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Effect of boundary conditions for functionally graded beams under uniformly distributed load

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Abstract. This article explores the impact of support conditions on a uniformly loaded FGM beam, focusing on hyperstaticity and material heterogeneity. It presents a mathematical formulation based on elasticity theory, considering edge conditions to predict stress and displacement distribution. Numerical tests validate the influence of these conditions on the FGM beam's supports, resulting in satisfactory resistance and stability. The study also presents numerical tests to validate the results, demonstrating the importance of considering the different parameters of variations in support conditions at the edge.

Keywords: boundary conditions; elastic properties; elasticity solution; functionally graded beams; static analysis

1. Introduction

Functionally graded materials (FGMs) are increasingly utilized in various fields like electronics, chemistry, optics, and biomedicine due to their superior mechanical properties compared to conventional fiber-matrix materials. However, new methodologies are needed to characterize and design structural components made from FGMs due to their continuously varying properties (Aman *et al*. 2024, Brahim *et al*. 2024, Nebab *et al*. 2024, Sankar 2001, Eiadtrong *et al.* 2024, Yu *et al*. 2007, Zahedinejad *et al*. 2020, Abdeljalil Meksi *et al*. 2021, Ismail *et al*. 2021a, Ismail *et al*. 2021b, Yahiaoui *et al*. 2018, Turan *et al*. 2023b, Turan *et al*. 2024a, Turan 2024b, Adiyaman *et al*. 2025, Turan *et al*. 2025, Ahmed *et al*. 1996, Gorik *et al*. 2020, Stanislav *et al*. 2024, Sun *et al*. 2023, Yang *et al*. 2014, Zheng *et al*. 2007). Limited publications exist on the FGM beam response to mechanical and other loading types. Benguediab *et al.* (2017) and colleagues studied FGM beam responses, while Youcef *et al*. (2024) and colleagues developed analytical methods for thermo-mechanical and contact analysis of FGM beams and sandwich beams. Sid Ahmed *et al.* (2024) used a combined Fourier series-Galerkin method to solve two-dimensional elasticity equations for an FGM beam subjected to transverse loading. They assumed that the thermo-mechanical properties of the FGM would vary exponentially through the

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thickness, with Young's modulus and Poisson's ratio assumed constant. Saidi *et al*. (2024) developed a new beam element using first-order shear deformation theory to study the thermo-elastic behavior of FGM beam structures. The method achieves exact solutions through continuous iteration and high-precision approximate solutions after a few iterations. However, it lacks explicit exact formulas for general field function solutions. Further studies are needed to provide explicit formulas. The method uses exponential and power-law variations of material property distributions, presents two-dimensional analytical solutions for plates and beams (Bourada *et al.* 2018), develops higher-order shear deformation theories, and refines trigonometric higher-order beam theory for bending and vibrations of FGM beams, including the thickness stretching effect. A critical review of recent research on FGM plates was also presented (Changwei *et al*. 2025, Chu *et al.* 2015, Volkan Kahya *et al.* 2017, Volkan Kahya *et al.* 2018, Muhittin Turan *et al*. 2021, Turan 2022, Turan *et al.* 2023a, Emrah *et al*. 2024, Hela *et al*. 2024, Mahmoud *et al*. 2023, Huang *et al*. 2007, Ma 2020, Reddy *et al*. 2021, Songsuwan *et al*. 2022, Sun *et al*. 2023, Wang *et al.* 2010, Xu *et al.* 2014, Laoud *et al*. 2024, Abdeljalil Meksi *et al*. 2024, Ouahiba *et al.* 2024, Mohamed Sekkal *et al.* 2024, Abdeljalil Meksi *et al*. 2023, Sekkal *et al*. 2023, Yang *et al.* 2013, Mohammadimehr 2024, Meiqin *et al*. 2013, Meiqin *et al.* 2010). The stress function approach is used to study the problem of an exponential FGM cantilever beam under boundary conditions for functionally graded beams under uniformly distributed load, resulting in an analytical solution easily degenerated to a homogeneous beam.

