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Dynamic response of pipe pile embedded in layered visco-elastic media with radial inhomogeneity under vertical excitation

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Abstract. A new mechanical model for predicting the vibration of a pipe pile embedded in longitudinally layered visco-elastic media with radial inhomogeneity is proposed by extending Novak's plain-strain model and complex stiffness method to consider viscous-type damping. The analytical solutions for the dynamic impedance, the velocity admittance and the reflected signal of wave velocity at the pile head are also derived and subsequently verified by comparison with existing solutions. An extensive parametric analysis is further performed to examine the effects of shear modulus, viscous damping coefficient, coefficient of disturbance degree, weakening or strengthening range of surrounding soil and longitudinal soft or hard interbedded layer on the velocity admittance and the reflected signal of wave velocity at the pile head. It is demonstrated that the proposed model and the obtained solutions provide extensive possibilities for practical application compared with previous related studies.

Keywords: analytical solution; dynamic response; inhomogeneous soil; viscous-type damping; pile-soil interaction; pipe pile

1. Introduction

Studying the coupled vibration of pile-soil interaction systems is of great significance for geomechanics, soil dynamics and civil engineering. In recent decades, more attention has been directed to the dynamic vibration of piles embedded in soil subjected to vertical excitation(Han Das 2011, Biswas 2013, Sinha 2015). In the dynamic system of pile-soil interaction, a number of simplified models have been developed by researchers, which require less numerical consumption compared with the FE models in frequency domain. The Winkler model has been extensively employed due to its simplicity, in which soil layers are represented by equivalent spring-dashpot elements (Shadlou 2014). However, the Winkler model has limitations when describing the mechanisms associated with wave propagation within the pile-soil system (Nogami 1987, Han 1992, Anoyatis 2012). Novak *et al.* (1978) presented a plane-strain model and considered the soil as a series of linear visco-elastic thin layers with hysteretic-type damping. In fact, the wave propagation in the horizontal direction is not considered in the plane strain model. Subsequently, Mylonakis (2001) investigated possible reasons for unsatisfactory performance of Novak's model. Thus, Hu *et al.* (2004) extended the pile-soil system by accounting for both the radial and vertical displacement of the surrounding soil layer, in which the surrounding soil is modeled as a three-dimensional axisymmetric continuum. Wu *et al.* (2014, 2016) developed a new pile-soil model to take account of the wave propagation effect based on a fictitious soil pile method. Furthermore, the effect of liquid-saturated media on the pile-soil interaction has been investigated by several researchers (Fattah 2017). Zhou *et al.* (2009), Zheng *et al.* (2015) and Cai and Hu (2010) examined the dynamic behavior of a foundation in a saturated media subjected to transient vertical loading by adopting Biot's theory. Based on the theory of porous media, proposed by De Boer *et al.*(1990), some substantial developments have also been made by Liu *et al.*(2009), Yang and Pan (2010) and Cui *et al.* (2016, 2018).

The soil surrounding the pile in the above studies of the pile-soil system is assumed to be a radially homogeneous medium. In practice, however, there may exist a disturbed zone in the soil with radial inhomogeneity immediately surrounding the pile due to construction disturbance (Novak 1990, Dotson 1990, Vaziri 1993, Ghazavi 2013). Novak and Sheta (1980) investigated the vertical and torsional vibration of a footing in radially inhomogeneous soil by developing a simple massless boundary zone model. Subsequently, Veletsos and Dotson (1986, 1988) proposed an extended boundary zone model accounting for the inertia effect of the mass and investigated the vertical and torsional vibration of foundations in this weakened inhomogeneous media. In order to eliminate the wave reflection from the interface with the boundary zone, Han and Sabin (1995)

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proposed a simplified model for this weakened boundary zone without any reflective effect, in which the material modulus and damping coefficient were assumed to vary parabolically. In addition, EI Naggar (2000) proposed a method of investigating the strengthening effect on the complex stiffness function of the soil layer by dividing the boundary zone into concentric annular sub-zones. However, Yang *et al.* (2009) found that EI Naggar's model had limitations when estimating the complex stiffness, which assumed each sub-zone to be modeled as multiple springs connected in series. Hence, Wang *et al.* (2012) proposed a new model to consider the continuity of displacement and stress at the sub-zone interface by combining it with the complex stiffness method. Subsequently, Li *et al.* (2016) extended Wang's model to analyze the vertical vibration of a large-diameter monopile in radially inhomogeneous material.

Recently, different types of pipe piles have been extensively applied in civil engineering. Unlike solid piles, both the effects of the outer and inner soil on the pile shaft should be considered to analyze the dynamic interaction behavior of the pile and soil. Some substantial developments of piles embedded in radially homogeneous soils have been made by some investigators. For example, Ding *et al.*(2011) derived an analytical solution for the vibration characteristics of large-diameter pipe piles in a visco-elastic soil under vertical excitation. This was used to investigate the effect of vibratory modes on the high-frequency wave components of low strain testing considering the material damping as viscous type. Subsequently, Zheng *et al.* (2016a, b) derived an analytical solution for the vertical dynamic response along the cross-section of pipe piles in homogeneous visco-elastic soil accounting for the three-dimensional wave effect and viscous-type damping. Moreover, Ding *et al.* (2014, 2015) proposed a new model to describe the wave propagation in a large-diameter pipe pile under an axial point load by using Winkler's model with a viscous-type damping in the surrounding soil. As for studies on the dynamic behavior of pipe piles in radially homogeneous soil, Li *et al.* (2017) derived an analytical solution for the dynamic impedance at the head of large-diameter pipe piles in a soil with radial inhomogeneity by using a hysteretic-type damping model.

It is noted that most of the aforementioned studies employ hysteretic-type damping to represent the material damping, which is independent of frequency, and it has limitations in describing the dynamic response of related problems subjected to non-harmonic loads in the time domain. In contrast, the viscous-type damping is suitable for describing the dynamic response of pile vibrations subjected to generalized modes of dynamic load in the time domain(Nogami 1976, Militano 1999).

Based on an extensive review of the literature, it is evident that little work has been devoted to the dynamic response of pipe piles in a visco-elastic soil with radial inhomogeneity accounting for viscous-type damping. The main purpose of this paper is to propose a new mechanical model for the vertical dynamic response of a pipe pile considering the radial inhomogeneity of longitudinally layered visco-elastic soil by extending Novak's plane-strain model and complex stiffness method based on viscous-type damping. In addition, the corresponding analytical solutions for the dynamic impedance, the velocity admittance and the reflected signal of the wave velocity at the pile head are determined and verified by the comparison with existing solutions. An extensive parametric analysis has also been performed to discuss the effects of shear modulus, viscous damping coefficient, the coefficient of degree of disturbance, the weakening or strengthening range of the surrounding soil and the longitudinal soft or hard interbedded layer on the velocity admittance and the reflected signal of the wave velocity at the pile head.

