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Optimal design of shape of a working in cracked rock mass

Vagif M. Mirsalimov*[1,2]

[1]*Department of Mechanics, Azerbaijan Technical University, Baku, H. Javid av, 25, Azerbaijan*
[2]*Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Baku, B. Vahabzade, 9, Azerbaijan*

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Abstract. A criterion and a method for solving a problem on the prevention of mine working fracture under the action of tectonic and gravitational forces are offered. Based on minimal criterion, theoretical analysis of the definition of the optimal shape of working in the rock mass weakened by arbitrarily located rectilinear cracks was carried out. A closed system of algebraic equations allowing to minimize the stress state and stress intensity factors depending on mechanical and geometrical characteristics of the rock, is constructed. The relation between the shape of the working and the stress intensity factors and also location and sizes of the cracks is obtained. The found optimal shape of working increases load-bearing capacity of the rock.

Keywords: isotropic rock mass; optimal working; rectilinear cracks; stress intensity factors; minimax criterion

1. Introduction

In the practice of mining and underground construction, there are extensive underground structures of which one of the characteristic linear sizes is much more than other (in cross section) sizes. Carrying out such underground workings in deep horizons in complex mining and geological conditions makes higher requirements to their fastening, constructions and support materials. Significant effect is achieved by giving such a form to extensive working that is most appropriate for the given mining-geological conditions. Designing of underground structures, i.e. creation of rather durable workings should be based on scientific developments. These scientific developments not only show under which conditions would be the working, but also indicate the way for increasing its strength. The conditions under which may be the working here we understand stress and strength conditions of the rock in the vicinity of the working. These conditions are determined by solving a problem of optimal designing of working in the rock mass. As is known, one of the methods for increasing strength of working is the stress reduction in the rock mass. Observations and special calculations show that the stress state and strength of the working are dependent not only with natural factors (stress in the untouched rock, elastic and strength properties of rocks), but also on the form of the working section shape. It is appropriate to quote the words of the famous Spanish architect Eduardo Torroja: “The best construction is the one whose reliability is provided mainly due to its shape, but not due to the strength of the material. The latter is achieved simply, while the first one, vice versa, with great difficulty” (Andreev 1986).

The problems of determining optimal workings in the general case are reduced to solving variational problems with unknown boundaries. In some cases, in order to determine the shape of the hole on which technologically inevitable stress concentration would be reduced to minimum, compared with all other possible shapes of holes, we come to an elasticity theory problem with an unknown boundary.

The elasticity and plasticity theory problems with unknown boundaries were considered in (Cherepanov 1963, 1974, 2015, Vigdergauz 1976, 1977, 2006, 2016, 2017, 2018, Banichuk 1977, 1980, Kurshin and Onoprienko 1976, Mirsalimov 1974, 1975, 1977, 1979, 2019, 2020, Wheeler 1976, 1992, Savruk and Kravets 2002, Burchill and Heller 2004, Kirilyuk and Levchuk 2008, Mir-Salim-zade 2007, 2019, Wu 2009, Gogolauri 2012, Odishelidze *et al.* 2016, Kalantarly 2017, Wang *et al.* 2018, Zeng *et al.* 2020). A review of the papers on determining equal strength holes in defect-free constructions was given in (Cherepanov 2015). The shape of the equally strong outlines of holes on which the technologically inevitable stress concentration would be least as compared with all other outlines is determined for the first time by Cherepanov (1963). Cherepanov (1974) proposed an effective exact solution of some inverse plane problems of the theory of elasticity concerning the determination of equally strong outlines of holes. The case of one and two holes as well as the case of periodic and doubly-periodic series of holes was considered. A formulation of the problem is given first and the fundamental relationships are presented. The general problem for any number of holes in an infinite plane was reduced to a standard Dirichlet problem for the exterior of the same number of parallel slits on a parametric plane. The problem of finding the optimum shape of holes in a perforated plate weakened by a triangular or square lattice of holes and subject to bending is considered (Mirsalimov 1974) by methods based on the theory of functions of a complex variable. Two-dimensional problems are considered to find the “equally rigid” form of a hole in an

*Corresponding author, Professor
E-mail: mir-vagif@mail.ru

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anisotropic medium (Mirsalimov 1975). The paper (Wheeler 1976) is concerned with conditions under which surfaces of constant stress magnitude serve as optimal from the standpoint of minimizing stress. Such conditions are established for elastic solids in cases of antiplane shear deformation, axisymmetric torsional deformation, and plane deformation. The problem of determining the shape of transverse section of a prismatic bar with a prismatic longitudinal cavity (hole) of given shape, subjected to torsion, from the condition that the torsional stiffness would be maximal for a given cross-sectional area, is considered by Kurshin and Onoprienko (1976). The apparatus of complex variable function theory is used to determine the outline required. Examples of computing the outlines of the sections for elliptical, square, and rectangular hole shapes are presented. Vigdergauz (1976) reduced the initial inverse problem for an arbitrary, finitely connected region to a Fredholm-type equation relative to the density of integral representation of the function which maps conformally a plane with circles excluded, onto a plane of the same connectivity with an unknown boundary. The equation obtained is solved by the method of least squares. The coefficients of the corresponding algebraic system are determined and a one-parameter family of the contours sought is constructed for a plane, symmetrically periodic distribution of holes. Inverse doubly periodic problem of thermoelasticity is considered in (Mirsalimov 1977). The problem of determining the contours of a finite number of holes of equal strength in a statically loaded plane under condition that the given normal stresses on the outline of each hole take on constant but distinct values is considered by Vigdergauz (1977). To seek the optimal hole shapes in elastic bodies, which cause minimal stress concentration Banichuk (1977) considered minimax optimization problems with a local criterion within the framework of plane elasticity theory and proved that holes with equi-stressed boundaries are optimal. Wheeler (1992) studied the problem of determining the shapes of minimum stress concentration devoting special attention to the mathematically precise formulation of the problem and the study of cases in which the optimum shape can be found in a highly explicit form. For all the cases discussed, a mathematical proof of the optimality is given. Savruk and Kravets (2002) proposed the procedure for the determination of the contours of equistrong holes in elastic plates based on the method of singular integral equations. The problem is reduced to the solution of inverse two-dimensional problems of the theory of elasticity with unknown boundaries. The singular integral equations of the direct problems were solved numerically by the method of mechanical quadratures. The parameters of the contours of equistrong holes are determined for plates with one, two, or a system of periodic holes loaded by biaxial tension at infinity and internal pressure on the contours of holes. An iterative two-dimensional finite element gradientless shape optimization procedure was used by Burchill and Heller (2004) to obtain optimal shapes, which offer the lowest possible stress concentration, subject to geometric constraints. The optimal free-form shapes were seeking for holes in large and finite width plates under biaxial and uniaxial loading conditions respectively. The mapping function coefficients were used by Vigdergauz (2006) as design variables in the genetic-algorithm approach to find a piecewise smooth optimal shape of a single traction-free hole in an elastic plate that minimizes the local stresses under remote shear. Determination of equistrong hole shape in isotropic medium (infinite plate), reinforced by regular system of stringers is given in paper (Mir-Salim-zade 2007). The static inverse thermoelastic problem for an infinite elastic isotropic medium containing a cavity of unknown shape, under three-axes tension and given constant values of the pressure and the temperature on the cavity surface, is considered by Kirilyuk and Levchuk (2008). Nonlinear equations for determining the geometric cavity parameters, which lead to an equal-stress state along the cavity surface, were obtained. To obtain optimal hole shape for minimum stress concentration in two-dimensional finite plates the parameterized geometry models are used by Wu (2009). The boundary shape for a hole was described by two families of smooth curves. The geometries of the optimized holes were presented in a form of compact parametric functions. The problem of finding equistrong unknown holes in an elastic square and stressed state of the square under condition that the tangential normal stress at the hole boundaries takes constant value is considered by Gogolauri (2012). Using the methods of complex analysis, Odishelidze *et al.* (2016) studied the problem of plate bending for a square weakened with a full-strength hole. Specially constructed Kolosov–Muskhelishvili potentials are used (Vigdergauz 2016) to obtain a concise formulation of 2D elastostatic problems for general regular structures with multiphase nested inclusions in complex-variable terms. The existence of the equi-stress nested inclusions under the square symmetry of the structure is proved. Vigdergauz (2017) generalized the developments in elastostatics concerning the equi-stressness criterion of optimality for two-dimensional multi-connected unbounded solids under the bulk-dominating load toward the transient three-dimensional case with rotational symmetry. The layer potentials of two-dimensional linear elastostatics are applied by Vigdergauz (2018) for the semi-analytical design of non-standard arrangements of the equi-stress holes in an infinite plate under a given bulk-type loading. The main attention focused on periodic structures of low rotational symmetry which are hard to tackle by the conformal mapping technique. To minimize the stress concentration around the edge of a hole in an orthotropic plate Wang *et al.* (2018) first presented the analytical solution of the stress distribution around arbitrary holes using the complex variable method and then carried out the shape optimization using the mixed penalty function method. To minimize the maximum tangential stress around the boundaries two symmetrical and identical holes are mapped onto an annulus by Zeng *et al.* (2020). The coefficients of found mapping function which describe the boundary are calculated by differential-evolution algorithm.