2. Theoretical formulations

2.1 Properties of the FGM constituent materials

This paper discusses Functionally Graded Material (FGM), a material produced by continuously varying multi-phase materials' constituents in a predetermined profile. It highlights its unique features, including non-uniform microstructures and continuously graded macro properties, and focuses on FGM beams with sigmoid function. The study investigates an elastic cantilever FGM beam made of ceramic and metal, analyzing its properties like Young's modulus and Poisson's ratio on its upper and lower surfaces. The beams' properties vary continuously in the thickness direction, with Poisson's ratio assumed constant. The Young's modules in the thickness direction vary with exponential functions (E-FGM).

$$S_{ij}(y) = S_{ij}^{0}.e^{\lambda(\frac{y}{h}+\frac{1}{2})} \tag{1}$$

with*:* i,j=1, 3 and 4.

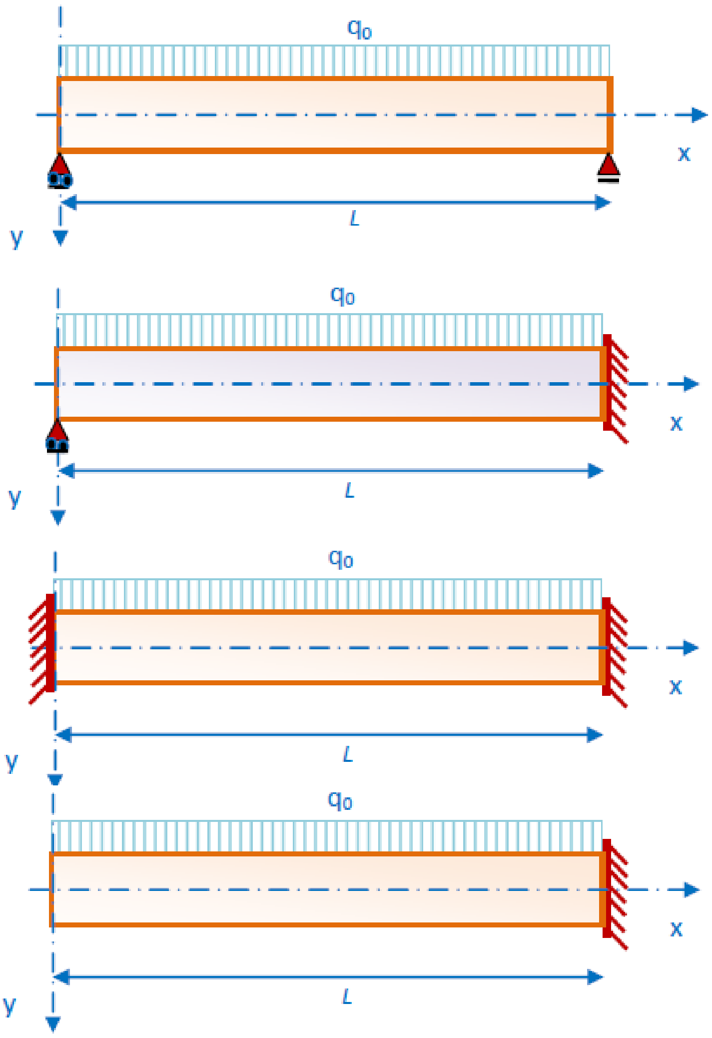
λ: The term "coefficient of degree of homogeneity" refers to the degree of material homogeneity.

2.2 Problem description and basic equations

A uniform thickness FGM beam is described using a cartesian coordinate system, with upper and lower surfaces in the plane y=h/2 and -y=h/2. The beam's edges are L and b, and it is assumed to be in a plane stress state normal to the x-y plane. Material properties vary in thickness direction.

In the absence of body forces the equilibrium equations are given as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \tag{2a}$$



(a)
FGM beam S-S

(b)
FGM beam S-C

(c)
FGM beam C-C

(d)
FGM beam F-C

Fig. 1 Geometries of FGM beam under uniformly distributed load with different boundary conditions

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{2b}$$

where σ_x, σ_y, τ_{xy} are stress components.