2. Mechanical model

The following assumptions are employed in this paper.

(1) The pile is linear elastic with a circular cross-section and the soil beneath the pile toe is simplified as a Kelvin-Voigt model.

(2) The outer surrounding soil of the interaction system consists of two annular zones, an inner disturbed zone and an outer undisturbed zone with semi-infinity.

(3) The inner annular zone of disturbed soil is divided into a series of sub-zones. For a given layer, the soil properties within different annular sub-zones are radially inhomogeneous.

(4) The frequency-dependent viscous-type damping is used to describe the inside and outside soil of pipe pile.

The plug effect is not considered in the model, which is suitable for cast-in-situ pipe pile.

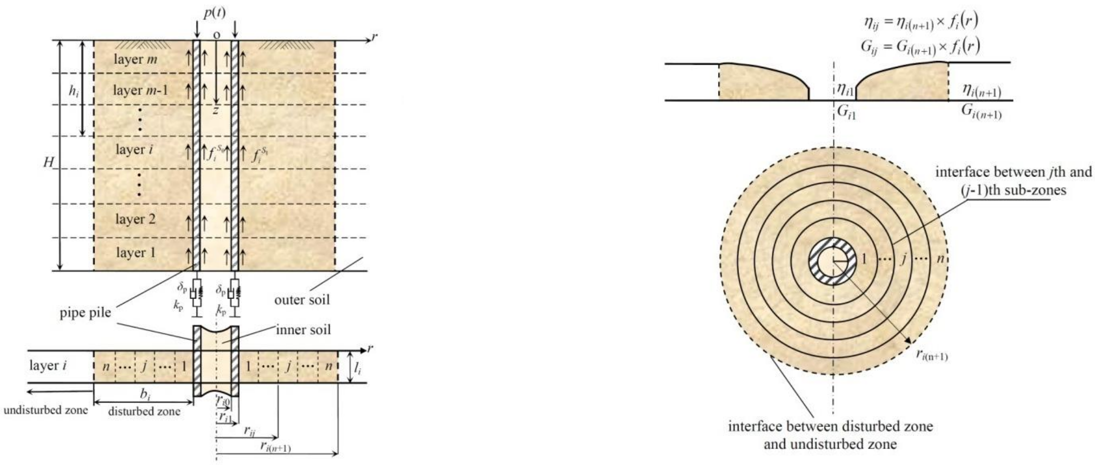
The deformations of the pile and surrounding soil are small. The equilibrium of shear stress and the continuity of displacement are both satisfied at the interfaces between pipe pile, adjacent annular zones and sub-zones of the surrounding soil.

The mechanical model of the pile-soil interaction system is shown in Fig. 1(a) and 1(b). The pile-soil system is divided into m layers or segments numbered 1, 2, . . ., i, . . ., m from the toe of the pile to the pile head. The thickness of the ith soil layer is l_i, and the depth of the upper interface of the ith soil layer is h_i. The inner and outer radius of the ith pile segment are r_{i0}, r_{i1}, respectively. The radius and radial thickness of the disturbed zone within the ith soil layer are $r_{i(n+1)}$ and b_i, respectively. The inner disturbed zone is subdivided into n concentric sub-zones numbered 1, 2, ..., j , ..., n along the radial direction. The radius of the interface between the (j-1)th and jth sub-zones within the ith soil layer is represented by r_{ij}.

The undisturbed zone of the surrounding soil is homogeneous, isotropic and visco-elastic with viscous-type damping. Within the disturbed zone of the ith layer, $G_{ij}(r)$ and $\eta_{ij}(r)$ are given by the expressions (1) and (2).

$$G_{ij}(r)=\begin{cases} G_{i1} & r=r_{i1} \\ G_{i(n+1)}\times f_i(r) & r_{i1}<r<r_{i(n+1)} \\ G_{i(n+1)} & r\geq r_{i(n+1)} \end{cases} \tag{1}$$

$$\eta_{ij}(r)=\begin{cases} \eta_{i1} & r=r_{i1} \\ \eta_{i(n+1)}\times f_i(r) & r_{i1}<r<r_{i(n+1)} \\ \eta_{i(n+1)} & r\geq r_{i(n+1)} \end{cases} \tag{2}$$



(a) (b)

Fig. 1 Mechanical model

where G_{ij}, η_{ij} are the shear modulus and viscous damping coefficient of the jth annular sub-zone within the ith layer, respectively. $f_i(r)$ denotes the parabolic variation of the material properties within the disturbed zone of the ith layer (Han 1995, EI Naggar 2000, Wang 2012). The shear modulus and viscous damping coefficient of each sub-zone are considered to be homogeneous, and can be determined by Eqs. (1) and (2).

The mechanical coefficients of the Kelvin-Voigt model are represented by δ_p and k_p,. The uniform distributed excitation at the pile head is denoted by $p(t)$.

Based on Novak's plane-strain model (Novak 1978), the governing equation for the jth inner annular sub-zone of disturbed zone within the ith layer of outer soil is given by Eq. (3).

$$G_{ij}\frac{\partial^2 u_{ij}^{S_1}(r,t)}{\partial r^2}+\eta_{ij}\frac{\partial^3 u_{ij}^{S_1}(r,t)}{\partial t\partial r^2}+\frac{G_{ij}}{r}\frac{\partial u_{ij}^{S_1}(r,t)}{\partial r}+\frac{\eta_{ij}}{r}\frac{\partial^2 u_{ij}^{S_1}(r,t)}{\partial t\partial r}=\rho_{ij}\frac{\partial^2 u_{ij}^{S_1}(r,t)}{\partial t^2} \tag{3}$$

where $u_{ij}^{S1}(r,t)$ represents the vertical displacement of the jth annular sub-zone within the ith layer of the disturbed soil. G_{ij}, η_{ij} and ρ_{ij} denote the shear modulus, the viscous damping coefficient and the mass density of the jth annular sub-zone within the ith layer of the disturbed zone, respectively.

Similarly, the governing equation for the ith layer of inside soil can be expressed as

$$G_{i0}\frac{\partial^2 u_{i}^{S_0}(r,t)}{\partial r^2}+\eta_{i0}\frac{\partial^3 u_{i}^{S_0}(r,t)}{\partial t\partial r^2}+\frac{G_{i0}}{r}\frac{\partial u_{i}^{S_0}(r,t)}{\partial r}+\frac{\eta_{i0}}{r}\frac{\partial^2 u_{i}^{S_0}(r,t)}{\partial t\partial r}=\rho_{i0}\frac{\partial^2 u_{i}^{S_0}(r,t)}{\partial t^2} \tag{4}$$

where $u_i^{S_0}(r,t)$ denotes the vertical displacement of inner soil within the ith layer. ρ_{i0}, G_{i0} and η_{i0} are the mass density, the shear modulus and the viscous damping coefficient of the ith layer of inner soil.