The problem of determining the working of uniform strength in the solid rock was considered in the work(Mirsalimov 1979).

In these works, the possibility of presence in the solid of

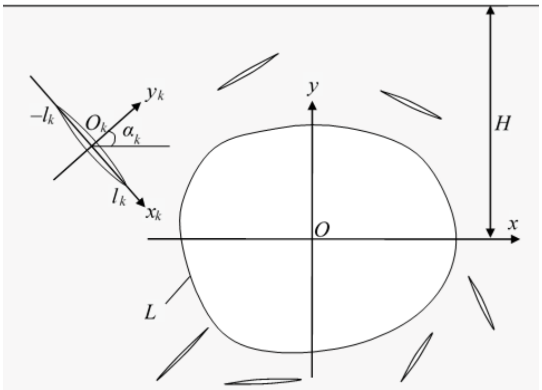

Fig. 1 Design scheme of the optimization problem for a working in a rock mass

a crack is not considered.

In the stage of designing of different structures and engineering constructions, it is necessary to take into account the cases when constructions have cracks.

The problem of finding the equally strong hole form in the crack tip and its influence on development of longitudinal shear crack are studied by Kalantarly (2017). A theoretical analysis to determine the optimal shape of a hole for an isotropic elastic plate weakened by an arbitrary system of cracks was performed (Mirsalimov 2019, 2020). The principle of equal strength and minimization of the stress intensity factors (Mirsalimov 2019) are used. Also, the minimax criterion is used (Mirsalimov 2020). A criterion and a method for solving the inverse problem of fracture prevention for the plate under the action of given system of external loads are proposed. The problem of minimizing the stress state on the contour of a hole in an unbounded isotropic stringer plate weakened by two rectilinear cracks is considered by Mir-Salim-zade (2019). Determined is the optimal hole contour, at which no crack growth occurs, and the maximum circumferential stress there on is minimal.

In the present article, problem of determining the working's shape that minimizes the stress state in the rock mass is considered taking into account the presence of the cracks.

We consider an elastic isotropic rock mass with working and arbitrarily located rectilinear cracks. Study on rock mass with cracks is of great interest (Chen *et al*. 2018, Zhou *et al*. 2018, Chang *et al*. 2018, Lee and Hong 2018, Zhu and Yang 2018, Lv *et al*. 2019, Sun *et al*. 2019). To increase the strength of the rock mass, it is necessary to determine the optimal shape of the working. The load-bearing capacity of the rock mass with working depends on stress concentrations on the contour of the working. In this connection, optimal design of a working in rock mass with cracks is of great significance. The load-bearing capacity of the rock with working may be reasonably managed with design and technological methods, in particular with the geometry of the working's shape. However, the solution of the problems of geomechanics by building such geometry of the surface of the working in the rock, so that the stress field created by it would prevent the crack growth (fracture) in the rock with working is still unknown. Changing the form of the working in the cracked rock mass one can lower the level of the stress state. The optimal design theory problem is in determination of characteristics of the rock mass so that the cracked rock mass would be the best rock mass among the remaining ones of the class under consideration under the action of tectonic and gravitational forces.

The goal of the paper is to develop a design model for a rock mass with the working and rectilinear cracks that allows to calculate the optimal shape of the working under the given loading conditions.

2. Problem statement

Let a rock mass be weakened by a working and arbitrarily located rectilinear cracks (Fig. 1).

Unfortunately, most of the rocks are anisotropic and heterogeneous (Hoek and Brown 1980, Barton and Quadros 2014). Therefore, solution of the problem becomes much more complicated. At the present level of solid mechanics, it is possible to use the anisotropic model of a rock mass (Mirsalimov 1975, Aitaliyev *et al*. 1986). The proposed method can be generalized and developed for anisotropic rock masses as well.