The relationships between strains and displacements are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \tag{3a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}, \tag{3b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3c}$$

Table 1 Value of the expressions for bending moments, shear force and displacement for a FGM beam as a function of boundary conditions

Boundary Conditions	N_0	M_0	T_0	u	v		$\frac{\partial v}{\partial x}$	
S-S	0	0	$-\frac{q_0 l}{2}$	0	for $x=0$	0	for $x=L/2$	0
S-C	0	0	$-\frac{3q_0 l}{8}$	0	for $x=0$	0	for $x=L$	0
C-C	0	$\frac{ql^2}{12}$	$-\frac{q_0 l}{2}$	0	for $x=0$	0	for $x=L/2$	0
F-C	0	0	0	0	for $x=L$	0	for $x=L$	0

where u, v are displacement components, ε_x, ε_y, γ_{xy} are strain components that should satisfy the following strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \tag{4}$$

The constitutive relations of orthotropic FGM are

$$\varepsilon_x = S_{11}\sigma_x + S_{12}\sigma_y, \tag{5a}$$

$$\varepsilon_y = S_{12}\sigma_x + S_{22}\sigma_y, \tag{5b}$$

$$\gamma_{xy} = S_{66}\tau_{xy} \tag{5c}$$

where S_{11}, S_{22}, S_{12}, S_{66} are the elastic compliance parameters given by

$$S_{11} = S_{22} = \frac{1}{E(y)}, \tag{6a}$$

$$S_{12} = -\frac{\nu}{E(y)}, \tag{6b}$$

$$S_{66} = \frac{2(1+\nu)}{E(y)} \tag{6c}$$

The boundary conditions of elasticity at the upper and lower surfaces are

$$\sigma_y(x, +h/2) = 0, \tag{7a}$$

$$\sigma_y(x, -h/2) = -q_0, \tag{7b}$$

$$\tau_{xy}(x, \pm h/2) = 0. \tag{7c}$$

The boundary conditions at ($x=0$; $L/2$ and L) of the FGM beam are, the boundary conditions for the simple supported beam are taken as:
where N_0, M_0 and T_0 denote the axial force, moment and shear force at $x=0$ and u, v are the displacements along x and y and is rotation around y.

2.3 Mathematical formulation of the general solution

In order to obtain a general solution to Eqs. (8)-(10), Airy stress function is introduced such that

$$\sigma_X = \frac{\partial^2 g}{\partial y^2}, \tag{8a}$$

$$\sigma_y = \frac{\partial^2 g}{\partial x^2}, \tag{8b}$$

$$\tau_{xy} = -\frac{\partial^2 g}{\partial x \, \partial y} \tag{8c}$$

So that Eq. (8) is satisfied automatically. We then assume that

$$g(x,y) = g_0(y) + x g_1(y) + x^2 g_2(y) \tag{9}$$

where $g_0(y)$, $g_1(y)$ and $g_2(y)$ are unknown functions to be determined. Substitution of Eq. (6) into Eq. (8) gives

$$\sigma_x = g_0^{''} + x g_1^{''} + x^2 g_2^{''}, \tag{10a}$$

$$\sigma_y = 2g_2, \tag{10b}$$

$$\tau_{xy} = -\left(g_1^{'} + 2x g_2^{'}\right) \tag{10c}$$

Substitution of Eq. (10) into Eq. (8), and then into Eq. (9), gives rise to

$$\left(S_{11} g_2^{''}\right)^{''} = 0 \tag{11a}$$

$$\left(S_{11} g_1^{''}\right)^{''} = 0 \tag{11b}$$

$$\left(S_{11} g_0^{''} + 2S_{12} g_2\right)^{''} + 2S_{12} g_2^{''} = 0. \tag{11c}$$

Integration of Eq. (11a) yields

$$\delta_2^{''} = \mu_1 \Delta_1 + \mu_2 \Delta_2 \tag{12a}$$

$$\delta_2^{'} = \mu_1 \Delta_1^0 + \mu_2 \Delta_2^0 + \mu_3 \tag{12b}$$

$$\delta_2 = \mu_1 \Delta_1^1 + \mu_2 \Delta_2^1 + \mu_3 z + \mu_9 \tag{12c}$$

where and hereafter μ_i $(i = 1,2,\ldots)$ are integral constants, and

$$\Delta_1 = \frac{y}{S_{11}}, \tag{13a}$$

$$\Delta_2 = \frac{1}{S_{11}}, \tag{13b}$$

$$\Delta_i^m(z) = \frac{1}{m!}\int_{-h/2}^{y} (y-\delta)^m \Delta_i \, d\delta \quad (i = 1,2; m = 0,1) \tag{13c}$$