The governing equation for the ith pile segment is expressed as

$$\frac{\partial^2 u_i^P(z,t)}{\partial z^2}-\frac{2\pi r_{i0} f_i^{S_0}}{E_i^P A_i^P}-\frac{2\pi r_{i1} f_i^{S_1}}{E_i^P A_i^P}=\frac{\rho_i^P}{E_i^P}\frac{\partial^2 u_i^P(z,t)}{\partial t^2} \tag{5}$$

where $u_i^P(z,t)$ denotes the vertical displacement of the ith pile segment. ρ_i^P, E_i^P and A_i^P are the mass density, the Young's modulus and the cross-sectional area, respectively, of the ith pile segment. $f_i^{S_0}$ and $f_i^{S_0}$ are the shear stresses exerted by the inner soil and surrounding soil on the ith pile segment, respectively.

The following assumptions are also used in this paper.

The displacement continuity and shear stress equilibrium at the interface between the pile shaft and the inner soil are written as

$$u_i^{S_0}(r_{i0},t)=u_i^P\left(r_{i0},t\right) \tag{6a}$$

$$f_i^{S_0}=\tau_i^{S_0}\left(r\right)\Big|_{r=r_{i0}} \tag{6b}$$

The displacement at the outermost zone of the outer soil diminishes at infinity within the ith layer can be expressed as

$$\lim_{r\to\infty} u_{i(n+1)}^{S_1}(r,t)=0 \tag{7}$$

The displacement continuity and shear stress equilibrium conditions at the interface between the pile shaft and the surrounding soil are given as

$$u_{i1}^{S_1}(r_{i1},t)=u_i^P\left(r_{i1},t\right) \tag{8a}$$

$$f_i^{S_1}=-\tau_{i1}^{S_1}\left(r\right)\Big|_{r=r_{i1}} \tag{8b}$$

The boundary condition at the pile head is

$$E_m^P A_m^P\frac{\partial u_m^P(z,t)}{\partial z}\bigg|_{z=0}=p(t) \tag{9}$$

where E_m^P and A_m^P are the Young's modulus and the cross-sectional area of the mth pile segment, respectively.

The boundary condition at the pile toe is

$$E_1^P \frac{\partial u_1^P(z,t)}{\partial z}\bigg|_{z=H} = -(k_p u_1^P(z,t) + \delta_p \frac{\partial u_1^P(z,t)}{\partial t}) \tag{10}$$

where E_1^p and A_1^p are the Young's modulus and the cross-sectional area of the first pile segment, respectively.

3. Solutions of the governing equations

3.1 Vibration of the surrounding soil

Applying the Laplace transform to Eq. (3) yields

$$G_{ij}\frac{\partial^2 U_{ij}^{S_1}(r,s)}{\partial r^2} + \eta_{ij} s \frac{\partial^2 U_{ij}^{S_1}(r,s)}{\partial r^2} + \frac{G_{ij}}{r}\frac{\partial U_{ij}^{S_1}(r,s)}{\partial r} + \frac{\eta_{ij} s}{r}\frac{\partial U_{ij}^{S_1}(r,s)}{\partial r} = \rho_{ij} s^2 U_{ij}^{S_1}(r,s) \tag{11}$$

where $U_{ij}^{S_1}(r,s)$ is the Laplace transform of $u_{ij}^{S_1}(r,t)$.

After rearranging the terms in Eq.(11), then Eq.(12) is obtained as

$$\frac{\partial^2 U_{ij}^{S_1}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial U_{ij}^{S_1}(r,s)}{\partial r} = \left(q_{ij}^{S_1}\right)^2 U_{ij}^{S_1}(r,s) \tag{12}$$

where $q_{ij}^{S_1} = \sqrt{\dfrac{\rho_{ij} s^2}{G_{ij} + \eta_{ij} s}}$.

The general solution of Eq.(12) can be obtained as

$$U_{ij}^{S_1}(r,s) = A_{ij}^{S_1} K_0(q_{ij}^{S_1} r) + B_{ij}^{S_1} I_0(q_{ij}^{S_1} r) \tag{13}$$

where $I_0(q_{ij}^{S_1} r)$ and $K_0(q_{ij}^{S_1} r)$ are the modified Bessel functions of the first and second kinds of zero order, respectively. $A_{ij}^{S_1}$ and $B_{ij}^{S_1}$ are undetermined coefficients. According to the boundary conditions and the continuity of displacement at the interfaces between sub-zones, the recursion formula of the vertical stiffness can be easily given by

$$KK_{ij}^{S_1} = 2\pi r_{ij} q_{ij}^{S_1}(G_{ij} + \eta_{ij} s)\frac{C_{ij}^{S_1} + E_{ij}^{S_1} KK_{i(j+1)}^{S_1}}{D_{ij}^{S_1} + F_{ij}^{S_1} KK_{i(j+1)}^{S_1}} \tag{14}$$

where

$$C_{ij}^{S_1} = 2\pi r_{i(j+1)} q_{ij}^{S_1}(G_{ij} + \eta_{ij} s)[I_1(q_{ij}^{S_1} r_{i(j+1)})K_1(q_{ij}^{S_1} r_{ij}) - K_1(q_{ij}^{S_1} r_{i(j+1)})I_1(q_{ij}^{S_1} r_{ij})],$$

$$D_{ij}^{S_1} = 2\pi r_{i(j+1)} q_{ij}^{S_1}(G_{ij} + \eta_{ij} s)[I_1(q_{ij}^{S_1} r_{i(j+1)})K_0(q_{ij}^{S_1} r_{ij}) + K_1(q_{ij}^{S_1} r_{i(j+1)})I_0(q_{ij}^{S_1} r_{ij})],$$

$$E_{ij}^{S_1} = I_0(q_{ij}^{S_1} r_{i(j+1)})K_1(q_{ij}^{S_1} r_{ij}) + K_0(q_{ij}^{S_1} r_{i(j+1)})I_1(q_{ij}^{S_1} r_{ij}),$$

$$F_{ij}^{S_1} = I_0(q_{ij}^{S_1} r_{i(j+1)})K_0(q_{ij}^{S_1} r_{ij}) - K_0(q_{ij}^{S_1} r_{i(j+1)})I_0(q_{ij}^{S_1} r_{ij}),$$

the corresponding derivation for $KK_{ij}^{S_1}$ is expressed in the Appendix I.

