abdel-Wahab, M., (2020), "A three-dimensional solution for free vibration and buckling of annular plate, conical, cylinder and cylindrical shell of FG porous-cellular materials using IGA", *Compos. Struct.*, **259**, 113216. https://doi.org/10.1016/j.compstruct.2020.113216.

Cuong-Le, T., Nguyen, T.N., Vu, T.H., Khatir, S. and Abdel-Wahab, M. (2020), "A geometrically nonlinear size-dependent hypothesis for porous functionally graded micro-plate", *Eng. Comput.*, 1-12. https://doi.org/10.1007/s00366-020-01154-0.

Demir., Ç. and Civalek, Ö. (2017), "On the analysis of microbeams", *Int. J. Eng. Sci.*, **121**, 14-33. https://doi.org/10.1016/j.ijengsci.2017.08.016.

Eisenberger, M. and Clastornik, J. (1987), "Vibrations and buckling of a beam on a variable Winkler elastic foundation", *J. Sound Vib.*, **115**, 233-241. https://doi.org/10.1016/0022-460X(87)90469-X.

Fariborz, J. and Batra, R.C. (2019), "Free vibration of bi-directional functionally graded material circular beams using shear deformation theory employing logarithmic function of radius", *Compos. Struct.*, **210**, 217-230. https://doi.org/10.1016/j.compstruct.2018.11.036.

Faroughi, S., Rahmani, A. and Friswell, M.I. (2020), "On wave propagation in two-dimensional functionally graded porous rotating nano-beams using a general nonlocal higher-order beam model", *Appl. Math. Model.*, **80**, 169-190. https://doi.org/10.1016/j.apm.2019.11.040.

Hadji, L. (2020), "Influence of the distribution shape of porosity on the bending of FGM beam using a new higher order shear deformation model", *Smart Struct. Syst.*, **26**(2), 253-262. http://doi.org/10.12989/sss.2020.26.2.25.

Hadji, L. and Avcar, M. (2021), "Free vibration analysis of FG porous sandwich plates under various boundary conditions", *J. Appl. Comput. Mech.*, **7**(2), 505-519. http://doi.org/10.22055/JACM.2020.35328.2628.

Hao, D. and Wei, C. (2016), "Dynamic characteristics analysis of bi-directional functionally graded Timoshenko beams", *Compos. Struct.*, **141**, 253-263. https://doi.org/10.1016/j.compstruct.2016.01.051.

Jalaei, M.H. and Civalek, Ö. (2019), "On dynamic instability of magnetically embedded viscoelastic porous FG nanobeam", *Int. J. Eng. Sci.*, **143**, 14-32. https://doi.org/10.1016/j.ijengsci.2019.06.013.

Karamanli, A. (2017a), "Elastostatic analysis of two-directional functionally graded beams using various beam theories and symmetric smoothed particle hydrodynamics method", *Compos. Struct.*, **160**, 653-669. http://doi.org/10.1016/j.compstruct.2016.10.065.

Karamanli, A. (2017b), "Bending behaviour of two directional functionally gradedsandwich beams by using a quasi-3d shear deformation theory", *Compos. Struct.*, **174**, 70-86. https://doi.org/10.1016/j.compstruct.2017.04.046

Kar, V.R. and Panda, S.K. (2020), "Nonlinear flexural vibration of shear deformable functionally graded spherical shell panel", *Steel Compos. Struct.*, **18**(3), 693-709. https://doi.org/10.12989/scs.2015.18.3.693.

Kiani, Y. (2019), "NURBS-based thermal buckling analysis of graphene platelet reinforced composite laminated skew plates", *J. Therm. Stresses*, 1-19. https://doi.org/10.1080/01495739.2019.1673687.

Li, J., Guan, Y., Wang, G., Zhao, G., Lin, J., Naceur, H. and Coutellier, D. (2018), "Meshless modeling of bending behavior of bi-directional functionally graded beam structures", *Compos. Part B Eng.*, **155**, 104-111. https://doi.org/10.1016/j.compositesb.2018.08.029.

Lu, Y. and Chen, X. (2020), "Nonlinear parametric dynamics of bidirectional functionally graded beams", *Shock Vib.*, 8840833, https://doi.org/10.1155/2020/8840833.

Merzoug, M., Bourada, M., Sekkal, M., Abir, A.C., Chahrazed, B., Benyoucef, S. and Benachour, A. (2020), "2D and quasi 3D computational models for thermoelastic bending of FG beams on variable elastic foundation: Effect of the micromechanical models", *Geomech. Eng.*, **22**(4), 361-374. http://doi.org/10.12989/gae.2020.22.4.361.

