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Reflection and propagation of plane waves at free surfaces of a rotating micropolar fibre-reinforced medium with voids

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Abstract. The present paper seeks to investigate propagation and reflection of waves at free surfaces of homogeneous, anisotropic and rotating micropolar fibre-reinforced medium with voids. It has been observed that, in particular when P-wave is incident on the free surface, there exist four coupled reflected plane waves traveling in the medium; quasi-longitudinal displacement (qLD) wave, quasi-transverse displacement (qTD) wave, quasi-transverse microrotational wave and a wave due to voids. Normal mode Analysis usually called harmonic solution method is adopted in concomitant with Snell's laws and appropriate boundary conditions in determination of solution to the micropolar fibre reinforced modelled problem. Amplitude ratios which correspond to reflected waves in vertical and horizontal components are presented analytically. Also, the Reflection Coefficients are presented using numerical simulated results in graphical form for a particular chosen material by the help of Mathematica software. We observed that the micropolar fibre-reinforced, voids and rotational parameters have various degrees of effects to the modulation, propagation and reflection of waves in the medium. The study would have impact to micropolar fibre-reinforecd rotational-acoustic machination fields and future works about behavior of seismic waves.

Keywords: rotation; micropolar; reflection; fibre-reinforced; voids; P-waves; reflection coefficients

1. Introduction

Oftentimes, fibre-reinforced composites and its analysis of stress and deformation is one positive area of research in solid mechanics. The light weight and high tensile strength of fibre reinforced materials are of prime importance to various structural designers mostly in the field of engineering and technology. Its distinctive characteristics as a single anisotropic composite unit cannot be under estimated. Hence, to examine and explain the mechanical composition of such materials, continuum models are exploited. Many researchers have investigated wave propagation in various media but particular investigations in isotropic generalized solid with voids paved way as an evidence to the existence of new or transformed waves, whose importance to seismologists in improving earthquake estimations cannot be over emphasized. In Geophysics, the reflection and refraction of seismic waves led credence for researchers to investigate into the earth's non-exterior and structures. Thus, it's obvious that most large bodies such as moon, planets and earth possess angular velocity, and this prompted or necessitated the study of rotational effects and reflection of plane waves. In the rotating frame of reference, the centrifugal acceleration and Coriolis effects are taken into considerations in the equation of motion.

Nevertheless, seismic wave is categorized majorly in two forms; body wave and surface wave. The former travels through the body of the medium while the latter travels through the surface of the medium. Body waves make ray paths refracted by the variation of the ratio of its mass and volume along with the stiffness of the earth's interior. P-waves are primary waves which are compressional in nature and travel faster than any other form of waves. It usually travels through any type of material with twice the speed of secondary waves (SV-waves). Subsequently, certain materials possess pores or voids and these are called porous materials which constitute attributes that influences the chemical reactivity of solids. Propagation of waves in the presence of voids is of great value to numerous fields; composite materials in aeronautics, astronautics, earthquake sciences and so on.

First grade micro-continuum which is a material property due to deformation play major role to wave propagation or reflection. It consists of microstretch, micropolar (involves rotation and translation of local points of the material) and micromorphic theories and which depend on the order to which the micro-degree is incorporated. These theories are a potential concept that characterizes the behavior of materials with complex structures. Eringen (1967) developed the theories of micropolar elasticity to micropolar linear constitutive theories with internal friction. McCarthy and Eringen (1969) investigated problem on micropolar viscoelastic waves. Tauchert (1971), investigated on thermal stresses in micropolar elastic solids. Schoenberg and Censor (1973) examined the propagation of waves in a rotating, homogeneous, isotropic, linear elastic medium. Cowin and Nunziato (1979-1983), discussed both the linear and nonlinear theories of elastic porous media. Puri and Cowin

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(1985) studied the effects of voids on plane waves in linear elastic media. Kumar, Gogna and Debnath (1990) envisaged problems in Lamb's plane micropolar viscoelastic medium with stretch. Biswas, Sengupta, and Debnath (1990), discussed on axisymmetric problems of wave propagation under the influence of gravity with micropolar viscolelastic half-space. Chattopadhyay and Choudhury (1990), obtained results on Propagation, reflection/transmission of waves under the magnetic effects in a self-reinforced medium. Kumar and Singh (1998), investigated on the amplitude ratios of reflection and refraction of plane waves between micropolar elastic solid and viscoelastic solid interfaces. Singh (2000), discussed on reflection and transmission of plane harmonic waves at an interface between liquid and micropolar viscoelastic solid with stretch. Sengupta and Nath (2001) discussed on Surface waves in fibre-reinforced media.

In addition, Chattopadhyay, Venkateswarlu, Saha (2002), investigated on reflection of quasi-SV waves and quasi-P using both free and rigid boundary conditions of a fibre-reinforced medium. Eigenvalue approach was utilized by Sinha and Bera (2003), to solve the fundamental equations of the problems of generalized thermo-elasticity with one relaxation parameter including heat source in an infinite rotating medium. Chaudhary, Kaushik and Tomar (2004) envisaged on reflection/transmission of plane wave between self-reinforced two elastic half-spaces. Nilratan and Manik (2011) discussed on reflection and refraction of a plane thermoelastic wave at a solid–solid interface under perfect boundary condition, in presence of normal initial stress. Khan, Anya and Hajra (2015), discussed on effects of surface waves under the influence of gravity in non-homogeneous fibre-reforced media with voids. Kumar, Sharma and Garg (2015) worked on reflection of plane waves in transversely isotropic micropolar viscoelastic media with thermal effects. Kumar, Sharma and Lata (2016) investigated on thermomechanical interactions in a transversely isotropic magneto-thermo-elastic with and without energy dissipation with combined effects of rotation, vacuum and two temperatures. Lata, Kuma and Sharma (2016), discussed on effects of hall current in a transversely isotropic magneto-thermo-elastic two temperature medium with rotation and with and without energy dissipation due to normal force. Furthermore, Baljeet Singh (2017) made account of reflection of elastic plane surface waves of a half-space with impedance boundary conditions. Abd-Alla, Abo-Dahab, Khan (2017), investigated the rotational behavior on magneto-thermoelastic surface waves in fibre-reinforced viscoelastic media of higher order. Lata (2018) studied reflection and refraction of plane waves in a layered nonlocal elastic and anisotropic thermoelastic medium. Lata (2018) studied the effect of energy dissipation on plane waves in sandwiched layered thermo-elastic medium. Sunita, Suresa and Kapil (2019), recently examined reflection at free surface of fibre-reinforced thermoelastic rotating medium with two-temperature and phase-lag.

In view of this, the present study is concerned with propagation and reflection of waves at free surfaces of plane rotating micropolar fibre-reinforced medium with porosity. The x_1x_2-plane is considered and the governing equations solved by using free boundary conditions to get reflection coefficients. We observed that four reflected waves exists when in particular P-wave is incident on the free surface or at the boundary x_2=0; quasi-longitudinal displacement (qLD) or P-wave, quasi-transverse displacement (qTD) or SV-wave, quasi-transverse microrotational (qTM) wave and a wave due to voids. Amplitude ratios or reflection coefficients which correspond to reflected waves in both vertical and horizontal components are presented analytically. Also, by using Mathematica Software, computed results are shown graphically. Some particular results can also be deduced in the absence of rotation and fibre-reinforced medium, yielding the results of micropolar isotropic with voids.

2. Formulation of the problem

The field relations for a micro-polar fibre-reinforced linearly elastic anisotropic medium with voids and reinforcement direction '$\bar{a}$' is specified by

$$\sigma_{ij} = M_{ijmn}E_{mn} + N_{ijmn}\psi_{mn} + \xi\phi\delta_{ij}, \tag{2.1}$$

$$m_{ij} = M_{jimn}\widehat{E}_{mn} + N_{mnji}\psi_{mn} \tag{2.2}$$

The deformations and wryness tensors are taken as

$$\left. E_{ij} = u_{j,i} + \varepsilon_{jim}\phi^*_m,\ \widehat{E}_{ij} = \phi^*_{j,i},\ \psi_{mn} = \phi^*_{m,n}, \right\} \tag{2.3}$$

The balance laws in the presence of rotation are written below

$$\left. \sigma_{ij,i} = \rho(\ddot{u}_j + \Omega_i u_i \Omega_j - \Omega^2 u_j + 2\varepsilon_{jik}\Omega_i \dot{u}_k) \right\} \tag{2.4}$$

$$m_{ij,i} + \varepsilon_{jmn}\sigma_{mn} = \rho J \ddot{\phi}_j^* \tag{2.5}$$

$$\xi_1(\phi_{,ii}) - \omega_o\phi - \varpi\dot{\phi} - \xi(u_{i,i}) = \rho\kappa\ddot{\phi}. \tag{2.6}$$