3.2 Vibration of the inner soil

Applying the Laplace transform to Eq. (4) yields

$$G_{i0}\frac{\partial^2 U_i^{S_0}(r,s)}{\partial r^2} + \eta_{i0} s \frac{\partial^2 U_i^{S_0}(r,s)}{\partial r^2} + \frac{G_{i0}}{r}\frac{\partial U_i^{S_0}(r,s)}{\partial r} + \frac{\eta_{i0} s}{r}\frac{\partial U_i^{S_0}(r,s)}{\partial r} = \rho_{i0} s^2 U_i^{S_0}(r,s) \tag{15}$$

where $U_i^{S_0}(r,s)$ is the Laplace transform of $u_i^{S_0}(r,t)$.

After rearranging the terms in Eq.(15), then Eq.(16) is obtained as

$$\frac{\partial^2 U_i^{S_0}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial U_i^{S_0}(r,s)}{\partial r} = \left(q_i^{S_0}\right)^2 U_i^{S_0}(r,s) \tag{16}$$

where $q_i^{S_0} = \sqrt{\dfrac{\rho_{i0} s^2}{G_{i0} + \eta_{i0} s}}$.

The general solution of Eq. (16) is given by

$$U_i^{S_0}(r,s) = A_i^{S_0} K_0(q_i^{S_0} r) + B_i^{S_0} I_0(q_i^{S_0} r) \tag{17}$$

where $A_i^{S_0}$ and $B_i^{S_0}$ are undetermined coefficients.

The displacement of inner soil is a limited value if r=0, namely, $u_i^{S_0}(r,t)\big|_{r\to 0} < \infty$. Hence, $A_i^{S_0}$=0.

Then, Eq.(17) can be reduced to

$$U_i^{S_0}(r,s) = B_i^{S_0} I_0(q_i^{S_0} r) \tag{18}$$

Applying the Laplace transform to Eq. (6a) and combining with Eq. (18) leads to

$$KK_i^{S_0} = -\frac{2\pi r_{i0}\tau_i^{S_0}(r_{i0})}{U_i^P} = -2\pi r_{i0}(G_{i0} + \eta_{i0} s) q_i^{S_0}\frac{I_1(q_i^{S_0} r_{i0})}{I_0(q_i^{S_0} r_{i0})} \tag{19}$$

where $U_i^P(r,s)$ is the Laplace transform of $u_i^P(r,s)$.

3.3 Vibration of the pipe pile

Performing the Laplace transform to Eq. (5) and combining with Eq.(14) and (19) gives

$$\frac{\partial^2 U_i^P(z,s)}{\partial z^2} - \alpha_i^2 U_i^P(z,s) = 0 \tag{20}$$

where $\alpha_i^2 = \dfrac{KK_{i1}^{S_1}}{E_i^P A_i^P} - \dfrac{KK_i^{S_0}}{E_i^P A_i^P} + \dfrac{\rho_i^P}{E_i^P} s^2$.

The general solution for Eq. (20) gives

$$U_i^P(z,s) = C_i^P e^{\bar{\alpha}_i z/l_i} + D_i^P e^{-\bar{\alpha}_i z/l_i} \tag{21}$$

where $\bar{\alpha}_i = \alpha_i l_i$, C_i^P and D_i^P are undetermined coefficients.

By using the recursion method of the transfer function, the complex impedance of the vertical displacement at the pile head is derived as

$$Z_m^P\big|_{z=h_m=0} = \frac{-E_m^P A_m^P \bar{\alpha}_m(\beta_m - 1)}{l_m(\beta_m + 1)} = \frac{-E_m^P A_m^P}{l_m} Z_m^{P'} \tag{22}$$

where $\beta_m = \dfrac{E_m^P A_m^P \bar{\alpha}_m - Z_{m-1}^P l_m}{E_m^P A_m^P \bar{\alpha}_m + Z_{m-1}^P l_m} e^{-2\bar{\alpha}_m h_{m-1}/l_m}$, $Z_m^{P'} = \dfrac{\bar{\alpha}_m(\beta_m - 1)}{\beta_m + 1}$.

$Z_m^{P'}$ can be further rewritten as

$$Z_m^{P'} = K_r + \mathrm{i}K_i \tag{23}$$

where K_r and K_i denote the true stiffness and the equivalent damping, respectively.

The corresponding derivation for the complex impedance of the vertical displacement is expressed in the Appendix II.

The frequency response function of the vertical displacement at the pile head can be easily given by

$$H_u(z,s) = 1/Z_m^P = -\frac{l_m(\beta_m+1)}{E_m^P A_m^P \bar{\alpha}_m(\beta_m-1)} \tag{24}$$

Given $s=i\omega$ (i^2=-1), the frequency response function of the vertical velocity at the pile head can obtained as

$$H_v = -\frac{i\omega l_m(\beta_m+1)}{E_m^P A_m^P \bar{\alpha}_m(\beta_m-1)} = -\frac{1}{\rho_m^P A_m^P V_m^P} H_v' \tag{25a}$$

$$H_v' = \frac{i\theta \bar{t}_m(\beta_m+1)}{\bar{\alpha}_m(\beta_m-1)} \tag{25b}$$

where H_v' is a dimensionless frequency response function of the vertical velocity at the pile head, $V_m^P = \sqrt{E_m^P/\rho_m^P}$, $\theta = \omega T_c$, $T_c = H/V_m^P$, $t_m = l_m/V_m^P$, $\bar{t}_m = t_m/T_c$.

Taking a transient semi-sine wave ($0 \le t \le T$) as the transient excitation applied on the pile head, the time-domain function of the velocity response at the pile head can be derived by inverse Fourier transform

$$\begin{aligned} g(t) &= \mathrm{IFT}[P(i\omega)\times H_v(i\omega)] \\ &= Q_{\max}\mathrm{IFT}[-\frac{1}{\rho_p A_p V_p} H_v' \frac{\pi T}{\pi^2 - T^2\omega^2}(1+e^{-i\omega T})] \\ &= -\frac{Q_{\max}}{\rho_p A_p V_p} V_v' \end{aligned} \tag{26}$$

where $V_v' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[H_v' \frac{\pi T'}{\pi^2 - T'^2\theta^2}\left(1+e^{-i\theta T'}\right)\right]e^{i\theta t'}d\theta$; $Q_{\max}$ and T are the excitation amplitude and impulse width, respectively; $T'=T/T_c$ denotes the dimensionless impulse width; $P(i\omega)$ is the Fourier transform of $p(t)$.