Integration of Eq. (11b) yields

$$\delta_1^{''} = \mu_4 \beta_4 + \mu_5 \beta_5 \tag{14a}$$

$$\delta_1^{'} = \mu_4\beta_4^0 + \mu_5\beta_5^0 + \mu_6 \tag{14b}$$

Where

$$\beta_1 = 0;\ \beta_2 = 0;\ \beta_3 = 0 \tag{15a}$$

$$\beta_4 = \Delta_1;\ \ \beta_5 = \Delta_2,\ \ , \tag{15b}$$

$$\beta_i^m(y) = \frac{1}{m!}\int_{-h/2}^{y}(y-\delta)^m\beta_i d\delta\ \ (i = 1,2,3,4,5)\ \text{ and } m = 0,1 \tag{15c}$$

Substituting Eq. (12) into Eq. (11c), and performing integration twice, we obtain

$$\delta_0^{''} = \sum_{i=1}^{9}\mu_i\gamma_i, \tag{16}$$

Where

$$\gamma_1 = \frac{1}{S_{11}}\left(-2\int_{-h/2}^{y}(y-\delta)S_{12}\Delta_1\,d\delta - 2\int_{-h/2}^{y}S_{66}\Delta_1^0\,d\delta - 2S_{12}\Delta_1^1\right) \tag{17a}$$

$$\gamma_2 = \frac{1}{S_{11}}\left(2\int_{-h/2}^{y}(y-\delta)S_{12}\Delta_2\,d\delta - 2\int_{-h/2}^{y}S_{66}\Delta_2^0\,d\delta - 2S_{12}\Delta_2^1\right) \tag{17b}$$

$$\lambda_3 = -\frac{1}{S_{11}}\left(-2S_{12}y - 2\int_{-h/2}^{y}S_{66}d\delta\right) \tag{17c}$$

$$\gamma_4 = 0;\ \ \gamma_5 = 0;\ \ \gamma_6 = 0; \tag{17d}$$

$$\gamma_7 = \Delta_1 \tag{17e}$$

$$\gamma_8 = \Delta_2, \tag{17f}$$

$$\gamma_9 = -2\frac{S_{12}}{S_{11}} \tag{17g}$$

And: $$\gamma_i^m(y) = \frac{1}{m!}\int_{-h/2}^{y}(y-\delta)^m\gamma_i d\delta\ \ (i = 1,2,\dots,10)\ \text{and } m = 0,1 \tag{17h}$$

The axial force N_0, bending moment M_0 and shearing force T_0 at the left end of the beam can be obtained by integrating Eqs. (10a) and (10c) according to:

For each condition:

$$N_0 = \int_{-h/2}^{h/2}\sigma_x(0,y)dy = \sum_{i=1}^{9}\mu_i\gamma_i^0(h/2), \tag{18a}$$

$$M_0 = \int_{-h/2}^{h/2}\sigma_x(0,y)ydy = \frac{h}{2}\sum_{i=1}^{9}\mu_i\gamma_i^0(h/2) - \sum_{i=1}^{9}\mu_i\gamma_i^1(h/2), \tag{18b}$$

$$T_0 = \int_{-h/2}^{h/2}\tau_{xy}(0,y)dy = -\sum_{i=1}^{9}\mu_i\beta_i(h/2) - \mu_6 h \tag{18c}$$

Substituting Eqs. (12) and (14) into Eqs. (19) respectively, we have

$$\mu_3 = \mu_6 = 0 \tag{19a}$$

$$\mu_9 = -\frac{q}{2} \tag{19b}$$

$$\sum_{i=1}^{2} \mu_i \Delta_i^0(h/2) = 0, \tag{19c}$$

$$\sum_{i=1}^{5} \mu_i \beta_i^0(h/2) = 0 \tag{19d}$$

$$\sum_{i=1}^{9} \mu_i \gamma_i^0(h/2) = 0 \tag{19e}$$

$$\sum_{i=1}^{2} \mu_i \Delta_i^1(h/2) = \frac{q}{2}, \tag{19f}$$

$$\sum_{i=1}^{5} \mu_i \beta_i^1(h/2) = 0 \tag{19g}$$

$$\sum_{i=1}^{9} \mu_i \gamma_i^1(h/2) = 0 \tag{19h}$$

Finally the integration of Eq. (8), we could obtain the displacement components as follows