Let the solid medium be rotating about x_3 − axis with uniform angular velocity Ω i.e., $\vec{\Omega} = \Omega(0,0,1)$, where σ_{ij}, m_{ij}, ϕ^*_j, u_j and ϕ are the stress tensor, couple stress, microrotation vector, displacement vector and volume fraction field respectively; ρ is the bulk mass density; ξ is void Parameter; J is the microinertia, M_{ijmn}, M_{jimn} are characteristics constants of material and also non symmetric properties of M_{ijmn}, M_{jimn} and N_{ijmn}, holds. For simplicity we chose $\bar{a} = (a_1, a_2, a_3)$ such that $\bar{a} = (1,0,0)$ is the fibre direction. δ_{ij}, is the Kronecker-delta function, ε_{jim} is the Levi-Civita tensor. Index after comma represents partial derivative with respect to coordinate and superscript dot specifies partial derivative with respect to time. Consider the deformation in x_1x_2-plane and thus ϕ^*=(0, 0, ϕ^*_3). Einstein summation convention over repeated indexes is used. Considering the fact the tensors are not symmetric in micropolar, equations (2.4-2.6) in component form can be written as follows

$$\begin{aligned} &B_1 u_{1,11} + (B_2 + B_3)u_{2,12} + B_4 u_{1,22} + B_1^* \phi^*_{3,2} \\ &+ \xi\phi_{,1} = \rho(\ddot{u}_1 - \Omega^2 u_1 - 2\dot{u}_2\Omega), \end{aligned} \tag{2.7}$$

$$B_5 u_{2,11} + B_2 u_{1,12} + B_6 u_{2,22} - B_5 \phi^*_{3,1} + \xi \phi_{,2} = \rho(\ddot{u}_2 - \Omega^2 u_2 + 2\dot{u}_1 \Omega), \tag{2.8}$$

$$B_5 \phi^*_{3,11} + B_4 \phi^*_{3,22} - 2B_4 \phi^*_3 + B_4(u_{2,1} - u_{1,2}) = \rho J \ddot{\phi}^*_3, \tag{2.9}$$

$$\xi_1(\phi_{,11} + \phi_{,22}) - \omega_o \phi - \varpi \dot{\phi} - \xi(u_{1,1} + u_{2,2}) = \rho\kappa\ddot{\phi}, \tag{2.10}$$

$$\left.\begin{array}{l} B_1 = (\lambda + \beta + 2\alpha - 2\mu_T + 4\mu_L), \\ B_2 = (\lambda + \alpha), \quad B_3 = 2(\mu_L - \mu_T), \\ B_4 = 2\mu_T, B_5 = 2\mu_L, B_6 = (\lambda + 2\mu_T), \\ B^*_1 = B_4 - B_3, \end{array}\right\} \tag{2.11}$$

3. Solution of the problem using normal mode analysis

Considering a homogeneous rotating micropalar fibre-reinforced anisotropic elastic medium with voids occupying the half-space $x_2 \le 0$ and plane waves incident at the free boundary $x_2=0$ at an angle θ with the x_2-axis, and let the normal mode analysis be applicable such that the incident waves have the displacement chosen as

$$u_1 = R e^{i\{k(x_1.\mathrm{Sin}\,\theta + x_2.\mathrm{Cos}\,\theta) - \omega t\}}, \tag{3.1}$$

$$u_2 = P e^{i\{k(x_1.\mathrm{Sin}\,\theta + x_2.\mathrm{Cos}\,\theta) - \omega t\}}, \tag{3.2}$$

$$\phi^*_3 = \phi^*_0 e^{i\{k(x_1.\mathrm{Sin}\,\theta + x_2.\mathrm{Cos}\,\theta) - \omega t\}}, \tag{3.3}$$

$$\phi^*_3 = \phi^*_0 e^{i\{k(x_1.\mathrm{Sin}\,\theta + x_2.\mathrm{Cos}\,\theta) - \omega t\}}, \tag{3.4}$$

where R, P, ϕ^*_0, and ϕ_0 are amplitudes of u_1, u_2, ϕ^*_3 and ϕ respectively. ω, is the angular velocity or frequency $c = \frac{\omega}{k}$ is the phase velocity of the wave, and k is the wave number and ($\sin\theta$, $\cos\theta$) represents the projection of wave normal onto the x_1x_2-plane.

We make use of equations (3.1-3.4) into equations (2.7-2.10) respectively, which yields

$$\{D_1 - (c^2 + \frac{\Omega^2}{k^2})\rho\}R + \{(B_2 + B_3)Cos\theta Sin\theta + 2ic\frac{\Omega}{k}\rho)\}P - i\{B^*_1 Cos\theta\}\frac{1}{k}\phi^*_0 - i\{\xi\frac{1}{k}Sin\theta\}\phi_0 = 0, \tag{3.5}$$

$$\{B_2 Cos\theta Sin\theta - 2ic\frac{\Omega}{k}\rho\}R + \{D_2 - (c^2 + \frac{\Omega^2}{k^2})\rho\}P + i\{B_5 Sin\theta\}\frac{1}{k}\phi^*_0 - i\{\xi\frac{1}{k}Cos\theta\}\phi_0 = 0 \tag{3.6}$$

$$\{ikB_4 Cos\theta\}R - \{ikB_4 Sin\theta\}P + \{D_3 + 2B_4 - \rho Jc^2\}\phi^*_0 = 0, \tag{3.7}$$

$$\{i\xi k Sin\theta\}R + \{i\xi k Cos\theta\}P + (\xi_1 k^2 + \omega_0 - \varpi ikc - \rho\kappa k^2 c^2)\phi_0 = 0, \tag{3.8}$$

where,

$$D_1 = B_1 Sin^2\theta + B_4 Cos^2\theta,$$
$$D_2 = B_5 Sin^2\theta + B_6 Cos^2\theta,$$
$$D_3 = B_5 Sin^2\theta + B_4 Cos^2\theta.$$

For non-trivial solution, the determinant of Equations (3.6-3.8) equals zero when (R, P, ϕ^*_0 and ϕ_0) is not equal to zero; which gives the equation for the speed and propagation of the waves, and thus becomes the quartic equation as follows

$$v^4 + H_1 v^3 + H_2 v^2 + H_3 v + H_4 = 0. \tag{3.9}$$

where $v=c^2$ This entails that the characteristic equation (3.9) with complex coefficients H_1, H_2, H_3, and H_4 (See appendix) yields four distinct complex roots; detailing that four waves propagates, with complex phase velocity: c_1, c_2, c_3 and c_4 and corresponding to the wave number k_1, k_2, k_3 and k_4 respectively. Hence, this also implies that the two dimensional model of rotational effects on reflection of plane surfaces waves in a micropolar fibre-reinforced half space medium with voids under consideration have four reflected waves. These are Quasi-P wave, Quasi-SV wave, Quasi-transverse microrotational wave and wave due to voids travelling in the medium if we assume any one of the four waves is incident at the free surface.

4. Reflection of Quasi-P waves at the free boundary

In Fig. 1, when quasi-P wave (P_0) is incident at the boundary $x_2=0$ of the rotating micropolar fibre-reinforced anisotropic semi-infinite medium with voids, then there exist reflected waves as quasi-P (P_1), Quasi-SV (P_2), quasi-TM (P_3) and due to void (P_4).

Thus, the total displacements can be assumed in the following form

$$u_1 = R_0 e^{(iQ_1)} + R_1 e^{(iS_1)} + R_2 e^{(iS_2)} + R_3 e^{(iS_3)} + R_4 e^{(iS_4)}, \tag{4.1}$$

$$u_1 = R_0 e^{(iQ_1)} + R_1 e^{(iS_1)} + R_2 e^{(iS_2)} + R_3 e^{(iS_3)} + R_4 e^{(iS_4)}, \tag{4.2}$$

$$\phi^*_3 = \phi^*_0\{e^{(iQ_1)} + e^{(iS_1)} + e^{(iS_2)} + e^{(iS_3)} + e^{(iS_4)}\} \tag{4.3}$$

$$\phi = \phi_0\{e^{(iQ_1)} + e^{(iS1)} + e^{(iS_2)} + e^{(iS_3)} + e^{(iS_4)}\}, \tag{4.4}$$

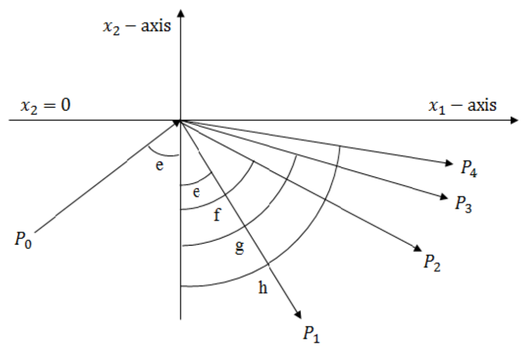


Fig. 1 Schematic of the problem showing incident and reflected waves

where,

$$Q_1 = k_0[x_1 \,\mathrm{Sin}(e) + x_2 \,\mathrm{Cos}(e) - c_1 t],$$
$$S_1 = k_1[x_1 \,\mathrm{Sin}(e) - x_2 \,\mathrm{Cos}(e) - c_1 t],$$
$$S_2 = k_2[x_1 \,\mathrm{Sin}(f) - x_2 \,\mathrm{Cos}(f) - c_2 t],$$
$$S_3 = k_3[x_1 \,\mathrm{Sin}(g) - x_2 \,\mathrm{Cos}(g) - c_3 t],$$
$$S_4 = k_4[x_1 \,\mathrm{Sin}(h) - x_2 \,\mathrm{Cos}(h) - c_4 t].$$

