Coupled Systems Mechanics, *Vol. 14, No. 2 (2025) 161-173*
https://doi.org/10.12989/csm.2025.14.2.161

Application of a new improved Airy polynomial function for FGM cantilever beams under concentrated load

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(Received December 27, 2024, Revised May 19, 2025, Accepted May 21, 2025)

Abstract. This paper addresses the plane stress problem of a cantilever functionally graded beam subjected to concentrated load. The material properties of the beam vary continuously through thickness, according to a power-law distribution of volume fraction. The existing solution focuses on resolving differential equations, but a new Airy function proposal is proposed. The paper examines the effect of material graduation on the degree of polynomial degree. A simple formulation developed to predict stress and displacement distributions relative to structural problems. The proposed model validated by comparison with existing literature, ensuring stability and resistance of the materials. A numerical example presented to show the effect of material inhomogeneity on the elastic field in a functionally graded cantilever beam.

Keywords: airy polynomial function; anisotropy; cantilever beams; functionally graded materials; static analysis

1. Introduction

Functionally graded materials (FGMs) are increasingly used in high-tech sectors like aerospace and nuclear reactors, and energy dynamics due to their unique advantages. The study of functionally graded beam theories, considering shear deformation, has rapidly progressed, with numerous scholars conducting research on various beams under different loads, using elasticity as an exact theory. Cantilever beams are common in frame structures, and their mechanical properties have been extensively studied by scholars. Methods like classical beam theory, Timoshenko's beam theory, and high order beam theory have been developed to study static displacement and stress distribution. Timoshenko's elasticity approach provides rigorous solutions, specifically elasticity solutions, in addition to approximate methods (Aman *et al.* 2024, Changwei *et al.* 2025, Chu *et al.* 2015, Emrah *et al.* 2024, Hela *et al.* 2024, Mahmoud *et al.* 2023, Mohammad *et al.* 2020). The Airy stress function method has been used to obtain elasticity solutions for various types of beams, including cantilever beams. Saidi *et al.* 2024 obtained elasticity solutions for cantilever beams. Hassaine Daouadji *et al.* (2013) obtained elasticity solutions for cantilever

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http://www.techno-press.org/?journal=csm&subpage=8 ISSN: 2234-2184 (Print), 2234-2192 (Online)

beams subjected to an arbitrarily directed force. Laoud *et al.* (2024) developed elasticity solutions for cantilever beams and rings with varying thickness, while Sid Ahmed *et al.* (2024) provided general elasticity solutions for orthotropic cantilever beams bent by couples at two ends and orthotropic cantilever beams bent by a force at the free end. Bourada *et al.* (2018) applied the elasticity solution to a full ring and a cantilever beam with a straight portion. Tlidji *et al.* (2024) used the elasticity solution to analyze debonding and buckling/wrinkling of cantilever sandwich beams (Mohammadimehr 2024, Meiqin *et al.* 2013, Nebab *et al.* 2024, Nie *et al.* 2013, Qing *et al.* 2013, Sun *et al.* 2023, Yang *et al.* 2014, Zheng *et al.* 2007). The development of functionally graded materials (FGMs) led to extensive research on FGM beams. Benguediab *et al.* (2017) developed the stress distribution of an inhomogeneous cantilever beam under pure bending, but the investigation on the FGM cantilever beam is insufficient. Further research is needed to understand the behavior of FGM beams. The study explores a new improved Airy polynomial function for FGM cantilever beams under concentrated load at the free end. The stress function method is used to derive stress and displacement expressions for a cantilever functionally graded beam, including two unknown functions for two general elastic compliance parameters. The paper examines the plane stress problem of a cantilever functionally graded beam subjected to concentrated load, focusing on the effect of material graduation on the degree of polynomial degree. A simple formulation is developed to predict stress and displacement distributions relative to structural problems. The proposed model is validated by comparison with existing literature, ensuring material stability and resistance.

2. Theoretical formulations

2.1 Properties of the material properties of P-FGM beams

The volume fraction of the P-FGM is assumed to obey a power law function

$$g(y) = (\frac{y}{h} + \frac{1}{2})^k \tag{1a}$$

where p is the material parameter that dictates the material variation profile through the thickness and h is the thickness of the beam. Once the local volume fraction $g(z)$ has been defined, the material properties of a P-FGM can be determined by the rule of mixture

$$E(y) = g(y)E_c + (1 - g(y))E_m \tag{1b}$$

where E_c and E_m are the Young modules of the lowest ($z=h/2$) and top surfaces ($z=-h/2$) of the FGM beam, respectively. The variation of Young's modulus in the thickness direction of the P-FGM beam is depicted in Fig. 1.