The working (tunnel) is far enough from the Earth's surface. We choose the origin of coordinates in the geometrical center of the working. Let a heavy elastic half-space $y < H$ be weakened by a tunnel that is a cylinder with an axis parallel to the surface of the half-space. We consider a plane problem of finding the working's shape minimizing the stress state in the rock mass. The stress state of the rock mass is formed mainly from the action of gravitational and tectonic forces. The tectonic forces are assumed constant over the depth of the rock mass. The stress distribution in the rock from the gravitational forces are accepted according to A.N. Dinnik's hypothesis

$$\sigma_x = -\lambda\rho_*(H-y)\,,\quad \sigma_y = -g\rho_*(H-y)\,,\quad \tau_{xy} = 0 \qquad (1)$$

where σ_x, σ_y are the horizontal and vertical normal stresses, respectively, τ_{xy} are tangential stresses, $\lambda=\nu/(1-2)$ is lateral thrust ratio of the rock; ν is the Poisson ratio of the rock; ρ_* is the mean density of the rock mass; g is gravity acceleration; $H-y$ is the depth of the considered point of the rock from the Earth surface.

It is considered that the rectilinear crack faces are free from external stresses. In the center of rectilinear cracks, we locate a local system of coordinates whose axis $x_kO_ky_k$ (k = 1,2,…,N_1) coincides with the crack line and makes an angle α_k with the axis x. To the still unknown surface of the working, we apply the normal pressure $\sigma_n=-p$ ($p > 0$) and tangential load equal to zero. Far from the working, all the elastic half-space is deformed by stresses (imitating tectonic forces):

$$\sigma_x = \sigma_x^\infty, \qquad \sigma_y = \sigma_y^\infty, \qquad \tau_{xy} = 0 \tag{2}$$

The surface of the half-space $y = H$ is subjected to the constant normal load $\sigma_y=\sigma^\infty_y$ and to tangential one equal to zero.

The boundary conditions in the problem under consideration have the form

$$\sigma_n = -p, \qquad \tau_{nt} = 0 \quad \text{on} \quad r = \rho(\theta)$$
$$\sigma_{y_k}=0, \quad \tau_{x_k y_k}=0 \ (k = 1,2,\dots,N_1) \quad \text{on the crack faces} \tag{3}$$

Here the function $\rho(\theta)$ characterizes the working's shape and is unknown in advance.

The problem is to determine the shape of the working (the function $\rho(\theta)$) in the rock mass so that the stress field formed by it would prevent the crack growth in the rock. To determine the sought-for function $\rho(\theta)$ (working's shape in the rock) the problem statement should be supplemented by the criterion for choosing the shape of the working. The lower the stress level in the rock mass the higher the load-bearing capacity of the rock. According to the Irwin-Orowan theory of brittle fracture, the stress intensity factor is a parameter characterizing the stress-strain state in the vicinity of the crack tips. Thus, the stress intensity factor in the vicinity of the crack tips may be considered responsible for the fracture of the rock material.

It is known that in the heavy rock mass there is no a hole on the contour of which the circumferential normal stress σ_t would be a constant (Mirsalimov 1979, 1987). Consequently, in the rock, there is no working that would not have a stress concentration. Therefore, it is natural to allow only a minimum stress concentration in the weakened rock mass. To determine the optimal shape of the working in the rock, as a criterion choosing the shape of the working, we introduce to the problem the condition of minimization of maximum circumferential normal stress σ_t on the contour of the working with additional equalities (restraints) to zero of stress intensity factors in the vertices of all the cracks. Without loss of generality of the stated problem, it is assumed that the sought-for function $\rho(\theta)$ can be represented in the form of trigonometric Fourier series.

Thus, the problem of optimal design of a rock mass with a working and cracks is reduced to determining the coefficients A^*_k (k = 1,2,…), B^*_k (k = 1,2,…) of expansion of the function $\rho(\theta)$ of the working's shape under which minimization of maximum value of the circumferential stress σ_t on the contour of the working is provided under the constraints

$$\min_{\eta\in C}\ \max_{\theta\in[0,2\pi]} \sigma_t(\theta,\eta) \tag{4}$$

where C is the collection of constraints to be determined; η= (A^*_k, B^*_k) are design parameters.

This extra condition (4) allows to find the sought-for function of the working shape.

3. Solution method

We assume that the working was located rather far from the surface of the half-space. On the boundary of the working and crack faces we will satisfy the boundary conditions exactly, while on the boundary of the half-space, approximately, asymptotically.

Analysis of the solution of plane problems of the elasticity theory for a hole near the boundary of a deformable solid (Mirsalimov 1979, 1987) shows that for that it is virtually enough that the depth $H \approx 2R$, where R is a characteristic linear size of the working. We will look for the unknown beforehand contour L of the working in the class of contours close to circular one. We represent the unknown contour L in the following form:

$$\rho(\theta) = R + \varepsilon H(\theta),$$
$$H(\theta) = \sum_{k=0}^{\infty}\left(A_k^* \cos k\theta + B_k^* \sin k\theta\right) \tag{5}$$

where the function $H(\theta)$ is to be determined. Here $\varepsilon = R^\circ / R$ is a small parameter; R° is the greatest height of the irregularities of the profile of the contour L of the working from the circle $r = R$.

The sought-for functions (stresses, displacement, and stress intensity factors) are found in the form of expansions in small parameter ε:

$$\sigma_n = \sigma_n^{(0)} + \varepsilon\sigma_n^{(1)}\dots, \qquad \sigma_t = \sigma_t^{(0)} + \varepsilon\sigma_t^{(1)}\dots,$$
$$\tau_{nt} = \tau_{nt}^{(0)} + \varepsilon\tau_{nt}^{(1)}\dots,$$
$$u = u^{(0)} + \varepsilon u^{(1)}\dots, \qquad v = v^{(0)} + \varepsilon v^{(1)}\dots,$$
$$K_{\mathrm{I}} = K_{\mathrm{I}}^{(0)} + \varepsilon K_{\mathrm{I}}^{(1)}\dots, \qquad K_{\mathrm{II}} = K_{\mathrm{II}}^{(0)} + \varepsilon K_{\mathrm{II}}^{(1)}\dots \tag{6}$$

where for simplicity the terms containing ε in the power higher than the first are ignored.

Each approximation satisfies the system of differential equations of the plane theory of elasticity. We get the values of the stress tensor components for $r = \rho(\theta)$, by expanding in series the expressions for stresses in the vicinity of $r = R$. By means of known formulas (Muskhelishvili 1977) for stress components σ_n, τ_{nt} and the perturbation method, we write the boundary conditions of the problem on the contour $r = R$ and crack faces in the form: for a zero approximation

$$\sigma_r^{(0)} = -p, \qquad \tau_{r\theta}^{(0)} = 0 \qquad \text{for} \quad r = R \tag{7}$$

$$\sigma_{y_k}^{(0)} = 0, \qquad \tau_{x_k y_k}^{(0)} = 0 \qquad (k = 1,2,\dots,N_1) \quad \text{on the crack faces.}$$

For a first approximation

$$\sigma_r^{(1)} = N, \qquad \tau_{r\theta}^{(1)} = T \qquad \text{for} \quad r = R$$
$$\sigma_{y_k}^{(1)} = 0, \qquad \tau_{x_k y_k}^{(1)} = 0 \qquad (k = 1,2,\dots,N_1) \quad \text{on the crack faces} \tag{8}$$