Incident and reflected waves given by (4.1)-(4.4) must satisfy equations (2.7- 2.8) of motion as a result of consistency conditions, thus equation (2.7) becomes

$$\left.\begin{aligned}
&\{(D_1(e)-\rho l_1\}R_0 + \{(B_2+B_3)Sin(e) \\
&\quad Cos(e)+2ic_1\frac{\Omega}{k_1}\rho)\}P_0 - i\{\frac{i}{k_0}B_1^* Cos(e)\}\phi_0^* - \{(\frac{i}{k_0}\xi Sin(e)\}\phi_0 = 0 \\
&\{(D_1(e)-\rho l_1\}R_1 - \{(B_2+B_3)Sin(e) \\
&\quad Cos(e)+2ic_1\frac{\Omega}{k_1}\rho)\}P_1 - i\{\frac{i}{k_1}B_1^* Cos(e)\}\phi_0^* - \{(\frac{i}{k_1}\xi Sin(e)\}\phi_0 = 0 \\
&\{(D_1(f)-\rho l_2\}R_2 - \{(B_2+B_3)Sin(f) \\
&\quad Cos(f)+2ic_2\frac{\Omega}{k_2}\rho)\}P_2 - i\{\frac{i}{k_2}B_1^* Cos(f)\}\phi_0^* - \{(\frac{i}{k_2}\xi Sin(f)\}\phi_0 = 0 \\
&\{(D_1(g)-\rho l_3\}R_3 - \{(B_2+B_3)Sin(g) \\
&\quad Cos(g)+2ic_3\frac{\Omega}{k_3}\rho)\}P_3 - i\{\frac{i}{k_3}B_1^* Cos(g)\phi_0^* - \{(\frac{i}{k_3}\xi Sin(g)\}\phi_0 = 0 \\
&\{(D_1(h)-\rho l_4\}R_4 - \{(B_2+B_3)Sin(h) \\
&\quad Cos(h)+2ic_4\frac{\Omega}{k_4}\rho)\}P_4 - i\{\frac{i}{k_4}B_1^* Cos(h)\}\phi_0^* - \{(\frac{i}{k_4}\xi Sin(h)\}\phi_0 = 0
\end{aligned}\right\} \tag{4.5}$$

Eq. (4.5) is made possible using Snell's law which states that the ratio of the sines of the angles of incidence and reflection of a wave is constant when it goes between two given media i.e.,

$k_0 Sin(e)=k_1 Sin(e)=k_2 Sin(f)=k_3 Sin(g)=k_4 Sin(h)$, and also $c_1 k_0=c_1 k_1=c_2 k_2=c_3 k_3=c_4 k_4$, at $x_2=0$.

Observe that $k_0=k_1$ and c_i, i= 1, 2, 3 and 4, are functions of the material parameters. Eq. (4.5) can be rewritten as;

$$\left.\begin{aligned}
R_0 &= -s_1 P_0 + s_2\phi_0^* + s_3\phi_0, \\
R_1 &= s_1 P_1 + s_4\phi_0^* + s_5\phi_0, \\
R_2 &= s_6 P_2 + s_7\phi_0^* + s_8\phi_0, \\
R_3 &= s_9 P_3 + s_{10}\phi_0^* + s_{11}\phi_0, \\
R_4 &= s_{12} P_4 + s_{13}\phi_0^* + s_{14}\phi_0
\end{aligned}\right\} \tag{4.6}$$

where

$$s_1 = \frac{((B_2+B_3)Sin(e)Cos(e)+2ic_1\frac{\Omega}{k_1}\rho)}{\{(D_1(e)-\rho l_1)\}},\quad s_2 = \frac{iB_1^* Cos(e)}{\{k_0(D_1(e)-\rho l_1)\}},$$

$$s_3 = \frac{i\xi Sin(e)}{\{k_0(D_1(e)-\rho l_1\}},\quad s_4 = \frac{i\xi B_1^* Cos(e)}{\{k_1(D_1(e)-\rho l_1)\}},\quad s_5 = \frac{i\xi Sin(e)}{\{k_1(D_1(e)-\rho l_1)\}},$$

$$s_6 = \frac{((B_2+B_3)Sin(f)Cos(f)+2ic_2\frac{\Omega}{k_2}\rho)}{\{(D_1(f)-\rho l_2)\}},\quad s_7 = \frac{iB_1^* Cos(f)}{\{k_2(D_1(f)-\rho l_2)\}},$$

$$s_8 = \frac{i\xi Sin(f)}{\{k_2(D_1(f)-\rho l_2)\}},\qquad s_9 = \frac{((B_2+B_3)Sin(g)Cos(g)+2ic_3\frac{\Omega}{k_3}\rho)}{\{(D_1(g)-\rho l_3)\}},$$

$$s_{10} = \frac{iB_1^* Cos(g)}{\{k_3(D_1(g)-\rho l_3)\}},\qquad s_{11} = \frac{i\xi Sin(g)}{\{k_3(D_1(g)-\rho l_3)\}},$$

$$s_{12} = \frac{((B_2+B_3)Sin(h)Cos(h)+2ic_4\frac{\Omega}{k_4}\rho)}{\{(D_1(h)-\rho l_4)\}},\quad s_{13} = \frac{iB_1^* Cos(h)}{\{k_4(D_1(h)-\rho l_4)\}},$$

$$s_{14} = \frac{i\xi Sin(h)}{\{k_4(D_1(h)-\rho l_4)\}},\qquad l_1 = (c_1^2+\frac{\Omega^2}{k_1^2}), l_2 = (c_2^2+\frac{\Omega^2}{k_2^2}), l_3 = (c_3^2+\frac{\Omega^2}{k_3^2}), l_4 = (c_4^2+\frac{\Omega^2}{k_4^2}).$$

Free boundary conditions

$$\sigma_{i2} = 0\ ,\ i = 1\ ,\ 2,\ m_{23} = 0 \Rightarrow \phi_{3,2}^* = 0,$$

and $\phi_{,2} = 0$ at $x_2 = 0$.

For i =1, $\sigma_{12} = 0, \Rightarrow$

$$a_{11}\frac{P_1}{P_0} + a_{12}\frac{P_2}{P_0} + a_{13}\frac{P_3}{P_0} + a_{14}\frac{P_4}{P_0} = a_{15}, \tag{4.8}$$

For i=2,

$$\sigma_{22} = 0, \Rightarrow a_{21}\frac{P_1}{P_0} + a_{22}\frac{P_2}{P_0} - a_{23}\frac{P_3}{P_0} + a_{24}\frac{P_4}{P_0} = -a_{25}, \tag{4.9}$$

$$\phi_{3,2}^* = 0, \Rightarrow a_{31}\frac{P_1}{P_0} - a_{32}\frac{P_2}{P_0} + a_{33}\frac{P_3}{P_0} + a_{34}\frac{P_4}{P_0} = -a_{35}, \tag{4.10}$$

$$\phi_{,2} = 0, \Rightarrow a_{41}\frac{P_1}{P_0} + a_{42}\frac{P_2}{P_0} + a_{43}\frac{P_3}{P_0} - a_{44}\frac{P_4}{P_0} = a_{45} \tag{4.11}$$