2.2 Problem description and basic equations

A cantilever FGM beam of uniform thickness h is described using a Cartesian coordinate system (Fig. 1), with the upper and lower surfaces in the plane $z=h/2$ and $-z=h/2$. The beam's edges are denoted by L and b, and it is assumed to be in a plane stress state normal to the x-z plane and subjected to a distributed concentrated load "P" on its upper surface. The material properties vary in the thickness direction.

In the absence of body forces the equilibrium equations are given as

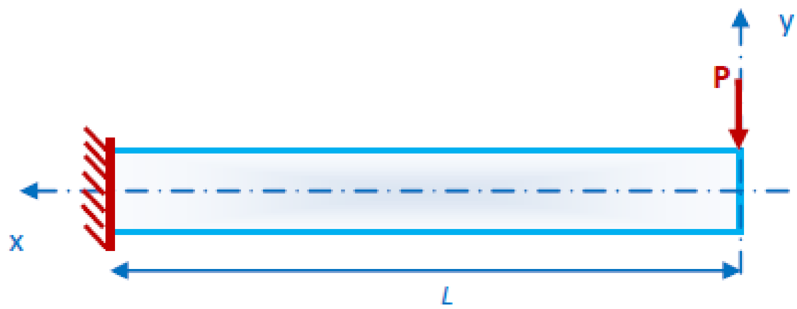

Fig. 1 Geometries and loading of the cantilever FGM beam

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \tag{2a}$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{2b}$$

where σ_x, σ_y, τ_{xy} are stress components.

The relationships between strains and displacements are

$$\varepsilon_x=\frac{\partial u}{\partial x}, \varepsilon_y=\frac{\partial v}{\partial y}, \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{3}$$

where u, v are displacement components, ε_x, ε_y, γ_{xy} are strain components that should satisfy the following strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}=0 \tag{4}$$

The constitutive relations of orthotropic FGM are

$$\varepsilon_x=S_{11}\sigma_x+S_{12}\sigma_y, \tag{5a}$$

$$\varepsilon_y=S_{12}\sigma_x+S_{22}\sigma_y, \tag{5b}$$

$$\gamma_{xz}=S_{66}\tau_{xy} \tag{5c}$$

where S_{11}, S_{22}, S_{12}, S_{66} are the elastic compliance parameters given by

$$S_{11}=S_{22}=\frac{1}{E(y)}, \tag{6a}$$

$$S_{12}=-\frac{\nu}{E(y)}, \tag{6b}$$

$$S_{66}=\frac{2(1+\nu)}{E(y)} \tag{6c}$$

The boundary conditions of elasticity at the upper and lower surfaces are

$$\sigma_y(x, \pm h/2)=0, \tag{7a}$$

$$\tau_{xy}(x, \pm h/2)=0. \tag{7b}$$

The boundary conditions at the left end (free) of the FGM beam are

$$N_0 = 0, \tag{8a}$$

$$M_0 = 0, \tag{8b}$$

$$Q_0 = -P \tag{8c}$$

where N_0 and M_0 denote the axial force, moment and shear force at $x = L$. The boundary conditions for the fixed end at the right end (fixed) of the beam are taken as

$$u = v = 0,\ \frac{\partial v}{\partial x} = 0 \ \text{ at } x = L,\ \ y = 0 \tag{9}$$

2.3 General solution

For the general solution we propose two new models based on the use of the Airy function, namely:

2.3.1 First present model

Solution in terms of unknown functions ψ_0 and ψ_1, according to the following expression

$$\psi(x, y) = \psi_0(y) + x.\psi_1(y) \tag{10}$$

In order to obtain a general solution to Eqs. (4)-(9), Airy stress function $\xi(x, z)$ is introduced such that

$$\sigma_X = \frac{\partial^2\psi}{\partial y^2}, \tag{11a}$$

$$\sigma_y = \frac{\partial^2\psi}{\partial x^2}, \tag{11b}$$

$$\tau_{xy} = -\frac{\partial^2\psi}{\partial x \partial y} \tag{11c}$$

So that Eq. (5) is satisfied automatically. We then assume that:
where $\psi_0(y)$ and $\psi_1(y)$ are unknown functions to be determined.