4. Results and discussions

In this section, numerical results are presented to demonstrate the validity of the obtained analytical solutions and to investigate the vertical dynamic response of a pipe pile embedded in longitudinally layered visco-elastic soil with radial inhomogeneity. It is observed that stable solutions can be derived if the number of the annular sub-zones n >10, by EI Naggar (2000) and Wang *et al.* (2012). To accurately describe the variation of radial inhomogeneity and reduce wave reflection from the interfaces of adjacent sub-zones, the number of the annular sub-zones n is taken as 20 in the following analyses. It follows the assumption that the shear velocity V_{ij} varies linearly from the outer undisturbed zone to the first sub-zone of the disturbed zone along the radial direction for the ith layer. Hence, the corresponding soil shear modulus $G_{ij}=V_{ij}^2\,\rho_{ij}$ changes in a quadratic sense. The variation in the viscous damping coefficient η_{ij} is also assumed to change quadratically from the outer undisturbed zone to the first sub-zone of the disturbed zone along the radial direction within the ith layer. In addition, the coefficient of degree of disturbance within the ith layer ξ_i is defined as

$$\xi_i = \sqrt{G_{i1}/G_{i(n+1)}} = \sqrt{\eta_{i1}/\eta_{i(n+1)}} = V_{i1}/V_{i(n+1)} \tag{27}$$

Unless otherwise specified, the following parameter values are used: r_{i0}=0.38 m, $r_{i1}=b_i$=0.5 m, ρ_i^P=2500 kg/m^3, E_i^P =25 GPa; H=6 m, k_p=1000 kN/m^3, δ_p=100 kN.s/m^2, ρ_{ij}=2000 kg/m^3, $V_{i(n+1)}$=50 m/s, $\eta_{i(n+1)}$=10 kN.s/m^2, ξ_i =2.0, m=5, n=20. The shear modulus G_{i0} and the viscous damping coefficient η_{i0} of the inner soil are identical with the corresponding parameters of the first annular sub-zone within the ith layer.

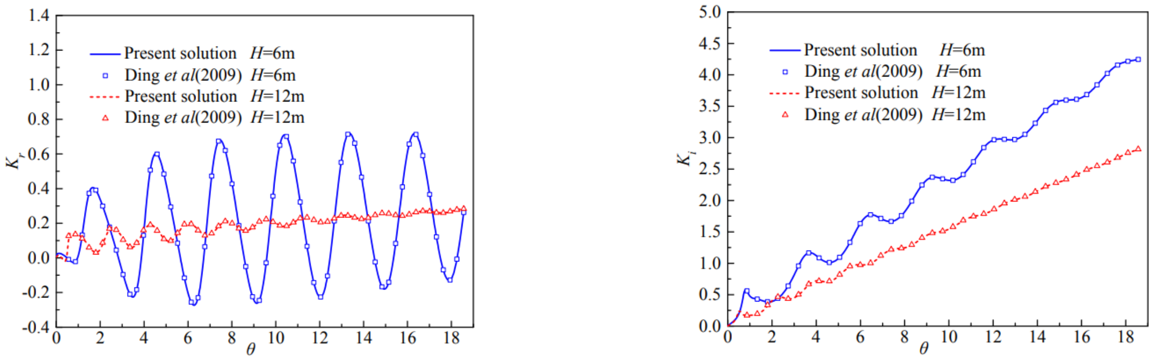
4.1 Comparison with existing solutions

The complex impedance solution expressed in Eq.(23) can be reduced to describe the vertical vibration of a pipe pile in homogeneous soil by setting $q_i \to 1$. Therefore, based on the same parameters, the solution of $Z_m^{P'}$ can be verified by comparing it with the existing solution of Ding *et al.* (2009). It is illustrated in Fig. 2 that the present solution of complex impedance with different values of pile length H is in very good agreement with that proposed by Ding *et al.* (2009). The solution of $Z_m^{P'}$ can also be reduced to describe the vertical vibration of a solid pile embedded in radially inhomogeneous soil by setting $r_{i0} \to 1$. It is noted that a viscous-type damping is adopted for the present solution, which is different from the hysteretic-type damping used for the solution of Yang *et al.* (2009). To perfectly match the parameters, the effect of material damping is not considered in the following comparison with the existing solution of Yang *et al.* (2009) as shown in Fig. 3. It is clear that the present solution with different values of pile length H agrees well with the solution of Yang *et al.* (2009). Therefore, the accuracy of the present solution is validated with these independent comparisons.

4.2 Parametric study and discussion

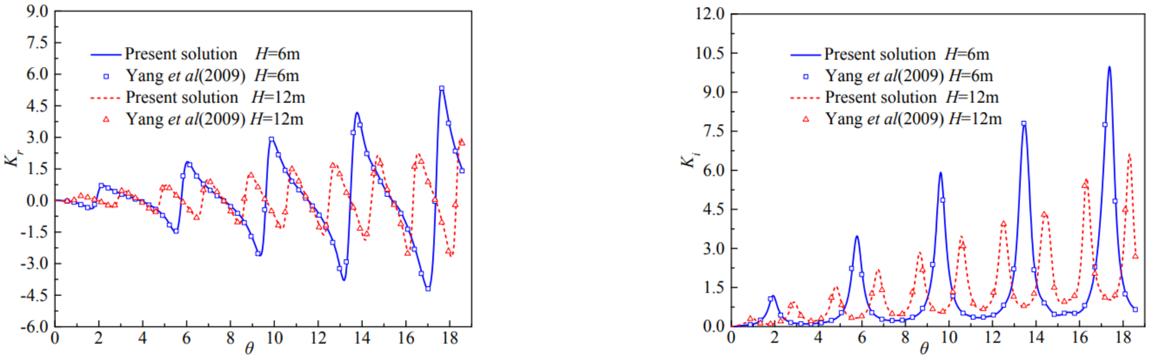
4.2.1 Effect of disturbance degree for radial inhomogeneity

The surrounding soil of pipe pile is radially inhomogeneous due to construction disturbance. The coefficient of degree of disturbance within the ith layer ξ_i (i=1, 2, ... , 5) is defined by Eq. (27). It is indicated that the soil layer is weakened due to construction disturbance within the disturbed zone if ξ_i<1. While, the soil layer is strengthened within the disturbed zone if ξ_i>1. Different coefficients of degree of disturbance corresponding to weakening cases W1-W4 are given in Table 1. In particular, case W4 indicates that the surrounding soil is radially homogeneous without construction disturbance. Fig. 4 shows the effect of soil weakening within the disturbed zone due to construction disturbance corresponding to cases W1-W4 listed in Table 1. It can be observed that the degree of weakening of the surrounding soil due to construction disturbance has an obvious effect on the dynamic response at the pile head. Specifically, with the degree of weakening of the disturbed zone increasing, the oscillation amplitudes and resonance frequencies of the velocity admittance, and the amplitudes of the reflected signal increase.