Thus, from equation (4.6), we have

$$\left.\begin{aligned}
\frac{R_1}{R_o} &= \frac{s_1 P_1/P_0 + s_4 \phi_0^*/P_0 + s_5 \phi_0/P_0}{s_3 \phi_0/P_0 + s_2 \phi_0^*/P_0 - s_1}, \\
\frac{R_2}{R_o} &= \frac{s_6 P_2/P_0 +_1 s_7 \phi_0^*/P_0 + s_8 \phi_0/P_0}{s_3 \phi_0/P_0 + s_2 \phi_0^*/P_0 - s_1},
\end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned}
\frac{R_3}{R_o} &= \frac{s_9 P_3/P_0 +_1 s_{10} \phi_0^*/P_0 + s_{11} \phi_0/P_0}{s_3 \phi_0/P_0 + s_2 \phi_0^*/P_0 - s_1}, \\
\frac{R_4}{R_o} &= \frac{s_{12} P_4/P_0 +_1 s_{13} \phi_0^*/P_0 + s_{14} \phi_0/P_0}{s_3 \phi_0/P_0 + s_2 \phi_0^*/P_0 - s_1}
\end{aligned}\right\} \tag{4.13}$$

where

$$a_{11} = k_1 Sin(e),\quad a_{12} = k_2 Sin(f),$$
$$a_{13} = k_3 Sin(g),\quad a_{14} = k_4 Sin(h),$$
$$a_{15} = -\{k_0 Sin(e) + \frac{i\phi_0^*}{P_0}\},$$
$$a_{21} = \{L_1 s_1 Sin(e) - L_2 Cos(e)\}k_1,$$
$$a_{22} = \{(L_1 s_6 Sin(f)) - L_2 Cos(f))\}k_2,$$
$$a_{23} = -\{(L_1 s_9 Sin(g) - L_2 Cos(g))\}k_3,$$

$$a_{24} = (L_1 s_{12} Sin(h) - L_2 Cos(h))k_4,$$

$$\begin{aligned}
a_{25} = &-\{L_1 s_1 Sin(e) - L_2 Cos(e)\}k_0 - (L_1)\{(s_2\frac{\phi_0^*}{P_0} + \\
&s_3\frac{\phi_0}{P_0})k_0 Sin(e) + (s_4\frac{\phi_0^*}{P_0} + s_5\frac{\phi_0}{P_0})k_1 Sin(e) \\
&+ (s_7\frac{\phi_0^*}{P_0} + s_8\frac{\phi_0}{P_0})k_2 Sin(f) + (s_{10}\frac{\phi_0^*}{P_0} + s_{11}\frac{\phi_0}{P_0})k_3 Sin(g) \\
&+ (s_{13}\frac{\phi_0^*}{P_0} + s_{14}\frac{\phi_0}{P_0})k_4 Sin(h)\} + i\xi\frac{\phi_0}{P_0},
\end{aligned}$$

$$\begin{aligned}
a_{31} = &\{-B_1 s_1 a_{11}^2 + \{(B_2+B_3)a_{11}k_1 Cos(e) + 2i\rho c_1 k_1\Omega\} \\
&- B_4 s_1 k_1^2 Cos^2(e) + \rho\{(k_1^2 c_1^2 + \Omega^2)s_1\},
\end{aligned}$$

$$\begin{aligned}
a_{32} = &-\{-B_1 s_6 a_{12}^2 + \{(B_2+B_3)a_{12}k_2 Cos(f) + 2i\rho c_2 k_2\Omega\} \\
&- B_4 s_6 k_2^2 Cos^2(f) + \rho\{(k_2^2 c_2^2 + \Omega^2)s_6\},
\end{aligned}$$

$$\begin{aligned}
a_{33} = &\{-B_1 s_9 a_{13}^2 + \{(B_2+B_3)a_{13}k_3 Cos(g) + 2i\rho c_3 k_3\Omega\} \\
&- B_4 s_9 k_3^2 Cos^2(g) + \rho\{(k_3^2 c_3^2 s_9 + \Omega^2)s_9\},
\end{aligned}$$

$$\begin{aligned}
a_{34} = &\{-B_1 s_{12} a_{14}^2 + \{(B_2+B_3)a_{14}k_4 Cos(h) + 2i\rho c_4 k_4\Omega\} \\
&- B_4 s_{12} k_1^2 Cos^2(h) + \rho\{(k_4^2 c_4^2 s_{12} + \Omega^2)s_{12}\},
\end{aligned}$$

$$a_{35} = -\{B_1\{(-s_1 + s_2\frac{\phi_0^*}{P_0} + s_3\frac{\phi_0}{P_0})k_0^2\sin^2(e)$$
$$+(s_4\frac{\phi_0^*}{P_0} + s_5\frac{\phi_0}{P_0})a_{11}^2 + (s_7\frac{\phi_0^*}{P_0} + s_8\frac{\phi_0}{P_0})a_{12}^2$$
$$+(s_{10}\frac{\phi_0^*}{P_0} + s_{11}\frac{\phi_0}{P_0})a_{13}^2 + (s_{13}\frac{\phi_0^*}{P_0} + s_{14}\frac{\phi_0}{P_0})a_{14}^2\}$$
$$+(B_2 + B_3)k_0^2Cos(e)Cos(e) + 2i\rho c_1\Omega$$
$$+B_4\{(-s_1 + s_2\frac{\phi_0^*}{P_0} + s_3\frac{\phi_0}{P_0})k_0^2Cos^2(e) + (s_4\frac{\phi_0^*}{P_0} + s_5\frac{\phi_0}{P_0})k_1^2Cos^2(e)$$
$$+(s_7\frac{\phi_0^*}{P_0} + s_8\frac{\phi_0}{P_0})k_2^2Cos^2(f) + (s_{10}\frac{\phi_0^*}{P_0} + s_{11}\frac{\phi_0}{P_0})k_3^2Cos^2(g)$$
$$+(s_{13}\frac{\phi_0^*}{P_0} + s_{14}\frac{\phi_0}{P_0})k_4^2Cos^2(h)\}$$

$$-\rho(-s_1 + s_2\frac{\phi_0^*}{P_0} + s_3\frac{\phi_0}{P_0})(k_0^2c_1^2 + \Omega^2) - \rho(s_4\frac{\phi_0^*}{P_0} + s_5\frac{\phi_0}{P_0})$$
$$(k_1^2c_1^2 + \Omega^2) - \rho(s_7\frac{\phi_0^*}{P_0} + s_8\frac{\phi_0}{P_0})(k_2^2c_2^2 + \Omega^2)$$
$$-\rho(s_{10}\frac{\phi_0^*}{P_0} + s_{11}\frac{\phi_0}{P_0})(k_3^2c_3^2 + \Omega^2) - \rho(s_{13}\frac{\phi_0^*}{P_0} + s_{14}\frac{\phi_0}{P_0})$$
$$(k_4^2c_4^2 + \Omega^2) - i\xi\frac{\phi_0}{P_0}\{\mathrm{k}_0\mathrm{Sin}(e) + a_{11} + a_{12} + a_{13} + a_{14}\}\},$$

$$a_{43} = \{-(B_5 + B_6)a_{13}^2 + (B_2a_{13}k_3Cos(g)$$
$$-2i\rho c_2k_3\Omega)s_9 + \rho(c_3^2k_3^2 + \Omega^2)\},$$
$$a_{44} = -\{-(B_5 + B_6)a_{14}^2 + B_2a_{14}k_4Cos(h)$$
$$-2i\rho c_2k_4\Omega)s_{12} + \rho(c_4^2k_4^2 + \Omega^2)\},$$
$$a_{45} = \{(B_5Sin^2(e) + B_6Cos^2(e))k_0^2 - \rho(c_1^2k_0^2 + \Omega^2)\}$$
$$+(s_2\frac{\phi_0^*}{P_0} + s_3\frac{\phi_0}{P_0})(B_2k_0^2Cos(e)Sin(e) - 2i\rho c_1k_1\Omega)$$
$$-(s_4\frac{\phi_0^*}{P_0} + s_5\frac{\phi_0}{P_0})(k_1B_2a_{11}Cos(e) - 2i\rho c_1k_1\Omega)$$
$$-(s_7\frac{\phi_0^*}{P_0} + s_8\frac{\phi_0}{P_0})k_2B_2a_{12}Cos(f) - 2i\rho c_2k_2\Omega)$$
$$-(s_{10}\frac{\phi_0^*}{P_0} + s_{11}\frac{\phi_0}{P_0})(B_2k_3a_{13}Cos(g) - 2i\rho c_3k_3\Omega)$$
$$-(s_{13}\frac{\phi_0^*}{P_0} + s_{14}\frac{\phi_0}{P_0})k_4B_2a_{15}Cos(h) - 2i\rho c_4k_4\Omega)$$
$$+iB_5\frac{\phi_0^*}{P_0}\{k_0Sin(e) + a_{11} + a_{12} + a_{13} + a_{14}\},\ L_1 = B_2,\ L_2 = B_6$$

Eqs. (4.8)-(4.11) can be solved for $W_i = \left|\frac{P_i}{P_0}\right|, i = 1,2,3,4,$ using Crammer's rule or any solver as the amplitude ratios (RC) along vertical reflection component and hence amplitude ratios $Z_i = \left|\frac{R_i}{R_0}\right|, i = 1,2,3,4,$ along horizontal reflection component can be obtained from equations (4.12-4.13).