$$\sigma_x = \psi_0'' + x\psi_1'', \tag{12a}$$

$$\sigma_y = 0, \tag{12b}$$

$$\tau_{xy} = -\psi_1' \tag{12c}$$

Substitution of Eq. (15) into Eq. (8), and then into Eq. (7), gives rise to

$$\left(S_{11}\psi_0''\right)'' = 0 \tag{13a}$$

$$\left(S_{11}\psi_1''\right)'' = 0 \tag{13b}$$

Integration of Eq. (16a) yields

$$\psi_0'' = a_1 A_1 + a_2 A_2 \tag{14}$$

where $\psi_0(y)$ and $\psi_1(y)$ are unknown functions to be determined. Substitution of Eq. (14) into Eq. (13) gives, where and hereafter a_i $(i = 1,2,\ldots)$ are integral constants, and

$$A_1 = \frac{y}{S_{11}}, \tag{15a}$$

$$A_2 = \frac{1}{S_{22}}, \tag{15b}$$

$$A_i^m(y) = \frac{1}{m!}\int_{-h/2}^{y}(y-\eta)^m A_i d\eta \ \ (i = 1,2; m = 0,1) \tag{15c}$$

Integration of Eq. (16b) yields

$$\psi_1^{''} = a_3 B_3 + a_4 B_4 \tag{16a}$$

$$\psi_1^{'} = a_3 B_3^0 + a_4 B_4^0 + a_5 \tag{16b}$$

Where

$$B_1 = B_2 = 0, \tag{17a}$$

$$B_3 = A_1, \tag{17b}$$

$$B_4 = A_2, \tag{17c}$$

$$B_i^m(y) = \frac{1}{m!}\int_{-h/2}^{y}(y-\eta)^m B_i d\eta \ \ (i = 1,4) \text{ and } m = 0,1 \tag{17d}$$

Substituting Eq. (17) into Eq. (16c), and performing integration twice, we obtain the axial force N_0, bending moment M_0 and shearing force Q_0 at the left end of the beam can be obtained by integrating Eqs. (15a) and (15c) according to

$$N_0 = \int_{-h/2}^{h/2}\sigma_x(0,y)dy = \psi_0^{'}(h/2) - \psi_0^{'}(-h/2), \tag{18a}$$

$$M_0 = \int_{-h/2}^{h/2}\sigma_x(0,y)ydy = \frac{h}{2}[\psi_0^{'}(h/2) + \psi_0^{'}(-h/2)] - \psi_0(h/2) + \psi_0(-h/2), \tag{18b}$$

$$Q_0 = \int_{-h/2}^{h/2}\tau_{xy}(0,y)dy = \psi_1(-h/2) - \psi_1(h/2) \tag{18c}$$

Substitution of Eq. (15) into Eq. (10) gives

$$\psi_1^{'}(-h/2) = 0, \tag{19a}$$

$$\psi_1^{'}(h/2) = 0, \tag{19b}$$

Finallym the integration of Eq. (8), we could obtain the displacement components as follows

$$u = S_{11}\psi_0^{''}x + S_{11}\psi_1^{''}\frac{x^2}{2} + f(y) \tag{20a}$$

$$v = \int_{-h/2}^{y} s_{12}\psi_0^{''}d\eta + x\int_{-h/2}^{y} s_{12}\psi_1^{''}d\eta + g(x) \tag{20b}$$

where

$$f(y) = -\int_{-h/2}^{y}(y-\eta)S_{12}\psi_1 d\eta - \int_{-h/2}^{y} S_{66}\psi_1^{'}d\eta - wy + u_0 \tag{21a}$$

$$g(x) = wx + v_0 - a_1\frac{x^2}{2} - a_3\frac{x^3}{6} \tag{21b}$$

Where: u_0 and v_0 are integral constants.

Substituting Eqs. (17b), (17c) and (19b) into Eqs. (32e) and (32c) respectively, we have

$$\textstyle\sum_{i=1}^{2} a_i A_i^0(h/2) = 0, \tag{22a}$$

$$\textstyle\sum_{i=1}^{4} a_i B_i^1(h/2) = P \tag{22b}$$

$$\textstyle\sum_{i=1}^{2} a_i A_i^1(h/2) = 0, \tag{22c}$$

$$\textstyle\sum_{i=1}^{4} a_i B_i^0(h/2) = 0 \tag{22d}$$

Where

$$a_5 = 0 \tag{23}$$

2.3.2 Second present improved model

Solution in terms of function of known polynomials according to the following expression: we propose fifth degree polynomials in y^i (in other words a quadratic graduation).

$$\psi(x,y) = \textstyle\sum_{i=1}^{6} a_i y^{i-1} + x\sum_{i=1}^{6} b_i y^{i-1} \tag{24}$$