Madenci, E. (2019), "A refined functional and mixed formulation to static analyses of fgm beams", *Struct. Eng. Mech.*, **69**(4), 427-437. https://doi.org/10.12989/sem.2019.69.4.427.

Madenci, E., Özkılıç, Y.O. and Gemi, L. (2020), "Experimental and theoretical investigation on flexure performance of pultruded GFRP composite beams with damage analyses", *Compos. Struct.*, **242**, 112162. https://doi.org/10.1016/j.compstruct.2020.112162.

Madenci, E., Özkılıç, T.O. and Gemi, L. (2020), "Buckling and free vibration analyses of pultruded GFRP laminated composites: Experimental, numerical and analytical investigations", *Compos. Struct.*, **254**, 112806. https://doi.org/10.1016/j.compstruct.2020.112806.

Madenci, E., Özkılıç, T.O. and Gemi, L. (2020), "Theoretical investigation on static analysis of pultruded GFRP composite beams", *Akademik Platform Mühendislik ve Fen Bilimleri Dergisi,* **8**(3), 483-490. https://doi.org/10.21541/apjes.734770.

Madenci, E. and Özütok, A. (2020), "Variational approximate for high order bending analysis of laminated composite plates", *Struct. Eng. Mech.*, **73**(1), 97-108. https://doi.org/10.12989/sem.2020.73.1.097.

Madenci, E. and Özütok, A. (2017), "Variational approximate and mixed-finite element solution for static analysis of laminated composite plates", *Solid State Phenom.*, **267**, 35-39. https://doi.org/10.4028/www.scientific.net/SSP.267.35

Madenci, E. and Gülcü, Ş. (2020), "Optimization of flexure stiffness of FGM beams via artificial neural networks by mixed FEM", *Struct. Eng. Mech.*, **75**(5), 633-642. https://doi.org/10.12989/sem.2020.75.5.633.

Nemati, A.R. and Mahmoodabadi, M.J. (2019), "Effect of micromechanical models on stability of functionally graded conical panels resting on Winkler-Pasternak foundation in various thermal environments", *Arch. Appl. Mech.*, 1-33. https://doi.org/10.1007/s00419-019-01646-6.

Nguyen, D.K., Nguyen, Q.H., Tran, T.T. and Bui, V.T. (2017), "Vibration of bi-dimensional functionally graded Timoshenko beams excited by a moving load", *Acta Mech.*, **228**(1), 141-155. http://doi.org/10.1007/s00707-016-1705-3.

Ouldlarbi, L., Kaci, A., Houari, M.S.A. and Tounsi, A., (2013), "An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams", *Mech. Based Des. Struct. Mach.*, **41**(4), 421-433. https://doi.org/10.1080/15397734.2013.763713.

Özütok, A. and Madenci, E. (2017), "Static analysis of laminated composite beams based on higher-order shear deformation theory by using mixed-type finite element method", *Int. J. Mech. Sci.*, **130**, 234-243. https://doi.org/10.1016/j.ijmecsci.2017.06.013.

Panjehpour, M., Loh, E.W.K. and Deepak, T.J. (2018), "Structural insulated panels: State-of-the-Art", *Trends Civ. Eng. Arch.*, **3**(1), 336-340. https://doi.org/10.32474/TCEIA.2018.03.000151.

Pradhan, S.C. and Murmu, T. (2009), "Thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations using differential quadrature method", *J. Sound Vib.*, **321**, 342-362. https://doi.org/10.1016/j.jsv.2008.09.018.

Pydah, A. and Sabale, A. (2017), "Static analysis of bi-directional functionally graded curved beams", *Compos. Struct.*, **160**, 867-876. https://doi.org/10.1016/j.compstruct.2016.10.120.

Rahmani, A., Faroughi, S. and Friswell, M.I. (2020), "The vibration of two-dimensional imperfect functionally graded(2D-FG) porous rotating nanobeams based on general nonlocaltheory", *Mech. Syst. Signal Process.*, **144**, 106854. https://doi.org/10.1016/j.ymssp.2020.106854.

Rahmani, M., Mohammadi, Y. and Kakavand, F. (2019), "Vibration analysis of sandwich truncated conical shells with porous FG face sheets in various thermal surroundings", *Steel Compos. Struct.*, **32**(2), 239-252. http://doi.org/10.12989/scs.2019.32.2.239.

Rajasekaran, S. and Khaniki, H.B. (2018), "Free vibration analysis of bi-directional functionally graded single/multi-cracked beams", *Int. J. Mech. Sci.*, **144**, 341-356. https://doi.org/10.1016/j.ijmecsci.2018.06.004.