$$u = x\left(S_{11}\delta_0'' + 2S_{12}\delta_2\right) + \frac{x^2}{2}(S_{11}\delta_1'') + \frac{x^3}{3}S_{11}\delta_2'' + \int_{-h/2}^{y}(-S_{66}\delta_1')d\delta - \int_{-h/2}^{y}(y - \delta)(S_{12}\delta_1'')d\delta - \rho\, \mathrm{y} + u_0, \tag{20a}$$

$$v = \int_{-h/2}^{y}\left(s_{12}\delta_0'' + 2s_{22}\delta_2\right)d\delta - \frac{\mu_1}{12}x^4 - \frac{\mu_4}{6}x^3 + \left(\int_{-h/2}^{y} S_{12}\delta_2'' d\delta - \frac{\mu_7}{2}\right)x^2 + \left(\int_{-h/2}^{y}(S_{12}\delta_1'')d\delta\right)x + \rho\, x + v_0 \tag{20b}$$

Where: u_0, v_0 and ρ are integral constants.
Effect of boundary conditions for functionally graded beams under uniformly distributed load.

3. Numerical results and discussion

In this section, numerical study of a FGM beam (*L*=1 m, height *h*=0.1 m) subjected to a uniform distribution load and various boundary conditions, whose amplitude $q_0=10^8$ GPa/ml. The analysis is performed for pure materials and different values of material parameter λ, for E-FGM materials in terms of flexibility. We assume the graded function as in E-FGM beam Eq (1). In present study, we take the graded index for different values of λ an s_{ij}^0 (*i*, *j*=1, 2, 6) in E-FGM. The material properties at *y=-h/2* are given as:

$$s_{11}^0 = 2.63 \times 10^{-12}/\mathrm{Pa},$$
$$s_{12}^0 = -0.71 \times 10^{-12}/\mathrm{Pa},$$
$$s_{22}^0 = 2.63 \times 10^{-12}/\mathrm{Pa},$$
$$s_{66}^0 = 6.684 \times 10^{-12}/\mathrm{Pa}.$$

Legend: S-S; C-C, S-C, F-C defined in Fig. 1

Table 2 reveals perfect agreement between the current solution and the simplified classical

Table 2 Validation by comparison of the deflections and stresses of the present model with the simplified Classical theory solution for the case of a beam made of uniformly loaded homogeneous material

Displacements $\bar{v} = \frac{v}{h}$	S-S	S-C	C-C	F-C
Present	0,42	0,173	0,0914	3,908
Classical theory	0,4109	0,164	0,0821	3,945
Relative error	2,16%	5,2%	10,1%	0,9%
Normal stress $\bar{\sigma}_x = \frac{\sigma_x}{q_0}$	S-S	S-C	C-C	F-C
Present	75,20	37,70	25,20	-74,999
Classical theory	75,00	37,50	25,00	-75,00
Relative error	0,26%	0,53%	0,8%	0,001%
Shear stress $\bar{\tau}_{xz} = \frac{\tau_{xz}}{q_0}$	S-S	S-C	C-C	F-C
Present	-7,499	-9,374	-7,499	-14,999
Classical theory	-7,70	-9,625	-7,70	15,40
Relative error	2,61%	2,60%	2,61%	2,60%

Table 3 Dimensionless comparison of stresses and displacements as a function of support conditions and the degree of graduation of a uniformly loaded beam

Boundary conditions	Degree of graduation λ	$\bar{\sigma}_x(\frac{L}{2},\frac{h}{2})$	$\bar{\tau}_{xy}(L,0)$	$\bar{v}(\frac{L}{2},0)$
S-S	0	75,20	-7,499	0,420
	-1	104,552	-7,406	0,256
	1	53,455	-7,406	0,698
S-C	0	37,70	-9,374	0,173
	-1	52,449	-9,257	0,105
	1	26,770	-9,257	0,288
C-C	0	25,20	-7,499	0,0729
	-1	35,081	-7,406	0,045
	1	17,875	-7,406	0,121
F-C	0	-74,799	-14,999	1,387
	-1	-103,859	-14,812	0,849
	1	-53,287	-14,812	2,306

solution in stability and stress, but some variation needs to be considered in this case.