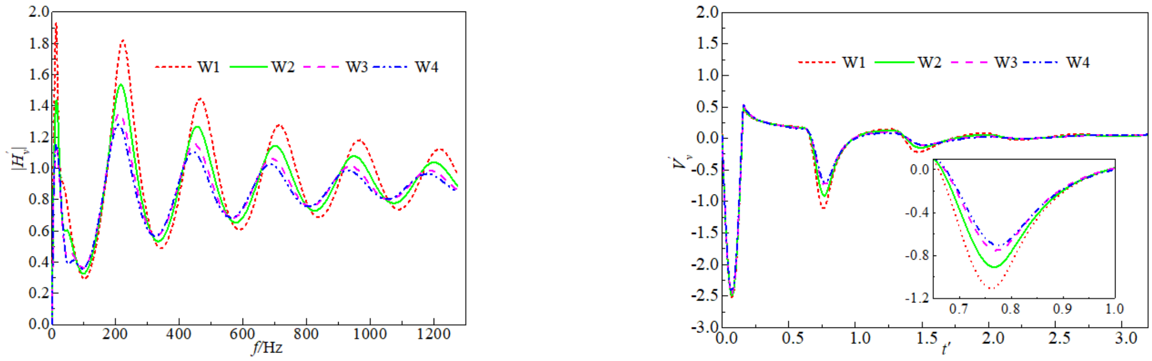
(a) True stiffness (b) Equivalent damping

Fig. 2 Comparison of the complex impedance in reduced form ($q_i \to 1$) with the solution in Ding *et al.* (2009)



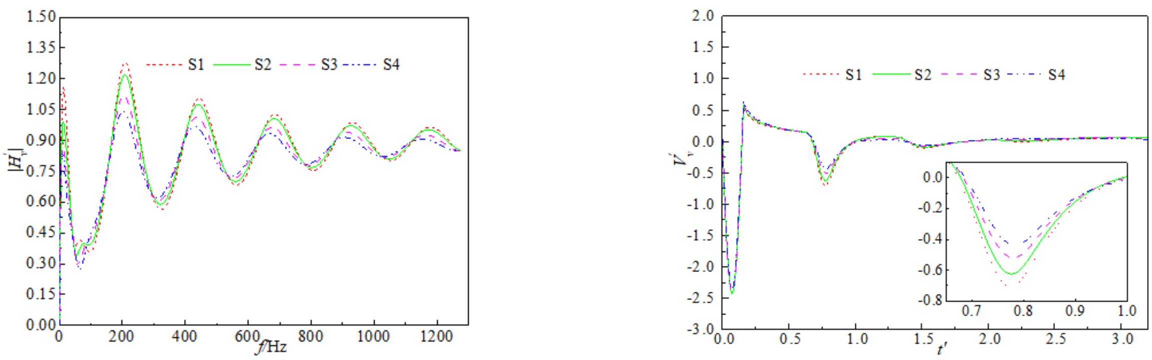
(a) True stiffness (b) Equivalent damping

Fig. 3 Comparison of the complex impedance in reduced form ($r_{i0} \to 0$) with the solution in Yang *et al.* (2009)



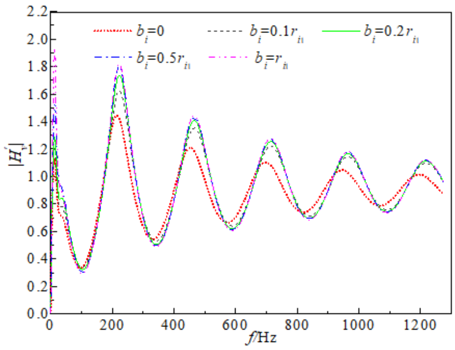
(a) Velocity admittance (b) Reflected signal

Fig. 4 Effect of soil weakening on the dynamic response at the pile head

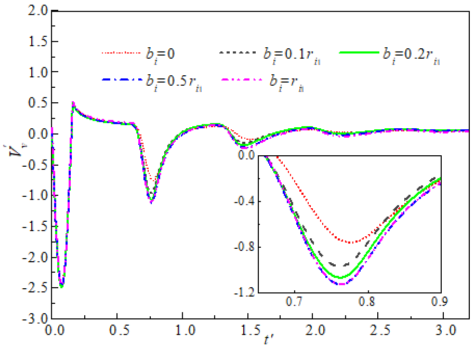


(a) Velocity admittance (b) Reflected signal

Fig. 5 Effect of soil strengthening on the dynamic response at the pile head

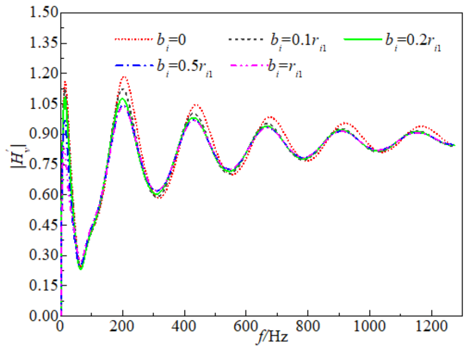


(a) Velocity admittance

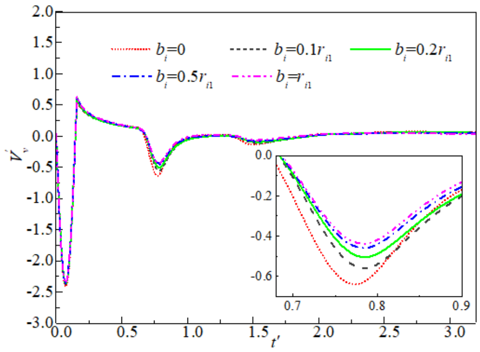


(b) Reflected signal

Fig. 6 Effect of the weakening zone of the surrounding soil on the dynamic response at the pile head

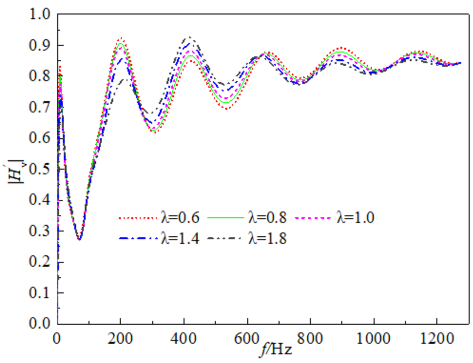


(a) Velocity admittance

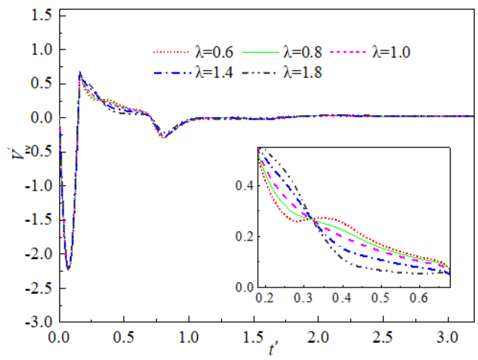


(b) Reflected signal

Fig. 7 Effect of the strengthening zone of the surrounding soil on the dynamic response at the pile head