Here

$$N=2\frac{\tau_{r\theta}^{(0)}}{R}\frac{dH(\theta)}{d\theta}-H(\theta)\frac{\partial\sigma_r^{(0)}}{\partial r}\qquad \text{for}\quad r=R$$
$$T=\frac{\sigma_\theta^{(0)}-\sigma_r^{(0)}}{R}\frac{dH(\theta)}{d\theta}-H(\theta)\frac{\partial\tau_{r\theta}^{(0)}}{\partial r}\tag{9}$$

We can represent the stresses in the form of the following formulas

$$\sigma_x+\sigma_y=\sigma_r+\sigma_t=$$
$$4\operatorname{Re}\Phi(z)+\sigma_x^\infty+\sigma_y^\infty+\frac{\rho_* g}{2i(1-\nu)}(z-\bar z-2iH)$$

$$\sigma_y-\sigma_x+2\tau_{xy}=(\sigma_t-\sigma_r+2i\,\tau_{rt})e^{-2i\theta}=$$
$$=2\left[\bar z\Phi'(z)+\Psi(z)\right]+\sigma_y^\infty-\sigma_x^\infty\tag{10}$$
$$+\frac{\rho_* g(1-2\nu)}{2i(1-\nu)}(z-\bar z-2iH)$$

Here $\Phi(z)$ and $\Psi(z)$ are the Kolosov-Muskhelishvili complex potentials (for $\Phi(z)=0$; $\Psi(z)=0$ formulas (10) give stresses in the untouched rock $y<H$); $i^2=-1$.

Using general representations (10), we write the boundary conditions of the problem in a zero approximation in the form: on the crack faces

$$\Phi_0(z)+\overline{\Phi_0(z)}-e^{2i\theta}\left[\bar z\Phi_0'(z)+\Psi_0(z)\right]=f_0\quad \text{for}\quad |z|=R$$
$$\Phi_0(x_k)+\overline{\Phi_0(x_k)}+x_k\overline{\Phi_0'(x_k)}+\overline{\Psi_0(x_k)}=F_0(x_k)\tag{11}$$
$$(k=1,2,\ldots,N_1)\quad \text{on the crack faces}$$

where

$$f_0=\frac{1}{2}\left(\sigma_x^\infty+\sigma_y^\infty\right)-p+\frac{\rho_* g}{4i(1-\nu)}(z-\bar z-2iH)-$$
$$-e^{2i\theta}\left\{\frac{1}{2}\left(\sigma_y^\infty-\sigma_x^\infty\right)+\frac{\rho_* g}{4i(1-\nu)}(z-\bar z-2iH)\right\}$$
$$F_0(x_k)=\sigma_y^\infty+2\rho_* g(x_k\sin\alpha_k+y_k^0-H)$$

$z_k^0=x_k^0+iy_k^0$ are the coordinates of the point O_k with respect to the system of coordinates $x_kO_ky_k$; x_k are the affixes of the points of the k-th crack in a zero approximation; $\Phi_0(z)$, $\Psi_0(z)$ are the complex potentials in a zero approximation.

The analytic functions $\Phi_0(z)$ and $\Psi_0(z)$ are found in the form:

$$\Phi_0(z)=\Phi_0^{(0)}(z)+\Phi_{01}^{(0)}(z)$$
$$\Psi_0(z)=\Psi_0^{(0)}(z)+\Psi_{01}^{(0)}(z)\tag{12}$$

$$\Phi_0^{(0)}(z)=\sum_{k=0}^{\infty}a_k^0z^{-k},\qquad \Psi_0^{(0)}(z)=\sum_{k=0}^{\infty}b_k^0z^{-k}\tag{13}$$

$$\Phi_{01}^{(0)}(z)=\frac{1}{2\pi}\sum_{k=1}^{N_1}\int_{-l_k}^{l_k}\frac{g_k^0(t)}{t-z_k}dt-\tag{14}$$

$$-\frac{1}{2\pi}\sum_{k=1}^{N_1}\int_{-l_k}^{l_k}\left[\left(\frac{1}{z}+\frac{\overline{T_k}}{1-z\overline{T_k}}\right)e^{i\alpha_k}g_k^0(t)-e^{-i\alpha_k}\overline{g_k^0(t)}\frac{1-T_k\overline{T_k}}{\overline{T_k}(1-z\overline{T_k})^2}\right]dt$$

$$\Psi_{01}^{(0)}(z)=\frac{1}{2\pi}\sum_{k=1}^{N_1}e^{-2i\alpha_k}\int_{-l_k}^{l_k}\left[\frac{\overline{g_k^0(t)}}{t-z_k}-\frac{\overline{T_k}g_k^0(t)e^{i\alpha_k}}{(t-z_k)^2}\right]dt+$$
$$+\frac{1}{2\pi z}\sum_{k=1}^{N_1}\int_{-l_k}^{l_k}\left\{\left[\frac{1}{zT_k}-\frac{2}{z^2}-\frac{\overline{T_k}}{z(1-z\overline{T_k})}+\frac{\overline{T_k}^2}{(1-z\overline{T_k})^2}\right]\times\right.\tag{14}$$
$$\times e^{i\alpha_k}g_k^0(t)+\left[\frac{1-T_k\overline{T_k}}{z\overline{T_k}(1-z\overline{T_k})^2}-\right.$$
$$\left.\left.\frac{1}{1-z\overline{T_k}}-\frac{2(1-T_k\overline{T_k})}{(1-z\overline{T_k})^3}\right]e^{-i\alpha_k}\overline{g_k^0(t)}\right\}dt$$

where $T_k=te^{i\alpha_k}+z_k^0$; $z_k=e^{-i\alpha_k}\left(z-z_k^0\right)$; $g_k^0(x_k)$ are sought-for functions characterizing the opening and shear, of crack faces

$$g_1^0(x_k)=\frac{2\mu}{i(1+\kappa)}\frac{d}{dx_k}\left[u_k^{0+}(x_k,0)-u_k^{0-}(x_k,0)+\right.$$
$$\left.+i\left(v_k^{0+}(x_k,0)-v_k^{0-}(x_k,0)\right)\right]$$

$$a_0^0=\frac{1}{2}\left(\sigma_x^\infty+\sigma_y^\infty\right)-p-\frac{\rho_* gH}{2(1-\nu)},$$
$$b_0^0=\frac{\rho_* gH(1-2\nu)}{2(1-\nu)}-\frac{1}{2}\left(\sigma_y^\infty-\sigma_x^\infty\right)$$

$$a_2^0=b_0^0+\overline{C}_2R^2,\qquad a_1^0=\frac{\overline{C}_1R}{1+\kappa},\qquad b_1^0=-\frac{\kappa\overline{C}_1R}{1+\kappa}$$
$$a_n^0=C_nR^n\qquad (n\geq 3)$$
$$b_n^0=(n-1)R^2a_{n-2}^0-R^nC_{-n+2}\qquad (n\geq 3)$$
$$C_0=\frac{1}{2}\left(\sigma_x^\infty+\sigma_y^\infty\right)-p-\frac{\rho_* gH}{2(1-\nu)}$$
$$C_1=-\frac{1}{2}i\rho_* gR,\qquad C_{-1}=-\frac{\rho_* gR}{4i(1-\nu)}$$
$$C_2=\frac{\rho_* gH(1-2\nu)}{2(1-\nu)}-\frac{1}{2}\left(\sigma_y^\infty-\sigma_x^\infty\right),$$
$$C_3=-\frac{\rho_* gR(1-2\nu)}{4i(1-\nu)}$$
$$C_{-n}=0\quad (n\geq 2),\qquad C_n=0\quad (n\geq 3)$$