Young's modulus varies gradually in PFGM according to

$$E(y) = (E_c - E_m)(\frac{y}{h} + \frac{1}{2})^k + E_m \tag{25}$$

where k=0, 1, 2

Using Eq. (10), the expressions for the stress components become

$$\sigma_x = \textstyle\sum_{i=1}^{6}(i-1)(i-2)a_i y^{i-3} + x\sum_{i=1}^{6}(i-1)(i-2)b_i y^{i-3} \tag{26a}$$

$$\sigma_y = 0 \tag{26b}$$

$$\tau_{xy} = -\textstyle\sum_{i=1}^{6}(i-1)b_i y^{i-2} \tag{26c}$$

State of deformations

$$\varepsilon_x = S_{11}\sigma_x, \tag{27a}$$

$$\varepsilon_y = S_{12}\sigma_x, \tag{27b}$$

$$\gamma_{xy} = S_{66}\tau_{xy} \tag{27c}$$

verification of the differential equation of compatibility of Eq. (3)

$$(S_{11}(y)\textstyle\sum_{i=1}^{6}(i-1)(i-2)a_i y^{i-3})'' + x(S_{11}(y)\sum_{i=1}^{6}(i-1)(i-2)b_i y^{i-3})'' = 0 \tag{28}$$

by integrating this last equation we obtain the following

$$\textstyle\sum_{i=1}^{6}(i-1)(i-2)a_i y^{i-3} = c_1E(y)y + c_2E(y) \tag{29a}$$

$$\textstyle\sum_{i=1}^{6}(i-1)(i-2)b_i y^{i-3} = c_2E(y)y + c_4E(y) \tag{29b}$$

namely for k=2, the expanded expression of the Young's modulus of the FGM material is written as

$$E(y) = (E_c - E_m)(\frac{y^2}{h^2} + \frac{y}{h} + \frac{1}{4}) + E_m \tag{30}$$

we introduce the graduation parameters for k=0, 1, 2 Eq. (31) can be written in terms of these parameters in the following form

$$E(y) = (E_c - E_m)(\mu_2 \frac{y^2}{h^2} + \mu_1 \frac{y}{h} + \frac{1}{2^k}) + E_m \tag{31a}$$

$$\text{for } k\text{=0:} \quad \mu_1 = \mu_2 = 0 \text{ and } E(y) = E_c \tag{31b}$$

$$\text{for } k\text{=1:} \quad \mu_1 = 1\,,\ \mu_2 = 0 \text{ and } E(y) = (E_c - E_m)(\frac{y}{h} + \frac{1}{2}) + E_m \tag{31c}$$

$$\text{for } k\text{=2:} \quad \mu_1 = 1\,,\ \mu_2 = 1 \text{ and } E(y) = (E_c - E_m)(\frac{y^2}{h^2} + \frac{y}{h} + \frac{1}{4}) + E_m \tag{31d}$$

where μ_1 and μ_2 are the graduation parameters

By simplifying Eq. (29), we derive the constants a_i and b_i (i=3, 4, 5 and 6) as a function of the new constants c_j (j=1, 2, 3 and 4) according to

$$a_3 = \frac{c_2}{2}(\frac{E_c - E_m}{2^k} + E_m) \tag{32a}$$

$$a_4 = \frac{1}{6}\left[c_1(\frac{E_c - E_m}{2^k} + E_m) + c_2\mu_1 \frac{E_c - E_m}{h}\right] \tag{32b}$$

$$a_5 = \frac{1}{12}\left[c_1\mu_1(\frac{E_c - E_m}{h}) + c_2\mu_2 \frac{E_c - E_m}{h^2}\right] \tag{32c}$$

$$a_6 = \frac{1}{20}\left[c_1\mu_2 \frac{E_c - E_m}{h^2}\right] \tag{32d}$$

likewise

$$b_3 = \frac{c_4}{2}(\frac{E_c - E_m}{2^k} + E_m) \tag{33a}$$

$$b_4 = \frac{1}{6}\left[c_3(\frac{E_c - E_m}{2^k} + E_m) + c_4\mu_1 \frac{E_c - E_m}{h}\right] \tag{33b}$$

$$b_5 = \frac{1}{12}\left[c_3\mu_1(\frac{E_c - E_m}{h}) + c_4\mu_2 \frac{E_c - E_m}{h^2}\right] \tag{33c}$$

$$b_6 = \frac{1}{20}\left[c_3\mu_2 \frac{E_c - E_m}{h^2}\right] \tag{33d}$$

by substituting Eqs. (32) and (33) into Eq. (26), the expressions of the constraints in terms of the constants c_i become as follows