Ramteke, P.M., Panda, S.K. and Sharma, N. (2019), "Effect of grading pattern and porosity on the eigen characteristics of porous functionally graded structure", *Steel Compos. Struct.*, **33**(6), 865-875. http://doi.org/10.12989/scs.2019.33.6.865.

Shahmohammadi, M.A., Azhari, M. and Saadatpou, M.M. (2020), "Free vibration analysis of sandwich FGM shells using isogeometric B-spline finite strip method", *Steel Compos. Struct.*, **34**(3), 361-376. http://doi.org/10.12989/scs.2020.34.3.361.

Selmi, A. (2020), "Exact solution for nonlinear vibration of clamped-clamped functionally graded buckled beam", *Smart Struct. Syst.*, **26**(3), 361-371. https://doi.org/10.12989/SSS.2020.26.3.361.

Shafiei, N., Mirjavadi, S.S., Mohasel Afshari, B., Rabby, S. and Kazemi, M. (2017), "Vibration of two-dimensional imperfect functionally graded (2D-FG) porous nano-/micro-beams", *Comput. Meth. Appl. Mech. Eng.*, **322**, 615-632. https://doi.org/10.1016/j.cma.2017.05.007.

Şimşek, M. (2010), "Fundamental frequency analysis of functionally graded beams by using different higher order beam theories", *Nucl. Eng. Des.*, **240**(4), 697-705. https://doi.org/10.1016/j.nucengdes.2009.12.013.

Şimşek, M. (2015), "Bi-directional functionally graded materials (BDFGMs) for free and forced vibration of Timoshenko beams with various boundary conditions", *Compos. Struct.*, **133**, 968-978. http://doi.org/10.1016/j.compstruct.2015.08.021.

Şimşek, M. (2016), "Buckling of Timoshenko beams composed of two-dimensional functionally graded material (2D-FGM) having different boundary conditions", *Compos. Struct.*, **149**, 304-314. https://doi.org/10.1016/j.compstruct.2016.04.034.

Si Tayeb, T., Zidour, M., Bensattalah, T., Heireche, H., Benahmed, A. and Adda Bedia, E.A. (2020), "Mechanical buckling of FG-CNTs reinforced composite plate with parabolic distribution using Hamilton's energy principle", *Adv. Nano Res.*, **8**(2), 135-148. https://doi.org/10.12989/anr.2020.8.2.135.

Sobhy, M. (2015), "Thermoelastic response of FGM plates with temperature-dependent properties resting on variable elastic foundations", *J. Appl. Mech.*, **7**(6), 1550082. https://doi.org/10.1142/S1758825115500829.

Timesli, A. (2020), "Prediction of the critical buckling load of SWCNT reinforced concrete cylindrical shell embedded in an elastic foundation", *Comput. Concrete*, **26**(1), 53-62. https://doi.org/10.12989/cac.2020.26.1.053.

Tang, Y. and Ding, Q. (2019), "Nonlinear vibration analysis of a bi-directional functionally graded beam under hygro-thermal loads", *Compos. Struct.*, **225**,111076. https://doi.org/10.1016/j.compstruct.2019.111076.

Tran, T.T. and Nguyen D.K. (2018), "Free vibration analysis of 2-DFGM beams in thermal environment based on a newthird-order shear deformation theory", *Vietnam J. Mech.*, **40**(2), 121-140. https://doi.org/10.15625/0866-7136/10503.

Vinyas, M. (2020), "On frequency response of porous functionally graded magneto-electro-elastic circular and annular plates with different electro-magnetic conditions using HSDT", *Compos. Struct.*, **240**, 112044. https://doi.org/10.1016/j.compstruct.2020.112044.

Wang, Z.H., Wang, X.H., Xu, G.D., Cheng, S. and Zeng, T. (2016), "Free vibration of two-directional functionally graded beams", *Compos. Struct.*, **135**, 191-198. http://doi.org/10.1016/j.compstruct.2015.09.013.

Yaghoobi, H. and Taheri, F. (2020), "Analytical solution and statistical analysis of buckling capacity of sandwich plates with uniform and non-uniform porous core reinforced with graphene nanoplatelets", *Compos. Struct.*,**252**, 112700. https://doi.org/10.1016/j.compstruct.2020.112700

Zhou, D. (1993), "A general solution to vibrations of beams on variable Winkler elastic foundation", *Comput. Struct.*, **47**(1), 83-90. https://doi.org/10.1016/0045-7949(93)90281-H.

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