By means of complex potentials (12)-(14), satisfying the boundary condition (11) on crack faces, after some transformations we get a system of N_1 integral equations with respect to unknown functions

$$\sum_{k=1}^{N_1}\int_{-l_k}^{l_k}\left[R_{nk}(t,x)g_k^0(t)+S_{nk}(t,x)\overline{g_k^0(t)}\right]dt=\pi F_n^0(x) \qquad (15)$$

$$|x_n|\le l_n \qquad (n=1,2,\ldots,N_1)$$

where

$$F_n^0(x)=-\left[\Phi_0^{(0)}(x_n)+\overline{\Phi_0^{(0)}(x_n)}+x_n\overline{\Phi_0^{(0)'}(x_n)}+\overline{\Psi_0^{(0)}(x_n)}\right]$$

x, t, z_n^0 and l_n are dimensionless variables referred to R; R_{nk}, S_{nk} are determined by the known formulas (Panasyuk, Savruk and Datsyshyn 1976, formulas (VI, 62)).

To the system of singular integral Eq. (15) it is necessary to add extra conditions

$$\int_{-l_k}^{l_k} g_k^0(t)dt=0 \quad (k=1,2,\ldots,N_1) \qquad (16)$$

providing uniqueness of displacements when tracing the contours of internal cracks.

The system of complex singular integral equations (15), under additional equalities (16) by means of algebraization process (Panasyuk *et al.* 1976, Mirsalimov 1987, Savruk and Kazberuk 2017) is reduced to the finite system of $N_1\times M$ algebraic equations with respect to approximate values of the sought-for functions $g_k^0(x_k)$ at the nodal points of partition of integration segment

$$\frac{1}{M}\sum_{k=1}^{N_1}\sum_{m=1}^{M} l_k\left[g_k^0(t_m)R_{nk}(l_kt_m,l_nx_r)\right.$$
$$\left.+\overline{g_k^0(t_m)}S_{nk}(l_kt_m,l_nx_r)\right]=F_n^0(x_r)$$
$$(r=1,2,\ldots,M-1) \qquad (17)$$
$$\sum_{m=1}^{M} g_n^0(t_n)=0 \quad (n=1,2,\ldots,N_1)$$

where

$$t_m=\cos\frac{2m-1}{2M}\pi \qquad (m=1,2,\ldots,M), \qquad x_r=\cos\frac{\pi r}{M}$$

If in system (17) we pass to complexly conjugated values, we get one more system of $N_1\times M$ linear algebraic equations.

For the stress intensity factors in a zero approximation, we have:

$$K_{\mathrm{I}}^{(0)}-iK_{\mathrm{II}}^{(0)}=\sqrt{\pi l_k}\sum_{m=1}^{M}(-1)^m g_k^0(t_m)\cot\frac{2m-1}{4M}\pi$$
$$(k=1,2,\ldots,N_1) \quad \text{for the cracks tips } x_k=l_k \qquad (18)$$
$$K_{\mathrm{I}}^{(0)}-iK_{\mathrm{II}}^{(0)}=\sqrt{\pi l_k}\sum_{m=1}^{M}(-1)^{m+M} g_k^0(t_m)\tan\frac{2m-1}{4M}\pi$$
$$\text{for the cracks tips } x_k=-l_k$$

The stress components in the rock mass are found from the relations (10) and (12) in a zero approximation. Knowing the stress state in a zero approximation, we find the functions N and T from formulas (9). Then we pass to the solution of the problem in a first approximation.

The boundary conditions of problem (8) are written in the form:

$$\Phi^{(1)}(z)+\overline{\Phi^{(1)}(z)}-e^{2i\theta}\left[\bar{z}\Phi^{(1)'}(z)+\Psi^{(1)}(z)\right]=N-iT$$
$$\text{for } |z|=R$$
$$\Phi^{(1)}(x_k)+\overline{\Phi^{(1)}(x_k)}+x_k\overline{\Phi^{(1)'}(x_k)}+\overline{\Psi^{(1)}(x_k)}=0 \qquad (19)$$
$$(k=1,2,\ldots,N_1) \quad \text{on the crack faces}$$

We look for the solution of problem (19) in the form

$$\Phi^{(1)}(z)=\Phi_0^{(1)}(z)+\Phi_1^{(1)}(z)$$
$$\Psi^{(1)}(z)=\Psi_0^{(1)}(z)+\Psi_1^{(1)}(z) \qquad (20)$$

where the complex potentials $\Phi_1^{(1)}(z)$ and $\Psi_1^{(1)}(z)$ are determined from the relations similar to (14), in which $g^0{}_k(t)$ should be replaced by $g^1{}_k(t)$, while analytic functions $\Phi_0^{(1)}(z)$ and $\Psi_0^{(1)}(z)$ are sought in the form of power series (13). The coefficients a_k and b_k of the analytic functions $\Phi_0^{(1)}(z)$ and $\Psi_0^{(1)}(z)$ are found from the formulas

$$a_0=0, \quad b_0=0, \quad a_1=-\frac{\overline{A}_1R}{1+\kappa}$$
$$b_1=-\frac{\kappa A_1R}{1+\kappa}, \quad a_n=\overline{A}_nR^n \quad (n\ge 2), \quad b_2=-A_0R^2$$
$$b_n=(n-1)R^2a_{n-2}-R^2A_{-n+2} \quad (n\ge 3) \qquad (21)$$
$$N-iT=\sum_{k=-\infty}^{\infty}A_ke^{ik\theta}$$