$$\sigma_x = \sum_{i=1}^{2} c_i\lambda_i(y) + x\sum_{i=3}^{4} c_i\lambda_i(y) \tag{34a}$$

$$\sigma_y = 0 \tag{34b}$$

$$\tau_{xy} = -b_2 - \sum_{i=3}^{4} c_i\gamma_i(y) \tag{34c}$$

where

$$\lambda_1(y) = \lambda_3(y) = (E_c - E_m)(\mu_2 \frac{y^3}{h^2} + \mu_1 \frac{y^2}{h} + \frac{y}{2^k}) + yE_m \tag{35a}$$

$$\lambda_2(y) = \lambda_4(y) = (E_c - E_m)(\mu_2 \frac{y^2}{h^2} + \mu_1 \frac{y}{h} + \frac{1}{2^k}) + E_m \tag{35b}$$

$$\gamma_3(y) = (E_c - E_m)(\mu_1 \frac{y^3}{3h} + \mu_2 \frac{y^4}{4h^2} + \frac{y^2}{2^{k+1}}) + \frac{y^2}{2} E_m \tag{35c}$$

$$\gamma_4(y) = (E_c - E_m)(\mu_1 \frac{y^2}{h} + \mu_2 \frac{y^3}{3h^2} + \frac{y}{2^k}) + yE_m \tag{35d}$$

Determination of the constants b_2, c_1, c_2, c_3 and c_4 from the loading conditions on the faces of the beam using the following equations

$$\tau_{xy}(x, -h/2) = b_2 + \sum_{i=3}^{4} c_i \gamma_i(-h/2) = 0 \tag{36a}$$

$$\tau_{xy}(x, +h/2) = b_2 + \sum_{i=3}^{4} c_i \gamma_i(+h/2) = 0 \tag{36b}$$

for x=0

$$N_0 = -\sum_{i=1}^{2} c_i(\lambda_i^0(+h/2) - \lambda_i^0(-h/2)) = 0 \tag{36c}$$

$$M_0 = \sum_{i=1}^{2} c_i \frac{h}{2}\left[(\lambda_i^0(+h/2) + \lambda_i^0(-h/2)) - (\lambda_i^1(+h/2) - \lambda_i^1(-h/2))\right] = 0 \tag{36d}$$

$$Q_0 = \int_{-h/2}^{+h/2} \tau_{xy}(0, y)dy = b_2 h + \sum_{i=3}^{4} c_i(\gamma_i^0(+h/2) - \gamma_i^0(-h/2)) = -P \tag{36e}$$

By integrating the deformation Eq. (26) the displacement components are expressed by

$$u(x, y) = (c_1 y + c_2)x + (c_3 y + c_4)\frac{x^2}{2} + \nu c_4 \frac{y^2}{2} + \nu c_3 \frac{y^3}{6} - \int_{-h/2}^{y} S_{66}(-b_2 + \sum_{i=3}^{4} c_i \gamma_i)d\eta - v_0 y + u_0 \tag{37}$$

$$v(x, y) = -\nu(y + \frac{h}{2})(c_2 + c_4 x) - \nu(\frac{y^2}{2} - \frac{h^2}{8})(c_1 + c_3 x) + v_0 x - c_1 \frac{x^2}{2} - c_3 \frac{x^3}{6} + v_1 \tag{38}$$

The integration constants u_0, v_0 and v_1 are determined from the displacement conditions at the embedding level (x=L).

3. Numerical results and discussion

In this section, numerical study of a FGM cantilever beam (L=1 m, height h=0.1 m) subjected to a concentrated load P, with P=1 kN. The analysis is performed for pure materials and different values of material parameter, k, for P-FGM materials. The Young's modulus for ceramic E_c=380 GPa; aluminum E_m=70 GPa and poison's ratio are ν_m=0.3, respectively on the upper surface of the beam will be made based on the above solution. We assume the graded function as in P-FGM beam Eq. (1). In present study, we take the graded index for different values of k (0, 1 and 2).