- The stability of isostatic beams, S-S and F-C, is satisfactory, but not for hyperstatic beams S-C and C-C, where the error varies from 5 to 10%. This is due to the formulation of the two approaches, where the present model is based on exact solutions, while the classical solution is based on simplified equations or underestimates displacement values.
- The comparison of stresses is perfect, with results very close for the four conditions at the supports.

Table 3 compares stresses and displacements of a model based on support conditions and beam graduation. Results confirm the hypothesis that higher hyperstaticity leads to more rigid, stable, and non-deformable structures. In resistance, rigid materials give higher stress, making them more deformable. From material graduation, more flexible materials are found. The higher the degree of

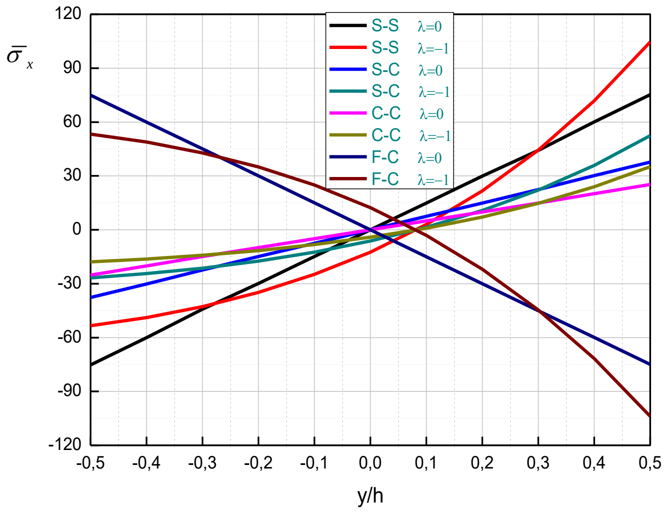


Fig. 2 Value of dimensionless normal stresses through the thickness and at mid-span of the FGM beam for different support conditions and with λ=0 and λ=-1

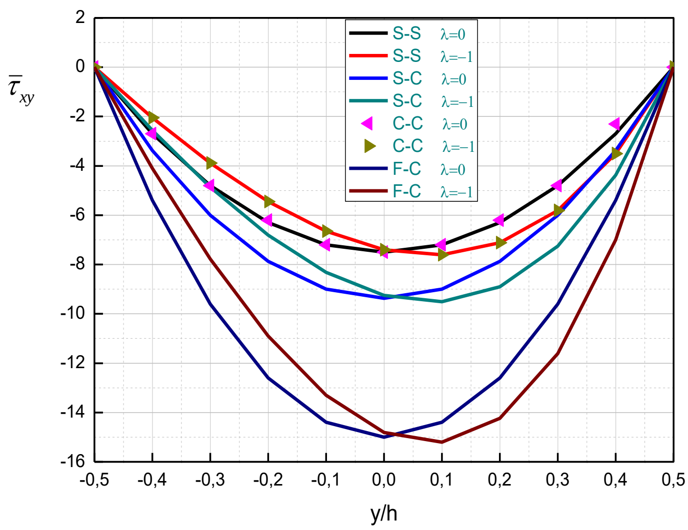


Fig. 3 Value of dimensionless shear stresses through the thickness and at the end (x=L) of the FGM beam for different support conditions and with λ=0 and λ=-1

hyperstaticity, the more rigid and stable the structure.

Fig. 2 shows the distribution of dimensionless normal stresses through the thickness of the E-FGM beam under different support conditions and material degree of graduation λ=0 and λ=-1. The homogeneous material maintains a linear appearance, aligning with structural analysis. However, the F-C combination's support conditions differ, resulting in inverted stresses. The non-homogeneous material exhibits a non-linear appearance, with maximum values for the isostatic beam (S-S) and minimum values at the level of the purely hyperstatic beam (C-C). This contrasts with the isostatic beam's linear appearance.