Satisfying the boundary conditions on the crack faces by the functions (20), after some transformations we get the system of N_1 integral equations with respect to unknown functions $g^1{}_k(x_k)$:

$$\sum_{k=1}^{N_1}\int_{-l_k}^{l_k}\left[R_{nk}(t,x)g_k^1(t)+S_{nk}(t,x)\overline{g_k^1(t)}\right]dt=\pi F_n^1(x) \quad \le l_n \qquad (22)$$
$$(n=1,2,\ldots,N_1)$$

where

$$F_n^1(x_k)=-\left[\Phi_0^{(1)}(x_k)+\overline{\Phi_0^{(1)}(x_k)}+x_k\overline{\Phi_0^{(1)'}(x_k)}+\overline{\Psi_0^{(1)}(x_k)}\right]$$

To the system of singular integral Eq. (22) it is necessary to add the conditions

$$\int_{-l_k}^{l_k} g_k^1(t)dt=0 \quad (k=1,2,\ldots,N_1) \qquad (23)$$

providing uniqueness of displacements when tracing the crack contours in a first approximation. Under the additional conditions (23), by means of algebraization procedure (Panasyuk *et al.* 1976, Mirsalimov 1987, Savruk and Kazberuk 2017), the system of complex singular integral Eq. (22) is reduced to the finite system of $N_1\times M$ linear algebraic equations with respect to approximate values of unknown functions $g^1{}_k(t_m)$ at the nodal points:

$$\frac{1}{M}\sum_{k=1}^{N_1}\sum_{m=1}^{M} l_k \Big[g_k^1(t_m)R_{nk}(l_k t_m, l_n x_r) + \overline{g_k^0(t_m)}S_{nk}(l_k t_m, l_n x_r)\Big] = F_n^1(x_r)$$
$$(r = 1,2,\ldots,M-1;\ n = 1,2,\ldots,N_1) \tag{24}$$
$$\sum_{m=1}^{M} g_k^1(t_m) = 0 \qquad (k = 1,2,\ldots,N_1)$$

If in the system (24) we pass to complexly conjugated values, we get one more system of $N_1\times M$ linear algebraic equations. Under the given function $H(\theta)$ of the shape of the working in rock the obtained linear algebraic equations will be closed. In this case, by solving the algebraic system of equations, one can study stress-strain state of the rock mass with a working and cracks and determine the stress intensity factors in the vicinity of all crack tips. For stress intensity factors in the first approximation, we have the following formulas: for the crack tips $x_k = l_k$

$$K_{\rm I}^{(1)} - iK_{\rm II}^{(1)} = \sqrt{\pi l_k}\sum_{m=1}^{M}(-1)^m g_k^1(t_m)\cot\frac{2m-1}{4M}\pi$$
$$(k = 1,2,\ldots,N_1) \quad \text{for the cracks tips } x_k = l_k \tag{25}$$
$$K_{\rm I}^{(1)} - iK_{\rm II}^{(1)} = \sqrt{\pi l_k}\sum_{m=1}^{M}(-1)^{m+M} g_k^1(t_m)\tan\frac{2m-1}{4M}\pi$$
$$\text{for the cracks tips } x_k = -l_k$$

Finally, the stress intensity factors are found by formulas (6).

4. The solution of optimal designing problem

In the case of unknown shape of the working (unknown function $H(\theta)$) the system of algebraic linear equations is not closed. To construct the missing equations, it is necessary to find the circumferential stress σ_t for $r = \rho(\theta)$. By means of the found solution we find σ_t for $r = \rho(\theta)$ within to the first order variable with respect to the small parameter ε

$$\sigma_t = \sigma_t^{(0)}(\theta)\Big|_{r=R} + \varepsilon\left[H(\theta)\frac{\partial\sigma_t^{(0)}}{\partial r} + \sigma_t^{(1)}(\theta)\right]\Bigg|_{r=R} \tag{26}$$

For the function $\sigma_\theta(\theta, A_k^*, B_k^*)$ we find its maximum value on the contour $r = R$

$$\max\sigma_t(\theta^*, A_k^*, B_k^*) \tag{27}$$

Here the value θ^* is the solution of the equation

$$\frac{d\sigma_t(\theta)}{d\theta} = 0 \tag{28}$$

The maximum of the function $\sigma_t(\theta)$ is found by ordinary methods of differential calculus.

To construct the missing equations, we require minimization of maximum values of the circumferential normal stress on the working contour (4) under the following constraints

$$\sum_{m=1}^{M}(-1)^m\big[g_k^0(t_m) + \varepsilon g_k^1(t_m)\big]\cot\frac{2m-1}{4M}\pi = 0$$
$$\sum_{m=1}^{M}(-1)^{m+M}\big[g_k^0(t_m) + \varepsilon g_k^1(t_m)\big]\tan\frac{2m-1}{4M}\pi = 0 \quad (k = 1,2,\ldots,N_1) \tag{29}$$
$$\max\sigma_t < [\sigma]$$

where $[\sigma]$ is admissible circumferential stress for the material of the rock, determined experimentally.

The design function $H(\theta)$ must be managed so as to ensure the minimization of the maximum value σ_t (minimax criterion).

Since the stresses $\sigma_t(\theta, A^*_k, B^*_k)$ (quality index of control) and $\max\sigma_\theta(A^*_k, B^*_k)$ linearly depend on the sought-for coefficients A^*_k, B^*_k then the stated optimal design problem is reduced to a linear programming problem. Thus, it is required to find the values of variables A^*_k, B^*_k satisfying the obtained system of equations (constraints) (21), (24), (29) and in addition converting the linear function max σ_t (objective function) to minimum.

Numerical methods for solving linear programming problems may be used for minimizing the stress state. In the problem under consideration the simplex algorithm is the most effective method. Thus, it is required to determine such values of variables A^*_k, B^*_k that satisfy the obtained system of equations (constraints) (21), (24), (29) and in addition convert the linear function $\max_t$ (the objective function) to minimum.

If a part of the crack N_2 crops out with one end, then equalities (16) and (23) at each approximation are replaced by additional conditions. These conditions express finiteness of stresses at the crack ends on the working contour in the rock.

Changing the values of the parameters α_k and z_k^0 ($k = 1,2,\ldots,N_1$) we can study different cases of crack location in the rock mass and their influence on optimal shape of the working.

The results of calculations of the functions $H(\theta)$ of the working shape in the rock mass are given in Table 1. Numerical calculation was carried out for the case when the rock mass has three rectilinear cracks

$$\alpha_1 = 15° \quad l_1/R = 0.075 \quad z_1^0 = 1.25Re^{i\pi/12}$$
$$\alpha_2 = 30° \quad l_2/R = 0.10 \quad z_2^0 = 1.30Re^{i\pi/8}$$
$$\alpha_3 = 60° \quad l_3/R = 0.15 \quad z_3^0 = 1.45Re^{i\pi/6}$$

The optimal solution, i.e. the found design parameters A^*_k and B^*_k of the design function $H(\theta)$ of the shape of the working in the rock mass, increases the load-bearing capacity of the rock.

Let us consider some partial cases.