Calculation parameters

$$\bar{\sigma}_x = \frac{\sigma_x h}{P} \tag{38a}$$

Table 1a Validation by comparison of the present method with the classical theory of beams: Case of homogeneous material (*k*=0)

k	Theory	$\bar{v}(0,0)$	$\bar{\sigma}_x(L/2\,,h/2)$	$\bar{\tau}_x(L/2\,,0)$
0	Classical theory	7,364	30	1,50
	Present Model 2	7,368	30	1,5

Table 1b Comparison of beam behavior for the two models

k	Theory	$\bar{v}(0,0)$	$\bar{\sigma}_x(L/2\,,h/2)$	$\bar{\tau}_x(L/2\,,0)$
0	Present Model 1	7,368	30	1,50
	Present Model 2	7,368	30	1,5
1	Present Model 1	14,782	46,366	1,499
	Present Model 2	14,782	46,366	1,5
2	Present Model 1	18,944	54,141	1,421
	Present Model 2	18,944	54,141	1,421

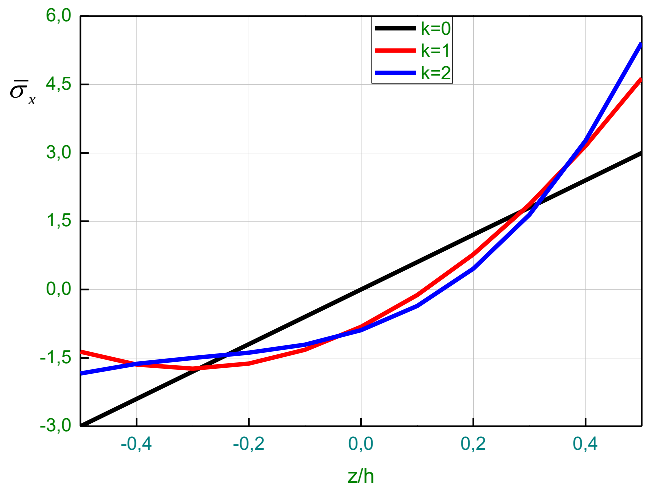

Fig. 2 Distribution of normal stresses through the thickness and at mid-span of the beam

$$\bar{\tau}_x = \frac{\tau_x h}{P} \tag{38b}$$

$$\bar{v} = 10\frac{vE_m h^3}{PL^3} \tag{38c}$$

Model 1 represents the Airy function's expression in terms of unknown functions, inspired by the old model (Hassaine Daouadji 2013). Model 2 represents a new configuration, substituting unknown functions with well-defined polynomials. Comparing the two models, it was found that the agreement is perfect for constraints and displacements according to the three graduations (0, 1, 2), due to the good choice of polynomial degree for this type of problem. This validates the good proposition of the new model 2 (Table 1a and 1b).

Fig. 2 shows normal stresses distribution through the beam's thickness and mid-span. According to Bernoulli's theory, the homogeneous model (*k*=0) is linear, while the FGM models

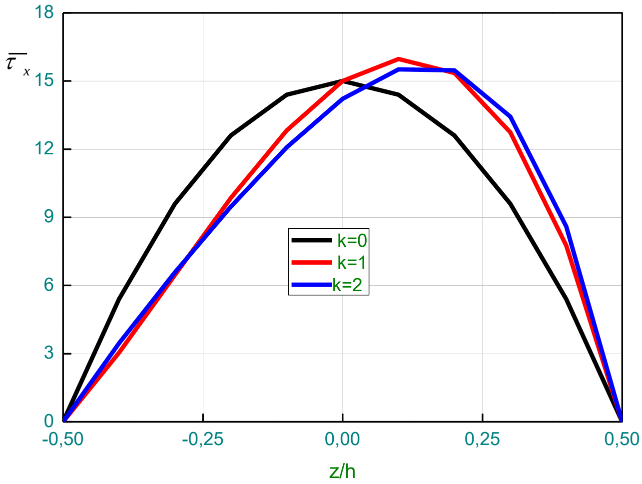

Fig. 3 shear stress distribution through the thickness and at mid-span of the beam

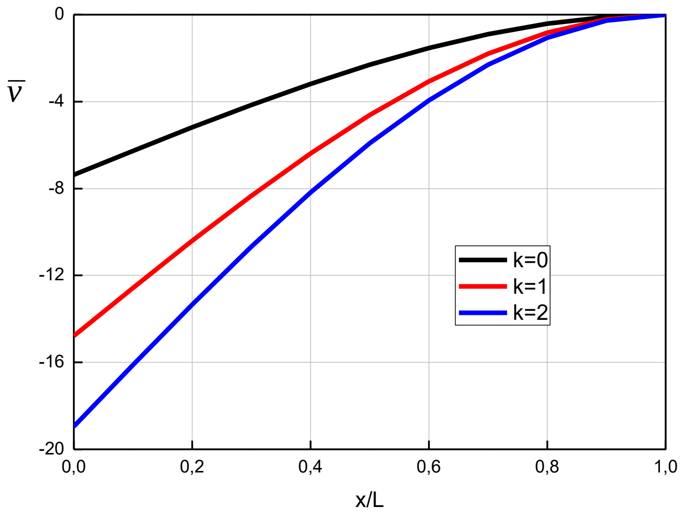

Fig. 4 Distribution of displacements along the beam for different graduations

(k=1 and k=2) show non-linearity, with the highest value in ceramic (k=2). This confirms the structural logic in compression and tension, thus validating the new model.