(a) Velocity admittance



(b) Reflected signal

Fig. 8 Effect of longitudinal interbedded layer on the dynamic response at the pile head

Table 1 Coefficients of degree of disturbance corresponding to weakening cases W1-W4

Case	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
W1	0.60	0.55	0.50	0.45	0.40
W2	0.80	0.75	0.70	0.65	0.60
W3	1.0	0.95	0.90	0.85	0.80
W4	1.0	1.0	1.0	1.0	1.0

Table 2 Coefficients of degree of disturbance corresponding to strengthening cases S1-S4

Case	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
S1	1.0	1.0	1.0	1.0	1.0

Different coefficients of degree of disturbance corresponding to the strengthening cases S1-S4 are given in Table 2. The coefficients of degree of disturbance for a homogeneous surrounding soil are applied in case S1. The effect of soil strengthening within the disturbed zone due to construction disturbance corresponding to cases S1-S4 is depicted in Fig. 5. It can be observed that the dynamic response at the pile head depends significantly on the degree of strengthening of the surrounding soil due to construction disturbance. In contrast, with the degree of strengthening of the disturbed zone increasing, the oscillation amplitudes and the resonance frequencies of the velocity admittance, and the amplitudes of the reflected signal decrease.

4.2.2 Effect of disturbance zone for radial inhomogeneity

Fig. 6 shows the effect of the weakening zone of the construction disturbance on the velocity admittance and the reflected signal of wave velocity at the pile head. The corresponding coefficients of degree of disturbance for an inhomogeneous surrounding soil are applied in case W1. The weakening zone of the surrounding soil b_i(i=1, 2, ... , 5) is taken as the values proportional to the outer radius r_{i1} of the pipe pile, in which b_i=0 means that the surrounding soil is homogeneous in the radial direction. It can be seen that the oscillation amplitudes of the velocity admittance and the reflected signal increase with an increase in the weakening zone of the surrounding soil. In contrast, the change in the resonance frequencies of the velocity admittance can be pactically ignored when b_i is increasing. Furthermore, the more the weakening zone of surrounding soil, the corresponding effect on the dynamic response is less at the pile head.

Fig. 7 shows the effect of the strengthening zone of the surrounding soil on the velocity admittance and the reflected signal of wave velocity at the pile head. The corresponding coefficients of degree of disturbance for an inhomogeneous surrounding soil are applied in case S4. This indicates that the oscillation amplitudes of the velocity admittance and the reflected wave signal increase with a decrease in the strengthening zone of the surrounding soil. In contrast, the effect of the strengthening zone on the resonance frequencies of the velocity admittance can also be practically ignored. In addition, the larger the strengthening zone of the surrounding soil, the corresponding effect on the dynamic response is less at the pile head.

4.2.3 Effect of longitudinal interbedded layer

It is assumed that there exists a longitudinal interbedded layer (e.g. layer 3) in which the shear wave velocity is different from that in the corresponding sub-zone of the other layers, that is, the velocity ratio of shear wave is defined as $\lambda = V_{3j}/V_{ij}$ (i=1, 2, 4, 5; j=1, 2, ... , 20). If λ<1, layer 3 is a soft interbedded layer compared with the other layers; if λ>1, then a hard layer exists. Fig. 8 shows the effect of longitudinal soft or hard interbedded layer on the velocity admittance and the reflected signal of the wave velocity at the pile head. It can be seen that the longitudinal soft or hard interbedded layers have little effect on the resonance frequency of the velocity admittance and the reflected wave at the pile head. In contrast, the oscillation amplitudes of the velocity admittance decrease with the increasing λ. Moreover, it is also shown that the signal phase of the reflected wave from the soft interbedded layer (λ<1) is identical to that reflected from the pile toe. As for the case with a hard interbedded layer (λ>1), the reflected wave signal from the interbedded layer displays an opposite signal phase to that reflected from the pile toe.

5. Conclusions

A new mechanical model for the vertical vibration of a pipe pile embedded in longitudinally layered visco elastic soil with radial inhomogeneity is proposed by extending Novak's plain strain model and complex stiffness method to consider viscous-type damping. The corresponding analytical solutions for the dynamic impedance, the velocity admittance and the reflected signal of the wave velocity at the pile head are also derived and subsequently verified by comparing it with existing solutions.

The results of an extensive parametric analysis are then presented to investigate the effects of shear modulus, viscous damping coefficient, coefficient of degree of disturbance, weakening or strengthening zone of the surrounding soil and longitudinal soft or hard interbedded layers on the velocity admittance and the reflected signal of wave velocity at the pile head. The parametric analysis show that:

• with increasing elastic modulus of the pipe pile, the oscillation amplitudes and the resonance frequencies of velocity admittance increase, but the amplitudes of the reflected wave signal decrease;

• the larger the viscous damping coefficient, the less the oscillation amplitudes and the resonance frequencies of the velocity admittance, and the amplitudes of the reflected wave signal become;

• the oscillation amplitudes and the resonance frequencies of velocity admittance, and the amplitudes of the reflected wave signal decrease with an increase in the coefficient of degree of disturbance (strengthening or weakening);

• the oscillation amplitudes of the velocity admittance and the reflected wave signal increase with an increase in the weakening zone and a decrease in the strengthening zone of the surrounding soil, respectively. Furthermore, the effect of the disturbance (strengthening or weakening) zone on the resonance frequencies of the velocity admittance can be practically ignored.

The proposed model and obtained analytical solutions provide extensive scope of application, compared with the relevant existing solutions. The present solutions can also be reduced to analyze the vertical vibration problem of a solid pile in a visco-elastic soil with radial inhomogeneity and pipe piles embedded in radially homogeneous visco-elastic soil described in previously related studies. In addition, the obtained solution can be conveniently further extended to investigate the vertical vibration problem of a pipe pile embedded in finite soil layers or in poro-visco-elastic half-space, by combining it with different functions of complex stiffness of the soil beneath the pile toe.