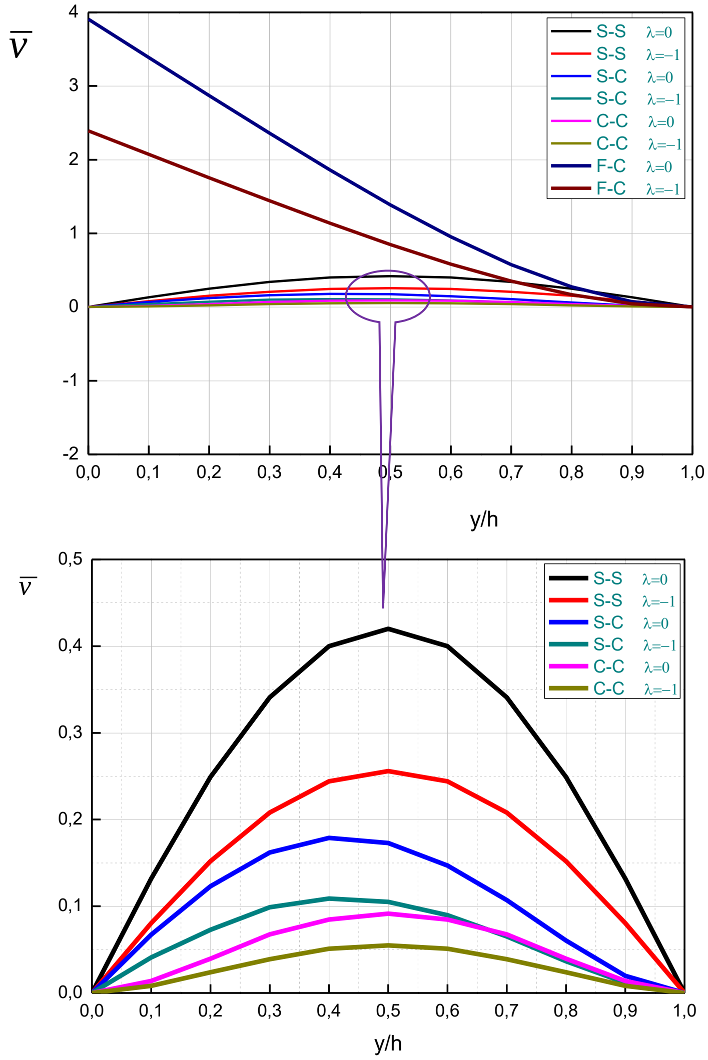


Fig. 4 Variation of displacements along of the FGM beam for different support conditions and with λ=0 and λ=-1

Fig. 3 shows the variation of dimensionless shear stresses through the beam's thickness and end for different support conditions and materials. The homogeneous material has symmetrical parabolic shapes with a maximum value at the middle fiber level. The same shear stress values were obtained for the two symmetric conditions S-S and C-C, due to the same shear force value. For the FGM material, the shapes remain parabolic but not symmetrical, with their maximum value tending towards the less flexible face. The study presents the variation of displacements along and at the average fiber of a beam under different support conditions Fig. 4 and two types of material: homogeneous λ=0 and FGM λ=-1. The displacement value is largely higher under the F-C condition (Fig. 4(a)). The maximum value is reached for homogeneous (Fig. 4(b)) and flexible material (condition S-S), while the minimum value is reached for the least flexible FGM material

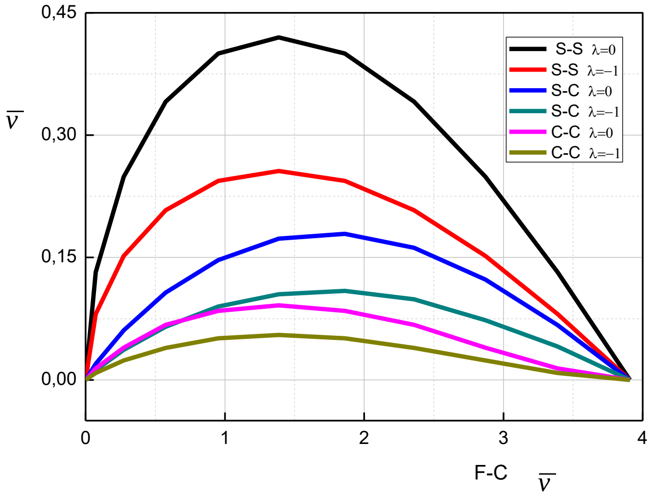


Fig. 5 Relationship of displacements between support conditions with non-free end as a function of the free end condition of the FGM beam and with λ=0 and λ=-1