Fig. 3 shows the variation of tangential stresses through the beam's thickness and mid-span. The shear stresses' nullity is well recorded, with a symmetrical parabolic distribution for homogeneous materials and a peak at mid-height. For FGM, a deviation is observed towards the face richest in ceramic, with a maximum value for the stiffest material (k=2). These results confirm the validity of the new model.

The analysis of the distribution of displacements along the beam for different graduations reveals that the deflection is lower for homogeneous material (k=0) compared to FGM material. This is due to the richness of homogeneous ceramic, which is rigid and non-deformable. As the graduation of FGM towards the metal increases, the deflection increases, as shown in Fig. 4.

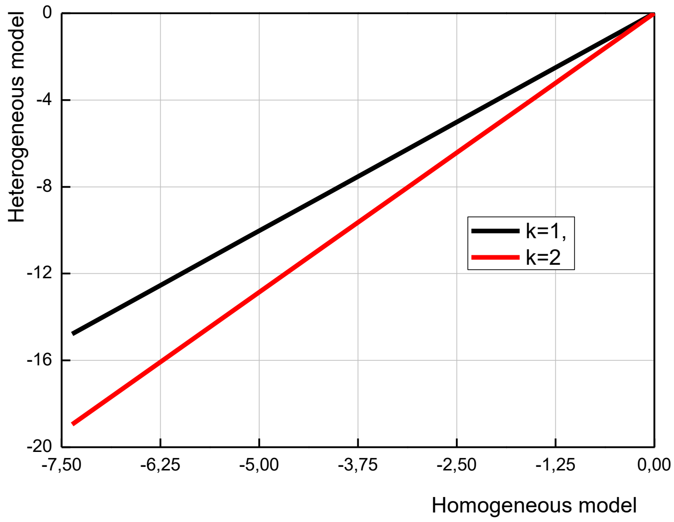

Fig. 5 Relationship of the homogeneous model as a function of the FGM model for displacements

The study tests the linearity of the homogeneous model's relationship with the FGM model for displacements, demonstrating the reliability of the results. The test, presented in Fig. 5, clearly demonstrates the linearity of the FGM model as a function of the homogeneous model. The new proposed model is found to be perfectly linear, indicating good agreement with the existing model and offering simplicity in formulation and resolution.

4. Conclusions

A plane elasticity solution for functionally graded beams developed using a semi-inverse method and an improved Airy polynomial function. The mechanical properties of the material are assumed to depend on the thickness-coordinate, and a general two-dimensional solution is obtained for FGM cantilever beams under concentrated load at the free end. The novelty lies in the proposal of a simpler Airy function, which simplifies formulation and resolution by reducing unknowns and combining them with integration constants of the differential equation. The degree of the polynomial depends on the degree of material graduation, which indicates the stability and behavior of the beam material. Tests have been conducted to validate this new model, including linearity of the FGM model as a function of the homogeneous model. The new improved Airy polynomial function is valid for arbitrary graded variations of material distribution and could serve as a basis for simplified FGM theories or a benchmark for assessing other approximate methodologies.

References

Ahmed, S.R., Idris, A.B.M. and Uddin, M.W. (1996), "Numerical solution of both ends fixed deep beams", *Comput. Struct.*, **61**(1), 21-29. https://doi.org/10.1016/0045-7949(96)00029-6.

Benguediab, S., Tounsi, A., Abdelaziz, H.H. and Meziane, M.A.A. (2017), "Elasticity solution for a cantilever beam with exponentially varying properties", *J. Appl. Mech. Tech. Phys.*, **58**(2), 354-361. https://doi.org/10.1134/S0021894417020213.

Bourada, F., Amara, K., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2018), "A novel refined plate theory for stability analysis of hybrid and symmetric S-FGM plates", *Struct. Eng. Mech.*, **68**(6), 661-675. https://doi.org/10.12989/sem.2018.68.6.661.