5. Computational results and discussion

To study the effects of rotation, reinforcement and voids parameters on reflection coefficients of plane waves in micropolar fibre-reinforced material, the numerical constants are used.

$$\lambda = 7.59\times10^9\,Nm^{-2},\quad \mu_T = 1.89\times10^9\,Nm^{-2},$$
$$\mu_L = 2.45\times10^9\,Nm^{-2},\quad \rho = 7800\,Kgm^{-3},$$
$$\alpha = -1.28\times10^9\,Nm^{-2},\quad \beta = 220.90\times10^9\,mN^{-2},$$
$$\xi = 3.35\times10^9\,Nm^{-2}, k_0 = 1, \Omega = 0.3$$

The angle of reflections and the wave numbers are calculated using Snell's law. Also the speeds of the waves are obtained from the quartic equation (3.9). Thus, the graphs are presented in Figs. 2-6. In the graphs (Figs. 2-7), we considered variations between the Horizontal reflection coefficients (RC or Z_i) such that P-wave is incident with angle "*e*", for different fibre reinforced α, β, parameters, voids ξ parameter and rotation Ω parameter (angular velocity). This is to ascertain the effects of these parameters on reflection coefficients of the waves.

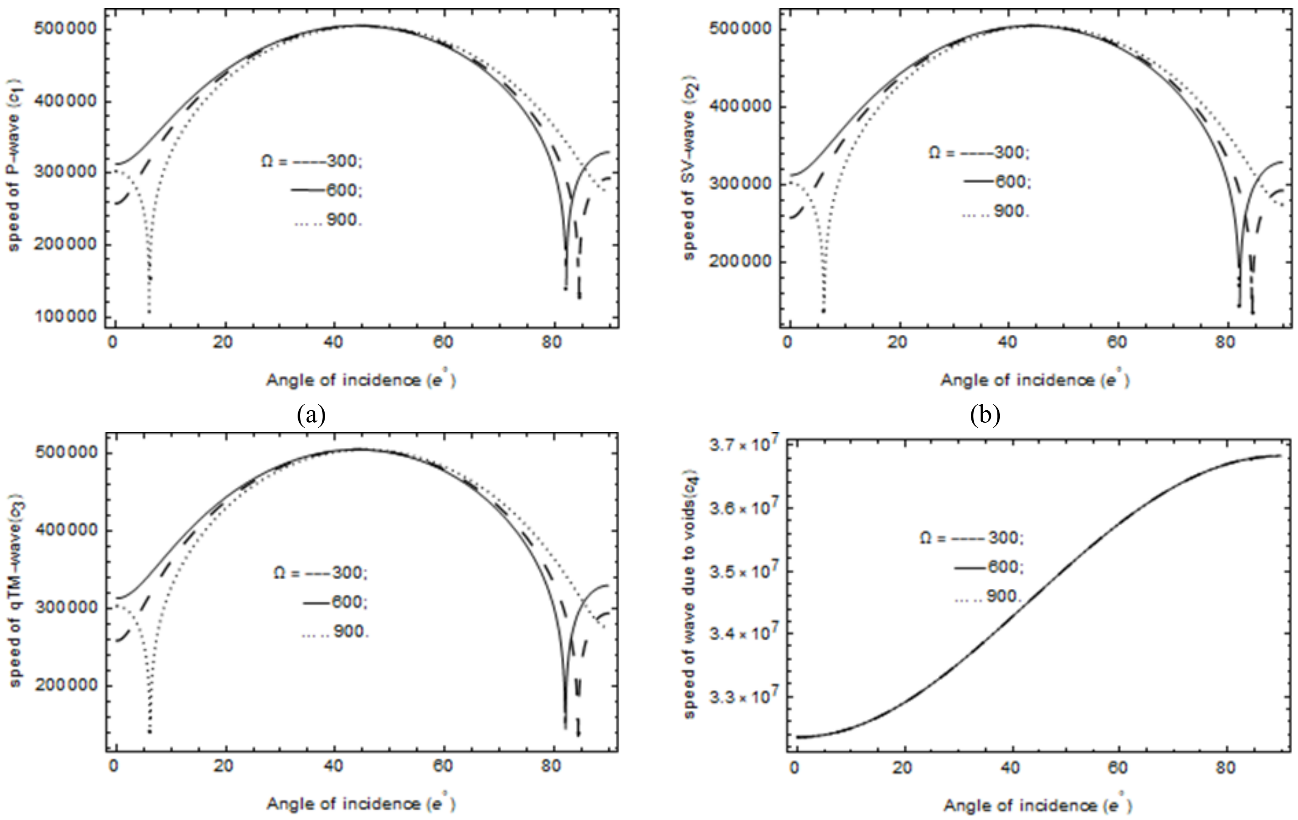
5.1 Propagation and speed of waves on the medium

Fig. 2 illustrates typical well-known phenomena called the Coriolis effects. That is, it depicts variation of speed of waves in the micropolar fibre-reinforced elastic medium with voids versus propagation angle for given distinct values of rotation parameter Ω. Fig. 2(a)-2(c) have maximum at angle of propagation of 45°. Mixed behaviors are observed in Fig. 2(a)-2(c). This is such that the speed of waves; c_1, c_2 and c_3 decreases when the propagation angle is increased, and when the rotation parameter is increased, decreasing and increasing effects are observed. However, this means that for a fast rotating medium there is a counter-clockwise direction on for the speed of the waves in Fig.2 (a-c) as observed when the rotation parameter is increased to 900. The speed of wave due to voids c_4 travels from normal incidence of propagation and attains maximum at angle near 90° and it is unaffected by the chosen rotation parameter. Generally, the propagation of the waves in the medium have virtually same pattern.

5.2 Discussion of reflection coefficients with respect to Ω, ξ, β, and α

It is observed in Fig. 3(a)-3(d) that the Reflection Coefficients, $Z_i(RC)$, (i =1, 2, 3, 4) of qLD (P-wave), qTD (SV-wave), qTM (quasi-transverse microrotational wave) and wave due to voids respectively, decreases for increasing angle of incidence. They have increasing behavior with respect to an increase Ω, except Reflection Coefficient of wave due to voids Z_4 which show slight decreasing behavior for varying rotation parameter. In Fig. 3, $Z_i(RC)$, (i =1, 2, 3and 4), starts with their curves at maximum values close to normal incidence and afterwards decreases uniformly when angle of incidence varies near 40°and finally vanish at grazing incidence. In Fig. 4(a)-4(c), the Reflection Coefficients Z_i, i = 1, 2, and 3, behave alike in such that effects of rotation is pronounced for the chosen values of angular velocity Ω. Furthermore, this some worth physical in such that the smaller the angle of incidence and increased rotation, reflection of the wave deem possible and its reflection coefficient having high values in such a medium and the reverse is the case for increased incident angle and rotation.

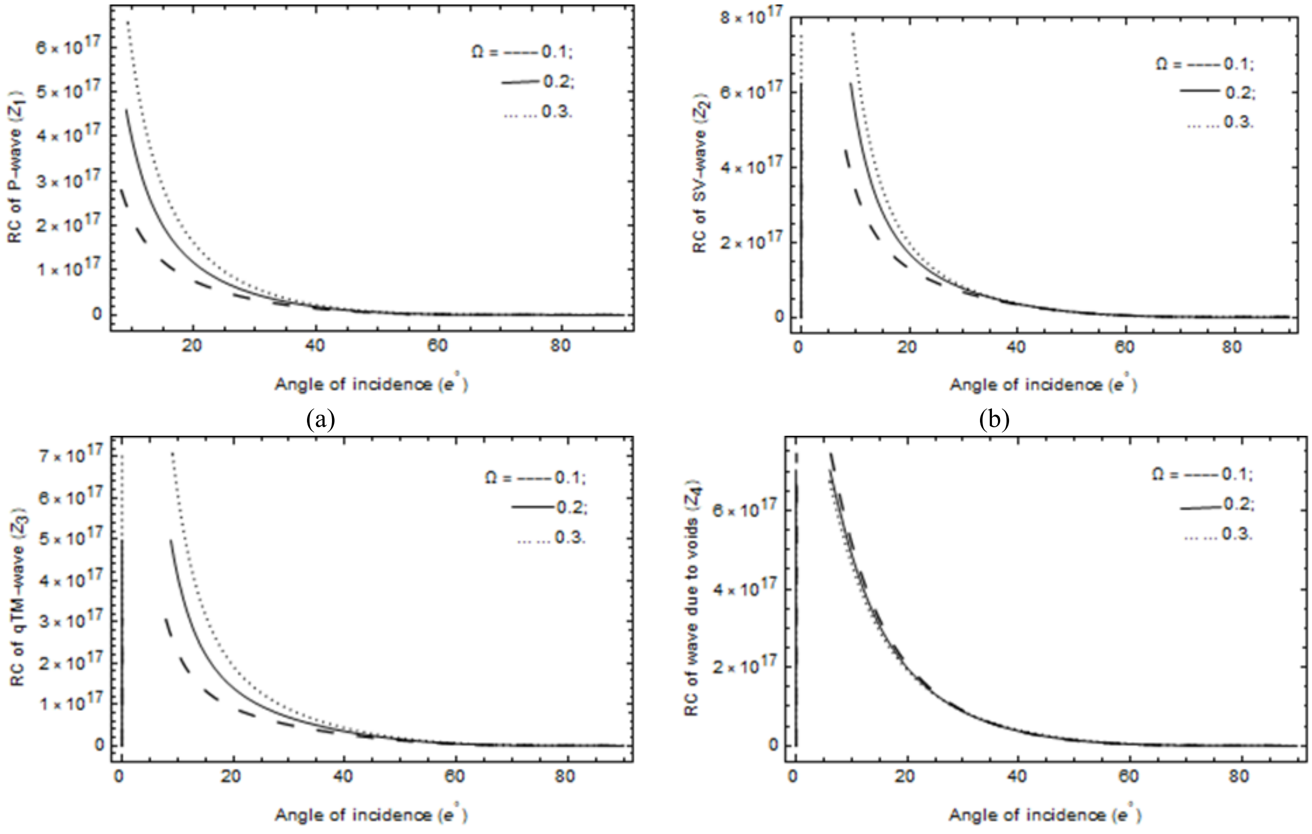
It is observed from Fig. 4(a)-4(d) that the reflection coefficients, $Z_i(RC)$, (i =1, 2, 3) of qLD (P-wave), qTD (SV-wave), qTM (quasi-transverse microrotational wave) and wave due to voids respectively, have increasing behavior for constant rotation parameter Ω=0.3, with respect to an increased voids parameter ξ and decreases