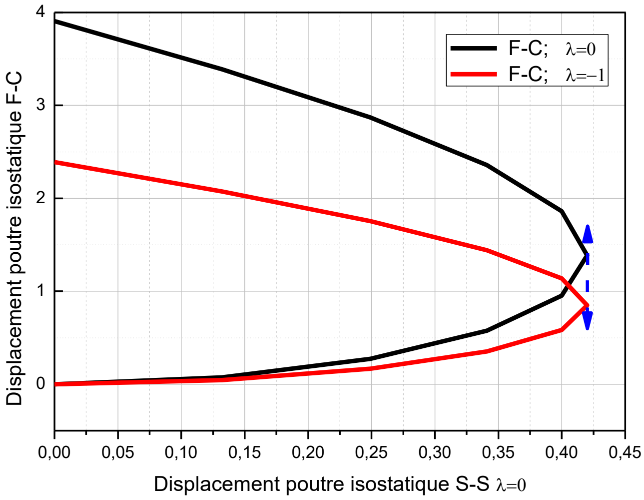


Fig. 6 Relationship of displacements between isostatic beams under condition F-C as a function of S-S

(condition C-C). The study concludes that the effect of support conditions and material graduation influences the deformation of beams, as demonstrated by the results obtained. The results are validated after this analysis.

The test in Fig. 5 demonstrates the relationship between displacements at non-free end supports and beam conditions at the free end (FGM) and two materials (homogeneous λ=0 and FGM λ=-1). The relationship forms a higher order shape with an amplitude closer to the free end, especially for the most flexible beam (S-S). The amplitude decreases with beam flexibility, moving from S-S to least flexible conditions (C-C), and the nullity of displacements at the support level is also examined. In Fig. 6, the study examines the relationship between homogeneous F-C and FGM isostatic beams and their displacements based on the homogeneous s-s beam. The results show parabolic shapes with different peaks concentrated at the non-free end of the F-C condition. The abscissa is identical, but the inordinate values vary, with maximum values in homogeneous F-C and minimum values in F-C beam FGM.

4. Conclusions

A plane elasticity formulation is developed for a functionally graded beam under various edge conditions has been developed using elasticity equations. The material is characterized by its own flexibility parameters in the form of an exponential distribution, justified by deformation compatibility equations. The formulation includes support conditions and material degree of graduation. The numerical calculation validates the calculation model, and conclusions can be drawn from this application.

- The present solution and the simplified classical solution in stability and stress have a perfect agreement.
- The study discusses the stability of isostatic beams, specifically the S-S and F-C beams, and the hyperstatic S-C and C-C beams, where the error ranges from 5 to 10%. The difference can be attributed to the formulation of the two approaches, where the present model relies on exact solutions, while the classical solution uses simplified equations or underestimates displacement values.
- The comparison of stresses is in perfect agreement, with results being very close for the four support conditions.
- The distribution of dimensionless normal stresses through the thickness of the E-FGM beam varies with different support conditions and material degree of graduation. Homogeneous materials maintain a linear appearance, compatible with structural analysis. Heterogeneous materials show non-linear appearances, with the maximum value remaining for the isostatic beam and the minimum value recorded at the level of the purely hyperstatic beam.
- The study examines the variation of dimensionless shear stresses through the beam FGM for different support conditions and materials. The results show that the shape of the homogeneous material is symmetrically parabolic, with a maximum value at the middle fiber level. However, the FGM material’s shapes remain parabolic but not symmetric, with their maximum value tending towards the less flexible face. This is due to the identical shear force values for both materials.
- The relationship between support conditions and non-free end displacements is influenced by beam FGM free end condition and materials like homogeneous and FGM. The relationship forms a higher order shape with an amplitude closer to the free end, especially for the most flexible beam. The amplitude decreases with beam flexibility, ranging from flexible to rigid conditions with zero displacement at the supports level.

The obtained solution is valid for arbitrary graded variations of the material distribution, so it could serve as a basis for establishing simplified FGM theories or as a benchmark result to assess other approximate methodologies.

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