1). *The rock mass is without cracks*. In this case, the resolving system of equations will consist of (4), (21). Solving it by the Gauss method with the choice of the main element, we find the required expansion coefficients A^*_k, B^*_k of the function $H(\theta)$. Comparison of these results with results obtained in (Cherepanov 1966, Mirsalimov 1979, 1987) by another method shows complete agreement.

2). *There are no gravitational forces in the rock mass*. In

Table 1 The values of Fourier coefficients for optimal shape of the working in rock mass for case of three cracks

A^*_0	A^*_1	A^*_2	A^*_3	A^*_4	A^*_5	A^*_6	A^*_7	A^*_8	A^*_9
0.4809	0.4257	0.4008	0.3761	0.3095	0.2657	0.2088	0.1719	0.1046	0.0573
	B^*_1	B^*_2	B^*_3	B^*_4	B^*_5	B^*_6	B^*_7	B^*_8	B^*_9
	0.4052	0.3611	0.3297	0.2569	0.2144	0.1832	0.1527	0.0873	0.0439

this case, in the obtained system of equations, all terms containing the factor $g\rho_*$ must be set equal to zero. Solving the minimization problem, we obtain the expansion coefficients A^*_k, B^*_k of the function $H(\theta)$ for this case. Comparison of the results with results given in the works (Mirsalimov 2019, 2020) shows complete agreement.

3). *The cracks are present in the rock mass and the shape of a working is known*. In this case the function $\rho(\theta)$ is known. The system of resolving equations is (21). Solving this system by the Gauss method with the choice of the main element, we find the expansion coefficients of the complex potentials. The results fully coincide with the corresponding results in the studies (Sheinin 1972, Givoli and Elishakoff 1992).

4). *The shape of a working is known*. In this case, the function $\rho(\theta)$ is known. The system of resolving equations consists of (21), (24). Using the Gauss method with the choice of the main element, we solve this system and obtain the expansion coefficients of the complex potentials and the stress intensity factors in the vicinity of the crack tips (6), (18), (25). The calculations were carried out for a circular and elliptical contour. The results are consistent with those previously obtained in (Savruk 1988).

5. Conclusions

A criterion and a method for solving a problem of prevention of fracture of the rock mass with a working and rectilinear cracks under the action of tectonic and gravitational forces are offered. A closed system of algebraic equations allowing to obtain the solution of the problem of optimal designing of the working shape in the cracked rock mass, depending on mechanical and geometrical characteristics of the rock, is obtained. The relation between the working's shape and stress intensity factors and also location and sizes of the cracks is obtained. The found shape of the working in the rock mass reduces stress concentration on the working's contour and minimizes stress intensity factors. This increases the load-bearing capacity of the rock. The results of the theoretical work under consideration open new opportunities for optimal design of the working's shape in the rock mass.

References

Andreev, L.V. (1986), *In the World of Shells*, Mir, Moscow, Russia.

Aitaliyev, Sh.M., Banichuk, N.V. and Kayupov, M.A. (1986), *Optimal Design of Extended Underground Structures*, Nauka, Alma-Ata, Kazakhstan.

Banichuk, N.V. (1977), "Optimality conditions in the problem of seeking the hole shapes in elastic bodies", *J. Appl. Math. Mech.*, **41**(5), 946-951. https://doi.org/10.1016/0021-8928(77)90179-4.

Banichuk, N.V. (1980), *Shape Optimization of Elastic Solids*, Nauka, Moscow, Russia.

Barton, N. and Quadros, E. (2014), "Most rock masses are likely to be anisotropic", *Proceedings of the ISRM Conference on Rock Mechanics for Natural Resources and Infrastructure-SBMR 2014*, Goiania, Brazil, September.

Burchill, M. and Heller, M. (2004), "Optimal free-form shapes for holes in flat plates under uniaxial and biaxial loading", *J. Strain Anal. Eng. Des.*, **39**(6), 595-614. https://doi.org/10.1243/0309324042379266.

Chang, X., Ma, W., Li, Z. and Wang, H. (2018), "Crack behaviour of top layer in layered rocks", *Geomech. Eng.*, **16**(1), 49-58. https://doi.org/10.12989/gae.2018.16.1.049.

Chen, Y., Zhang, X., Zhu, W. and Wang, W. (2018), "Modified discontinuous deformation analysis for rock failure: Crack propagation", *Geomech. Eng.*, **14**(4), 325-336. https://doi.org/10.12989/gae.2018.14.4.325.

Cherepanov, G.P. (1963), "An inverse elastic-plastic problem under plane strain", *Izvestija Akad. nauk SSSR. Otdelenie tehn. nauk. Mehanika i mashinostroenie*, (2), 57-60.

Cherepanov, G.P. (1966), "One inverse problem of elasticity theory", *Mech. Solids*, (3), 119-130.

Cherepanov, G.P. (1974), "Inverse problems of the plane theory of elasticity", *J. Appl. Math. Mech.*, **38**(6), 915-931. https://doi.org/10.1016/0021-8928(75)90085-4.

Cherepanov, G.P. (2015), "Optimum shapes of elastic bodies: Equistrong wings of aircrafts and equistrong underground tunnels", *Phys. Mesomech.*, **18**(4), 391-401. https://doi.org/10.1134/S1029959915040116.

Givoli, D. and Elishakoff, I. (1992), "Stress concentration at a nearly circular hole with uncertain irregularities", *J. Appl. Mech.*, **59**(2S), S65-S71. https://doi.org/10.1115/1.2899509.

Gogolauri, L. (2012), "The problem of finding equistrong holes in an elastic square", *Proc. A. Razmadze Mathematical Institute*, **158**, 25-31.

Hoek, E. and Brown, E.T. (1980), *Underground Excavations in Rock*, Institution of Mining and Metallurgy, London, England, U.K.

Kalantarly, N.M. (2017), "Equal strength hole to inhibit longitudinal shear crack growth", *J. Mech. Eng.*, **20**(4), 31-37. https://doi.org/10.15407/pmach2017.04.031.

Kirilyuk, V.S. and Levchuk, O.I. (2008), "On an inverse thermo-elasticity problem for an infinite medium containing a cavity of unknown shape", *J. Eng. Math.*, **61**(2-4), 219-229. https://doi.org/10.1007/s10665-007-9189-8.

Kurshin, L.M. and Onoprienko P.N. (1976), "Determination of the shapes of doubly-connected bar sections of maximum torsional stiffness", *J. Appl. Math. Mech.*, **40**(6), 1020-1026. https://doi.org/10.1016/0021-8928(76)90144-1.

Lee, J. and Hong, J.W. (2018), "Crack initiation and fragmentation processes in pre-cracked rock-like materials", *Geomech. Eng.*, **15**(5), 1047-1059. https://doi.org/10.12989/gae.2018.15.5.1047.