(a) (b)

(c) (d)

Fig. 2 Variations of speed of waves (c_1, c_2, c_3, c_4) versus incidence angle for given values of rotation parameter Ω



(a) (b)

(c) (d)

Fig. 3 Variations of reflection coefficients Z_i versus incidence angle for distinct values of rotation parameter Ω

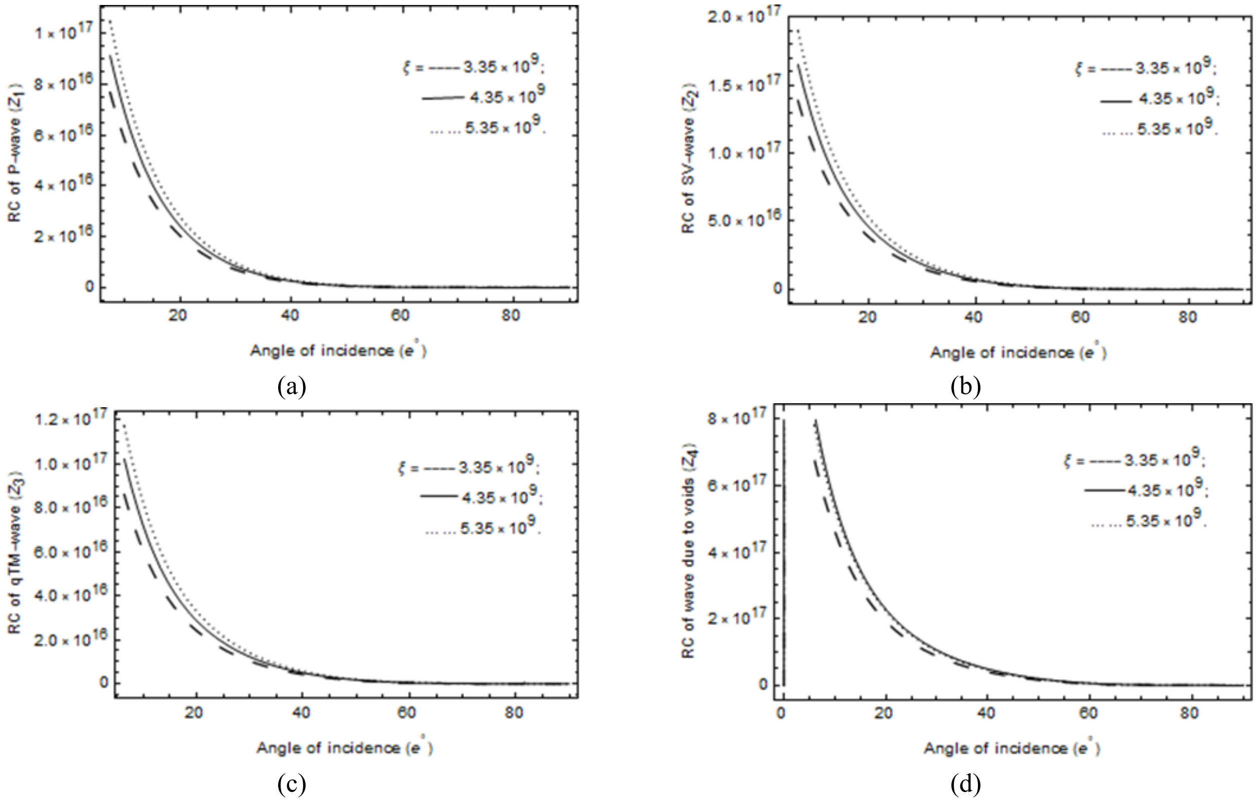


Fig. 4 Variations of reflection coefficients Z_i versus incidence angle for distinct value of voids parameter ξ

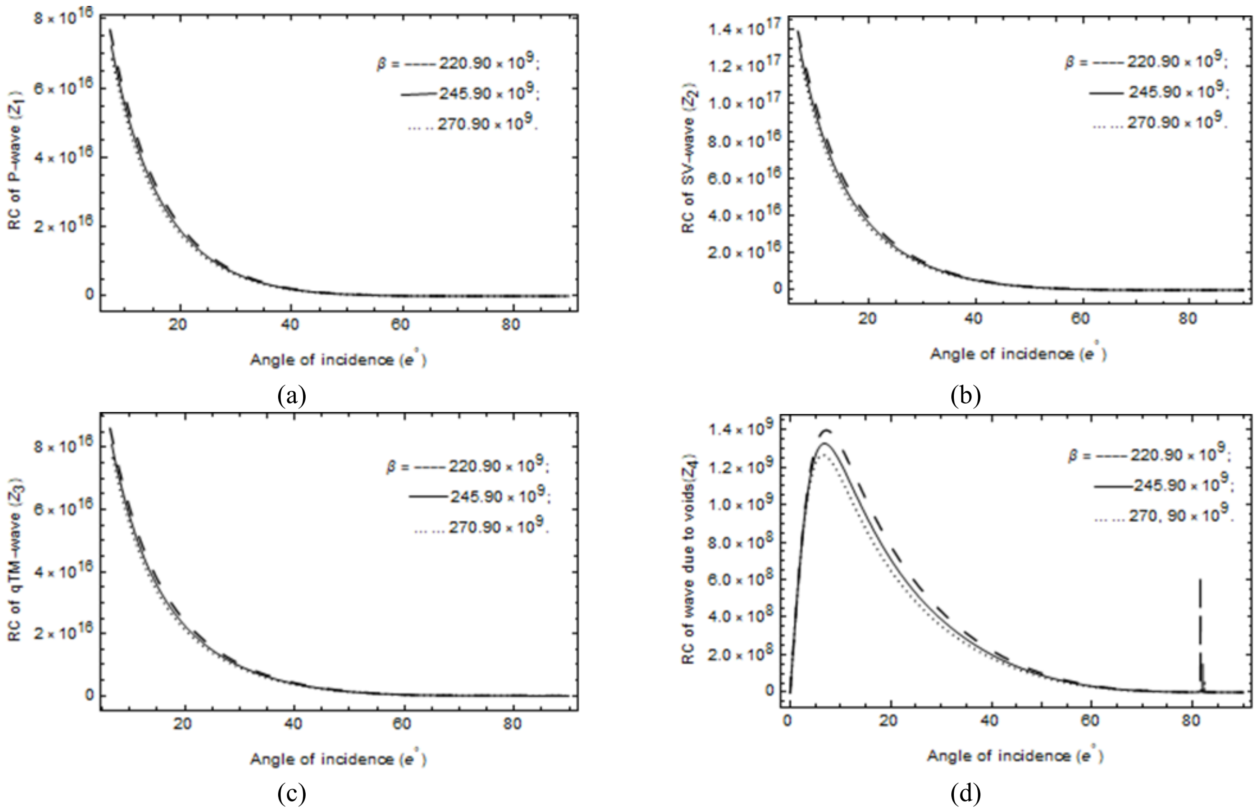


Fig. 5 Variations of reflection coefficients Z_i versus incidence angle for given values of fibre-reinforced parameter β

with increasing angle of incidence. Reflection Coefficient Z_4 of wave due to voids show slight mixed behavior with respect to increased voids parameter. In Fig. 4, $Z_i(RC)$, (i =1, 2, 3), starts with their curves at maximum values close to normal incidence and subsequently, decreases uniformly when angle of incidence varies near 60° and finally vanish