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Appendix I Derivation for computing KK_{ij}^{S1}

Performing the Laplace transform to Eq.(7) and substituting into Eq.(13) yields

$$B_{i(n+1)}^{S_1}=0 \qquad \text{(AP1-1)}$$

$$U_{i(n+1)}^{S_1}(r,s)=A_{i(n+1)}^{S_1}K_0(q_{i(n+1)}^{S_1}r) \qquad \text{(AP1-2)}$$

The vertical shear stress of the undisturbed zone within the *i*th layer can be expressed as

$$\begin{aligned}\tau_{i(n+1)}^{S_1}&=(G_{i(n+1)}+\eta_{i(n+1)}s)\frac{\partial U_{i(n+1)}^{S_1}(r,s)}{\partial r}\\&=-(G_{i(n+1)}+\eta_{i(n+1)}s)q_{i(n+1)}^{S_1}A_{i(n+1)}^{S_1}K_1(q_{i(n+1)}^{S_1}r)\end{aligned} \qquad \text{(AP1-3)}$$

Thus, the vertical complex stiffness at the interface between the disturbed zone and the undisturbed zone within the *i*th layer can be conveniently given by

$$\begin{aligned}KK_{i(n+1)}^{S_1}&=-\frac{2\pi r_{i(n+1)}\tau_{i(n+1)}^{S_1}(r_{i(n+1)})}{U_{i(n+1)}^{S_1}(r_{i(n+1)},s)}\\&=2\pi r_{i(n+1)}(G_{i(n+1)}+\eta_{i(n+1)}s)q_{i(n+1)}^{S_1}\frac{K_1(q_{i(n+1)}^{S_1}r_{i(n+1)})}{K_0(q_{i(n+1)}^{S_1}r_{i(n+1)})}\end{aligned} \qquad \text{(AP1-4)}$$

The shear stress of the *j*th annular sub-zone within the *i*th layer is expressed as

$$\begin{aligned}\tau_{ij}^{S_1}&=(G_{ij}+\eta_{ij}s)\frac{\partial U_{ij}^{S_1}(r,s)}{\partial r}\\&=-(G_{ij}+\eta_{ij}s)q_{ij}^{S_1}[A_{ij}^{S_1}K_1(q_{ij}^{S_1}r)-B_{1ij}I_1(q_{ij}^{S_1}r)]\end{aligned} \qquad \text{(AP1-5)}$$

Therefore, the vertical complex stiffness at the outer boundary ($r=r_{i(j+1)}$) of the *j*th sub-zone within the *i*th layer are written as the following form

$$\begin{aligned}KK_{i(j+1)}^{S_1}&=-\frac{2\pi r_{i(j+1)}\tau_{ij}^{S_1}(r_{i(j+1)})}{U_{ij}^{S_1}(r_{i(j+1)},s)}\\&=2\pi r_{i(j+1)}q_{ij}^{S_1}(G_{ij}+\eta_{ij}s)\frac{A_{ij}^{S_1}K_1(q_{ij}^{S_1}r_{i(j+1)})-B_{ij}^{S_1}I_1(q_{ij}^{S_1}r_{i(j+1)})}{A_{ij}^{S_1}K_0(q_{ij}^{S_1}r_{i(j+1)})+B_{ij}^{S_1}I_0(q_{ij}^{S_1}r_{i(j+1)})}\end{aligned} \qquad \text{(AP1-6)}$$

Following a similar procedure as above, the vertical complex stiffness at the inner boundary $r=r_{ij}$ of the *j*th sub-zone within the *i*th layer is obtained by

$$\begin{aligned}KK_{ij}^{S_1}&=-\frac{2\pi r_{ij}\tau_{ij}^{S_1}(r_{ij})}{U_{ij}^{S_1}(r_{ij},s)}\\&=2\pi r_{ij}q_{ij}^{S_1}(G_{ij}+\eta_{ij}s)\frac{A_{ij}^{S_1}K_1(q_{ij}^{S_1}r_{ij})-B_{ij}^{S_1}I_1(q_{ij}^{S_1}r_{ij})}{A_{ij}^{S_1}K_0(q_{ij}^{S_1}r_{ij})+B_{ij}^{S_1}I_0(q_{ij}^{S_1}r_{ij})}\end{aligned} \qquad \text{(AP1-7)}$$

Appendix II Derivation for computing Z_i^p

The dynamic impedance function of the vertical displacement at the pile toe is written as the following expression

$$\begin{aligned}Z_0^P\Big|_{z=h_0}&=\frac{-E_1^PA_1^P\frac{\partial}{\partial z}U_1^P(z,s)\Big|_{z=h_0}}{U_1^P(z,s)\Big|_{z=h_0}}\\&=\frac{-E_1^PA_1^P\bar{\alpha}_1\left(C_1^Pe^{\bar{\alpha}_1h_0/l_1}-D_1^Pe^{-\bar{\alpha}_1h_0/l_1}\right)}{l_1\left(C_1^Pe^{\bar{\alpha}_1h_0/l_1}+D_1^Pe^{-\bar{\alpha}_1h_0/l_1}\right)}=A_1^P\left(k_{\mathrm{p}}+\delta_{\mathrm{p}}s\right)\end{aligned} \qquad \text{(AP2-1)}$$

where $h_0=H$.

Thus, the dynamic impedance of the vertical displacement at the pile head of the first pile segment is expressed as

$$\begin{aligned}Z_1^P\Big|_{z=h_1}&=\frac{-E_1^PA_1^P\frac{\partial}{\partial z}U_1^P(z,s)\Big|_{z=h_1}}{U_1^P(z,s)\Big|_{z=h_1}}\\&=\frac{-E_1^PA_1^P\bar{\alpha}_1\left(C_1^Pe^{\bar{\alpha}_1h_1/l_1}-D_1^Pe^{-\bar{\alpha}_1h_1/l_1}\right)}{l_1\left(C_1^Pe^{\bar{\alpha}_1h_1/l_1}+D_1^Pe^{-\bar{\alpha}_1h_1/l_1}\right)}\end{aligned} \qquad \text{(AP2-2)}$$

Combining Eq. (AP2-1) and (AP2-2) yields

$$Z_1^P=\frac{-E_1^PA_1^P\bar{\alpha}_1\left(\beta_1e^{\bar{\alpha}_1h_1/l_1}-e^{-\bar{\alpha}_1h_1/l_1}\right)}{l_1\left(\beta_1e^{\bar{\alpha}_1h_1/l_1}+e^{-\bar{\alpha}_1h_1/l_1}\right)} \qquad \text{(AP2-3)}$$

where $\beta_1=\frac{E_1^PA_1^P\bar{\alpha}_1-Z_0^Pl_1}{E_1^PA_1^P\bar{\alpha}_1+Z_0^Pl_1}e^{-2\bar{\alpha}_1h_0/l_1}$.

Similarly, the dynamic impedance function of the vertical displacement at the pile head of the *i*th pile segment is written as

$$Z_i^P=\frac{-E_i^PA_i^P\bar{\alpha}_i\left(\beta_ie^{\bar{\alpha}_ih_i/l_i}-e^{-\bar{\alpha}_ih_i/l_i}\right)}{l_i\left(\beta_ie^{\bar{\alpha}_ih_i/l_i}+e^{-\bar{\alpha}_ih_i/l_i}\right)} \qquad \text{(AP2-4)}$$

where $\beta_i=\frac{E_i^PA_i^P\bar{\alpha}_i-Z_{(i-1)}^Pl_i}{E_i^PA_i^P\bar{\alpha}_i+Z_{(i-1)}^Pl_i}e^{-2\bar{\alpha}_ih_{i-1}/l_i}$.