Lv, H., Tang, Y., Zhang, L., Cheng, Z. and Zhang, Y. (2019), "Analysis for mechanical characteristics and failure models of coal specimens with non-penetrating single crack", *Geomech. Eng.*, **17**(4), 355-365. https://doi.org/10.12989/gae.2019.17.4.355.

Mirsalimov, V.M. (1974), "On the optimum shape of apertures for a perforated plate subject to bending", *J. Appl. Mech. Tech. Phys.*, **15**(6), 842-845. https://doi.org/10.1007/BF00864606.

Mirsalimov, V.M. (1975), "Converse problem of elasticity theory for an anisotropic medium", *J. Appl. Mech. Tech. Phys.*, **16**(4), 645-648. https://doi.org/10.1007/BF00858311.

Mirsalimov, V.M. (1977), "Inverse doubly periodic problem of thermoelasticity", *Mech. Solids*, **12**(4), 147-154.
Mirsalimov, V.M. (1979), "A working of uniform strength in the solid rock", *Soviet Mining*, **15**(4), 327-330. https://doi.org/10.1007/BF02499529.
Mirsalimov, V.M. (1987), *Non-one-dimensional Elastoplastic Problems*, Nauka, Moscow, Russia.
Mirsalimov, V.M. (2019), "Inverse problem of elasticity for a plate weakened by hole and cracks", *Math. Prob. Eng.* https://doi.org/10.1155/2019/4931489.
Mirsalimov, V.M. (2020), "Minimizing the stressed state of a plate with a hole and cracks", *Eng. Optimiz.*, **52**(2), 288-302. https://doi.org/10.1080/0305215X.2019.1584619.
Mir-Salim-zade, M.V. (2007), "Determination of equistrong hole shape in isotropic medium, reinforced by regular system of stringers", *Materialy, tehnologii, instrumenty*, **12**(4), 10-14.
Mir-Salim-zade, M.V. (2019), "Minimization of the stressed state of a stringer plate with a hole and rectilinear cracks", *J. Mech. Eng.*, **22**(2), 59-69. https://doi.org/10.15407/pmach2019.02.059.
Muskhelishvili, N.I. (1977), *Some Basic Problems of Mathematical Theory of Elasticity*, Springer, Dordrecht, The Netherlands.
Odishelidze, N., Criado-Aldeanueva, Criado, F.F. and Sanchez, J.M. (2016), "Stress concentration in an elastic square plate with a full-strength hole", *Math. Mech. Solids*, **21**(5), 552-561. https://doi.org/10.1177/1081286514530753.
Panasyuk, V.V., Savruk, M.P. and Datsyshyn, A.P. (1976), *Stress Distribution around Cracks in Plates and Shells*, Naukova Dumka, Kiev, Ukraine.
Savruk, M.P. (1988), *Stress Intensity Factors in Solids with Cracks*, Naukova Dumka, Kiev, Ukraine.
Savruk, M.P. and Kazberuk A. (2017), *Stress Concentration at Notches*. Springer, Cham, Switzerland.
Savruk, M.P. and Kravets, V.S. (2002), "Application of the method of singular integral equations to the determination of the contours of equistrong holes in plates", *Mater. Sci.*, **38**(1), 34-46. https://doi.org/10.1023/A:1020116613794.
Sheinin, V.I. (1972), "On the asymptotic method for calculating stress near rough surface of elastic solids", *Mech. Solids*, **7**(2), 94-102.
Sun, W., Du, H., Zhou, F. and Shao, J. (2019), "Experimental study of crack propagation of rock-like specimens containing conjugate fractures", *Geomech. Eng.*, **17**(4), 323-331. https://doi.org/10.12989/gae.2019.17.4.323.
Vigdergauz, S.B. (1976), "Integral equation of the inverse problem of the plane theory of elasticity", *J. Appl. Math. Mech.*, **40**(3), 518-522. https://doi.org/10.1016/0021-8928(76)90046-0.
Vigdergauz, S.B. (1977), "On a case of the inverse problem of two-dimensional theory of elasticity", *J. Appl. Math. Mech.*, **41**(5), 902-908. https://doi.org/10.1016/0021-8928(77)90176-9.
Vigdergauz, S. (2006), "The stress-minimizing hole in an elastic plate under remote shear", *J. Mech. Mater. Struct.*, **1**(2), 387-406. https://doi.org/2140/jomms.2006.1.387.
Vigdergauz, S. (2016), "A planar grained structure with a multiphase nested inclusion in a periodic cell: Elastostatic solution and the equi-stressness", *Math. Mech. Solids*, **21**(6), 709-724. https://doi.org/10.1177/1081286514536084.
Vigdergauz, S. (2017), "Equi-stress boundaries in two- and three-dimensional elastostatics: The single-layer potential approach", *Math. Mech. Solids*, **22**(4), 837-851. https://doi.org/10.1177/1081286515615001.
Vigdergauz, S. (2018), "Simply and doubly periodic arrangements of the equi-stress holes in a perforated elastic plane: The single-layer potential approach", *Math. Mech. Solids*, **23**(5), 805-819. https://doi.org/10.1177/1081286517691807.
Wang, S.J., Lu, A.Z., Zhang X.L. and Zhang N. (2018), "Shape optimization of the hole in an orthotropic plate", *Mech. Based Des. Struct. Machines*, **46**(1), 23-37. https://doi.org/10.1080/15397734.2016.1261036.
Wheeler, L.T. (1976), "On the role of constant-stress surfaces in the problem of minimizing elastic stress concentration", *Int. J. Solids Struct.*, **12**(11), 779-789. https://doi.org/10.1016/0020-7683(76)90042-1.
Wheeler, L.T. (1992), "Stress minimum forms for elastic solids", *Appl. Mech. Rev.*, **45**(1), 1-12. https://doi.org/10.1115/1.3119743.
Wu, Z. (2009), "Optimal hole shape for minimum stress concentration using parameterized geometry models", *Struct. Multidisciplin. O.*, **37**(6), 625-634. https://doi.org/10.1007/s00158-008-0253-4.
Zeng, X., Lu, A. and Wang, S. (2020), "Shape optimization of two equal holes in an infinite elastic plate", *Mech. Based Des. Struct. Machines*, **48**(2), 133-145. https://doi.org/10.1080/15397734.2019.1620111.
Zhou, L., Zhu, Z., Liu, B. and Fan, Y. (2018), "The effect of radial cracks on tunnel stability", *Geomech. Eng.*, **15**(2), 721-728. https://doi.org/10.12989/gae.2018.15.2.721.
Zhu, J.Q. and Yang, X.L. (2018), "Probabilistic stability analysis of rock slopes with cracks", *Geomech. Eng.*, **16**(6), 655-667. https://doi.org/10.12989/gae.2018.16.6.655.

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