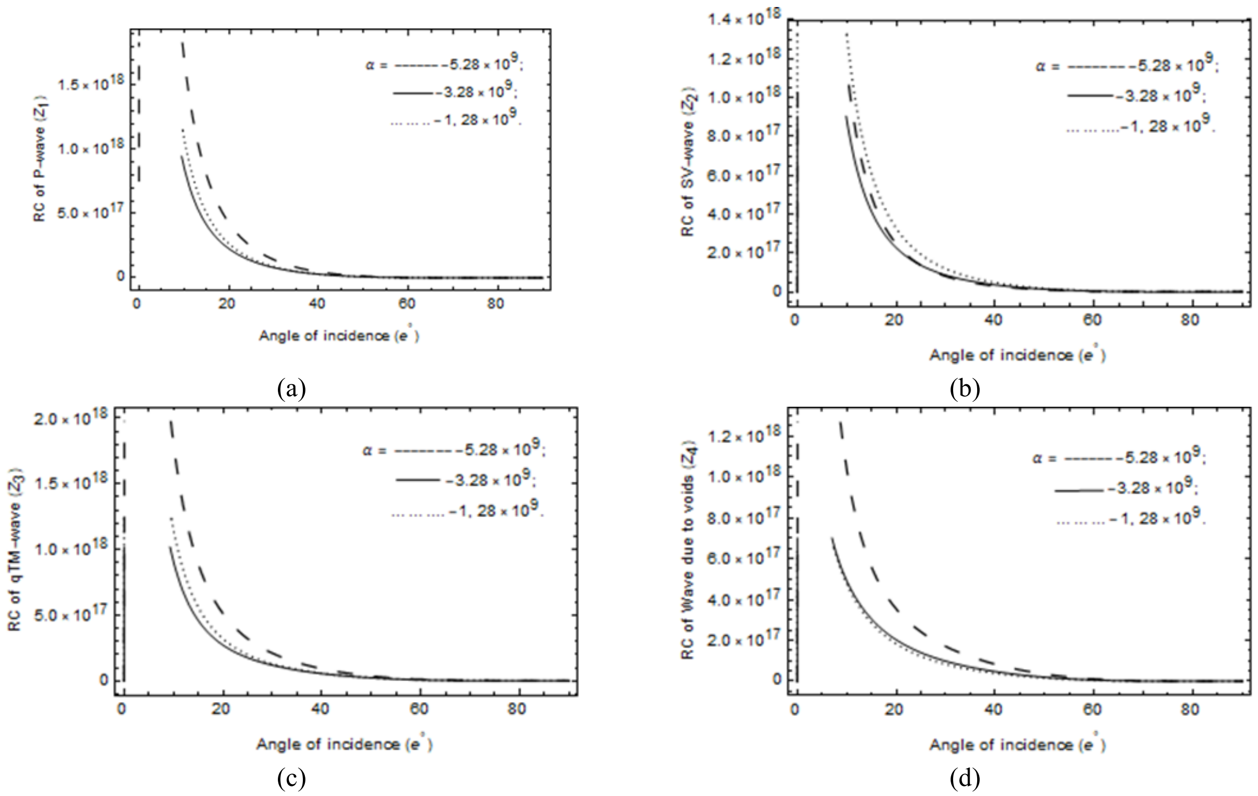


Fig. 6 Variations of reflection coefficients Z_i versus incidence angle for given values of fibre-reinforced parameter α

at grazing incidence.

Furthermore, we observed that in Fig. 5(a)-5(d), the Reflection Coefficients, $Z_i(RC)$, (i =1, 2, 3, 4) of qLD, (P-wave), qTD (SV-wave), qTM (quasi-transverse microrotational wave) and wave due to voids respectively, have slight decreasing behavior for constant rotation parameter Ω=0.3 with respect to an increased β parameter, and decreases for increased angle of incidence. In Fig. 5, $Z_i(RC)$, (i =1, 2, 3), starts with their curves at maximum values close to normal incidence and afterwards decreases uniformly when angle of incidence varies near 60°and finally vanish at grazing incidence. Contrarily, the reflection coefficients Z_i, i = 4, start with the value zero in all cases of given β parameter and decreases uniformly with the increase of angle of incidence and increased β to attain maximum value at 8° angle of incidence. It subsequently decreases uniformly to 82.5° angle of incidence where oscillation occurs and further decreases from 83° as it vanish at grazing incidence.

However, in Fig. 6(a)-6(d) we observed that the Reflection Coefficients, $Z_i(RC)$, (i =1, 2, 3) of qLD, (P-wave), qTD (SV-wave), qTM (quasi-transverse microrotational wave) and wave due to voids respectively, have more mixed behaviors for constant rotation parameter Ω=0.3, with respect to an increased α parameter and generally decreases for increasing angle of incidence. Though, the reflection coefficient Z_4 shows slight decreasing behavior for varying α parameter. In Fig. 6, $Z_i(RC)$, (i =1, 2, 3, and 4), starts with their curves at maximum values close to normal incidence and afterwards decreases uniformly when angle of incidence varies near 60°and finally vanish at grazing incidence.

7. Conclusions

We investigated propagation and reflection of plane Waves at free surfaces of rotating micropolar fibre-reinforced medium with porosity. The characteristics of the propagation and reflection of the waves were made up of angular velocity of the medium Ω, micropolar fibre-reinforced, and voids parameters. It was observed that these parameters greatly have various degrees of influence on reflection coefficients and modulation of waves as observed from the numerical simulated graphs. This is such that the micropolar fibre-reinforcement parameters tend to either increase or decrease the Reflection Coefficients in a medium with effects of rotation; one of the reinforced parameter, increases the reflection coefficient for increasing rotation near normal incidence and the other decreases it for increasing rotation. It's evident from this investigation that the propagation of waves and the reflection coefficients at some points on the medium decreases for an increased incident angle owing to the micropolar reinforcement. Moreover, voids caused an increasing behavior to the Reflection Coefficients of qLD, (P-wave), qTD (SV-wave) and qTM (quasi-transverse microrotational wave), while wave due to voids is affected in mixed behavior all with constant rotation. Also it is deduced that reflection may not occur for certain incident angles on the medium. The method used in this investigation is noteworthy for dealing with such problems for successful study. The study showed that for example the voids parameter and rotation parameter/angular velocity are proportional to each other in the modelled problem as portrayed from the graphs.

The information provided in this current investigation may be helpful for experimental based study involving rotational effects on propagation and reflection of plane waves at free surfaces of micropolar fibre- reinforced Medium with voids and also in mechanization fields of similar model such as seismic wave analysis.

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Appendix

$$H_1 = \rho^3 \begin{pmatrix} iJ\varpi\omega + 2J\kappa\rho\Omega^2 - Jk^2\kappa D_1 - Jk^2\kappa D_2 \\ -2k^2\kappa B_4 D_3 - Jk^2\xi_1 - J\omega_0 \end{pmatrix} / Jk^2\kappa\rho^4$$

$$H_2 = \frac{1}{k^2}\rho^2(2iJ\varpi\rho\omega\Omega^2 + J\kappa\rho^2\Omega^4 + i^2Jk^2\xi^2\cos[\theta]^2$$
$$+4i^2J\kappa\rho^2\omega^2\Omega^2\cos[\theta]\sin[\theta]$$
$$-i^2Jk^2\xi^2\sin[\theta]^2 - Jk^4\kappa\cos[\theta]^2\sin[\theta]^2 B_2^2$$
$$+iJk^2\kappa\rho\omega\Omega\sin[2\theta]B_3 -$$
$$\frac{1}{4}Jk^2\kappa\sin[2\theta]B_2\begin{pmatrix} 2i\rho\omega\Omega(-2+\sin[2\theta]) \\ +k^2\sin[2\theta]B_3 \end{pmatrix}$$
$$+i^2k^4\kappa\sin[\theta]^2 B_4B_5$$
$$-iJk^2\varpi\omega D_1 - Jk^2\kappa\rho\Omega^2 D_1 - iJk^2\varpi\omega D_2 -$$
$$Jk^2\kappa\rho\Omega^2 D_2 + Jk^4\kappa D_1D_2 - 2ik^2\varpi\omega B_4D_3 -$$
$$4k^2\kappa\rho\Omega^2 B_4D_3 + 2k^4\kappa B_4D_1D_3 + 2k^4\kappa B_4D_2D_3$$
$$-2Jk^2\rho\Omega^2\xi_1 + Jk^4D_1\xi_1 + Jk^4D_2\xi_1$$

$$H_3 = \frac{1}{k^4}\rho(iJ\varpi\rho^2\omega\Omega^4 + i^2Jk^2\xi^2\rho\Omega^2\cos[\theta]^2 -$$
$$i^2Jk^2\xi^2\rho\Omega^2\sin[\theta]^2 + 2i^3Jk^2\xi^2\rho\omega\Omega\cos[\theta]^2\sin[\theta]^2$$
$$+i^3Jk^2\xi^2\rho\omega\Omega\sin[2\theta] + i^2Jk^4\xi^2\cos[\theta]^2\sin[\theta]^2 B_3 +$$
$$iJ\varpi\omega\cos[\theta]\sin[\theta]\left(2i\rho\omega\Omega - k^2\cos[\theta]\sin[\theta]B_2\right)$$
$$\left(2i\rho\omega\Omega + k^2B_2 + k^2B_3\right) + i^3k^4\varpi\omega\sin[\theta]^2 B_4B_5 +$$
$$i^2k^4\kappa\rho\Omega^2\sin[\theta]^2 B_4B_5 + 2i^3k^4\kappa\rho\omega\Omega\cos[\theta]^2\sin[\theta]^2 B_4B_5 +$$
$$i^2k^6\kappa\cos[\theta]^2\sin[\theta]^2 B_2B_4B_5 + i^2k^6\kappa\cos[\theta]^2\sin[\theta]^2 B_3B_4B_5$$
$$-iJk^2\varpi\rho\omega\Omega^2 D_1 - i^2Jk^4\xi^2\cos[\theta]^2 D_1 - i^2k^6\kappa\sin[\theta]^2 A_4A_5D_1$$
$$-iJk^2\varpi\rho\omega\Omega^2 D_2 + i^2Jk^4\xi^2\sin[\theta]^2 D_2 + iJk^4\varpi\omega D_1D_2 -$$
$$4ik^2\varpi\rho\omega\Omega^2 B_4D_3 - 2k^2\kappa\rho^2\Omega^4 B_4D_3 - 2i^2k^4\xi^2\cos[\theta]^2 B_4D_3 +$$
$$2i^2k^4\xi^2\sin[\theta]^2 B_4D_3 + 2k^2\kappa\cos[\theta]\sin[\theta]$$
$$\left(-2i\rho\omega\Omega + k^2\cos[\theta]\sin[\theta]B_2\right)\left(2i\rho\omega\Omega + k^2B_2 + k^2B_3\right)A_4D_3$$
$$+2ik^4\varpi\omega B_4D_1D_3 + 2k^4\kappa\rho\Omega^2 B_4D_1D_3 +$$
$$2ik^4\varpi\omega B_4D_2D_3 + 2k^4\kappa\rho\Omega^2 B_4D_2D_3 - 2k^6\kappa B_4D_1D_2D_3 -$$
$$Jk^2\rho^2\Omega^4\xi_1 + Jk^2\cos[\theta]\sin[\theta]$$
$$\left(-2i\rho\omega\Omega + k^2\cos[\theta]\sin[\theta]A_2\right)\left(2i\rho\omega\Omega + k^2B_2 + k^2B_3\right)\xi_1$$
$$-i^2k^6\sin[\theta]^2 B_4B_5\xi_1 + Jk^4\rho\Omega^2 D_1\xi_1 + Jk^4\rho\Omega^2 D_2\xi_1 -$$
$$Jk^6D_1D_2\xi_1 + 4k^4\rho\Omega^2 B_4D_3\xi_1 - 2k^6B_4D_1D_3\xi_1 - 2k^6B_4D_2D_3\xi_1 -$$
$$J\rho^2\Omega^4\omega_0 + J\cos[\theta]\sin[\theta]$$
$$\left(-2i\rho\omega\Omega + k^2\cos[\theta]\sin[\theta]A_2\right)\left(2i\rho\omega\Omega + k^2B_2 + k^2B_3\right)\omega_0$$
$$-i^2k^4\sin[\theta]^2 B_4B_5\omega_0 + Jk^2\rho\Omega^2 D_1\omega_0 + Jk^2\rho\Omega^2 D_2\omega_0$$
$$-Jk^4D_1D_2\omega_0 + 4k^2\rho\Omega^2 B_4D_3\omega_0 - 2k^4B_4D_1D_3\omega_0 - 2k^4B_4D_2D_3\omega_0$$
$$i^3k^4\varpi\omega\cos[\theta]^2 B_4(A_1)^* - i^2k^4\kappa\rho\Omega^2\cos[\theta]^2 B_4(B_1)^*$$
$$+i^3k^4\kappa\rho\omega\Omega\sin[2\theta]B_4(B_1)^* -$$
$$-i^2k^6\kappa\cos[\theta]^2\sin[\theta]^2 B_2B_4(B_1)^* + i^2k^6\kappa\cos[\theta]^2 B_4D_2(B_1)^*$$
$$+i^2k^6\cos[\theta]^2 B_4\xi_1(B_1)^* + i^2k^4\cos[\theta]^2 A_4\omega_0(B_1)^*)/Jk^2\kappa\rho^4,$$

$$H_4 = \frac{1}{4k^4}B_4(4i^2k^2\sin[\theta]^2 B_5(i^2\varpi\rho\omega^2\Omega + i\varpi\rho\omega\Omega^2$$
$$+i^2k^2\xi^2\cos[2\theta] + i^2\varpi\rho\omega^2\Omega\cos[2\theta] - ik^2\varpi\omega D_1 - ik^2\rho\omega\Omega\xi_1$$
$$-k^2\rho\Omega^2\xi_1 - ik^2\rho\omega\Omega\cos[2\theta]\xi_1 + k^4D_1\xi_1$$
$$-i\rho\omega\Omega\omega_0 - \rho\Omega^2\omega_0 - i\rho\omega\Omega\cos[2\theta]\omega_0 +$$
$$k^2D_1\omega_0 - k^2\cos[\theta]^2 B_2\left(-i\varpi\omega + k^2\xi_1 + \omega_0\right) - k^2\cos[\theta]^2$$
$$B_3\left(-i\varpi\omega + k^2\xi_1 + \omega_0\right)) - 2D_3(k^4\sin[2\theta]^2 B_2^2\begin{pmatrix} -i\varpi\omega + \\ k^2\xi_1 + \omega_0 \end{pmatrix}$$
$$+k^2\sin[2\theta]B_2\begin{pmatrix} 2i\rho\omega\Omega(-2+\sin[2\theta]) + \\ k^2\sin[2\theta]B_3 \end{pmatrix}$$
$$\left(-i\varpi\omega + k^2\xi_1 + \omega_0\right) + 4(i\varpi\rho^2\omega\Omega^4 +$$
$$i^2k^2\xi^2\rho\Omega^2\cos[\theta]^2 + 4i^3\varpi\rho^2\omega^3\Omega^2$$
$$\cos[\theta]\sin[\theta] - i^2k^2\xi^2\rho\Omega^2\sin[\theta]^2$$
$$+2i^3k^2\xi^2\rho\omega\Omega\cos[\theta]^2\sin[\theta]^2 + i^3k^2\xi^2\rho\omega\Omega\sin[2\theta]$$
$$-ik^2\varpi\rho\omega\Omega^2 D_2 + i^2k^4\xi^2\sin[\theta]^2 D_2 - k^2\rho^2\Omega^4\xi_1$$
$$-4i^2k^2\rho^2\omega^2\Omega^2\cos[\theta]\sin[\theta]\xi_1 + k^4\rho\Omega^2 D_2\xi_1$$
$$-\rho^2\Omega^4\omega_0 - 4i^2\rho^2\omega^2\Omega^2\cos[\theta]\sin[\theta]\omega_0 + k^2\rho\Omega^2 D_2\omega_0$$
$$+ik^2\cos[\theta]\sin[\theta]B_3\begin{pmatrix} 2i\varpi\rho\omega^2\Omega + ik^2\xi^2\cos[\theta]\sin[\theta] \\ -2k^2\rho\omega\Omega\xi_1 - 2\rho\omega\Omega\omega_0 \end{pmatrix} -$$
$$k^2D_1(i\varpi\rho\omega\Omega^2 + i^2k^2\xi^2\cos[\theta]^2 - k^2\rho\Omega^2 - \rho\Omega^2\omega_0 +$$
$$k^2D_2\begin{pmatrix} -i\varpi\omega + k^2\xi_1 \\ +\omega_0 \end{pmatrix}))) + i^2k^2(k^2\sin[2\theta]^2 A_2\begin{pmatrix} -i\varpi\omega + \\ k^2\xi_1 + \omega_0 \end{pmatrix} -$$
$$2\cos[\theta](i^2k^2\xi^2\cos[\theta] + 2i\varpi\rho\omega\Omega^2\cos[\theta] + i^2k^2\xi^2\cos[3\theta]$$
$$-4i^2\varpi\rho\omega^2\Omega\sin[\theta] - 2k^2\rho\Omega(\Omega\cos[\theta]$$
$$-2i\omega\sin[\theta])\xi_1 - 2\rho\Omega^2\cos[\theta]\omega_0 + 4i\rho\omega\Omega\sin[\theta]\omega_0 +$$
$$2k^2\cos[\theta]D_2\left(-i\varpi\omega + k^2\xi_1 + \omega_0\right)))(A_1)^*)/Jk^2\kappa\rho^4.$$