Chu, P., Li, X.F., Wang, Z.G. and Lee, K.Y. (2015), "Double cantilever beam model for functionally graded materials based on two-dimensional theory of elasticity", *Eng. Fract. Mech.*, **135**, 232-244. http://doi.org/10.1016/j.engfracmech.2015.01.008.

Daouadji, T.H., Henni, A.H., Tounsi, A. and El Abbes, A.B. (2013), "Elasticity solution of a cantilever functionally graded beam", *Appl. Compos. Mater.*, **20**, 1-15. https://doi.org/10.1007/s10443-011-9243-6.

Eiadtrong, S., Nguyen, T.N. and Wattanasakulpong, N. (2024), "Nonlinear vibration of sandwich beams made of FGM faces and FGP core under multiple moving loads using a quasi-3D theory", *Eng. Struct.*, **316**, 118575. https://doi.org/10.1016/j.engstruct.2024.118575.

El-Shahrany, H.D. and Zenkour, A.M. (2024), "Control of dynamic response of the functionally graded smart sandwich beam coupled variable Kelvin–Voigt–Pasternak's model", *Ain Shams Eng. J.*, **15**(3), 102476. https://doi.org/10.1016/j.asej.2023.102476.

Garg, A., Belarbi, M.O., Li, L., Chalak, H.D., Tounsi, A. and Zenkour, A.M. (2024), "Comparative study on the buckling analysis of exponential, power and sigmoidal sandwich FGM plates under hygro-thermal conditions", *Adv. Mater. Res.*, **13**(6), 431-462. https://doi.org/10.12989/amr.2024.13.6.431.

Gorik, A.V. and Koval'chuk, S.B. (2020), "Solving the problem of elastic bending of a layered cantilever under a normal load linearly distributed over longitudinal faces", *Int. Appl. Mech.*, **56**(1), 65-80. https://doi.org/10.1007/s10778-020-00997-w.

Houari, M.S.A., Bessaim, A., Merzouki, T., Daikh, A.A., Garg, A., Tounsi, A., ... & Belarbi, M.O. (2024), "Shear correction factors of a new exponential functionally graded porous beams", *Struct. Eng. Mech.*, **89**(1), 1-11. https://doi.org/10.12989/sem.2024.89.1.001.

Huang, D.J., Ding, H.J. and Chen, W.Q. (2007), "Piezoelasticity solutions for functionally graded piezoelectric beams", *Smart Mater. Struct.*, **16**(3), 687. https://doi.org/10.1088/0964-1726/16/3/015.

Koval'chuk, S., Goryk, O., Yakhin, S. and Antonets, A. (2024), "Exact analytical solution of the problem of elastic bending of a multilayer beam with a normal trapezoidal load", *Key Eng. Mater.*, **1005**, 107-119. https://doi.org/10.4028/p-MZJc71.

Laoud, B., Benyoucef, S., Bachiri, A., Bouiadjra, R.B., Tounsi, A., Selim, M.M. and Saad, H.A. (2024), "A comprehensive computational approach to assess the influence of the material composition on vibration, bending and buckling response of FG beam lying on viscoelastic foundation", *Steel Compos. Struct.*, **52**(1), 45-56. https://doi.org/10.12989/scs.2024.52.1.045.

Ma, H. (2020), "Rational approach for higher-order shear deformation beam theories", *Compos. Struct.*, **251**, 112599. https://doi.org/10.1016/j.compstruct.2020.112599.

Madenci, E., Gulcu, S. and Draiche, K. (2024), "Analytical nonlocal elasticity solution and ANN approximate for free vibration response of layered carbon nanotube reinforced composite beams", *Adv. Nano Res.*, **16**(3), 251-263. https://doi.org/10.12989/anr.2024.16.3.251.

Mahmoud, S.R., Ghandourah, E.I., Algarni, A.H., Balubaid, M.A., Tounsi, A., Tounsi, A. and Bourada, F. (2023), "Stability investigation of symmetrically porous advanced composites plates via a novel hyperbolic RPT", *Steel Compos. Struct.*, **46**(4), 471-483. https://doi.org/10.12989/scs.2023.46.4.471.

Malikan, M. and Eremeyev, V.A. (2020), "A new hyperbolic-polynomial higher-order elasticity theory for mechanics of thick FGM beams with imperfection in the material composition", *Compos. Struct.*, **249**, 112486. https://doi.org/10.1016/j.compstruct.2020.112486.

Mohammadimehr, M. (2024), "The effect of a nonlocal stress-strain elasticity theory on the vibration analysis of Timoshenko sandwich beam theory", *Adv. Nano Res.*, **17**(3), 275-284. https://doi.org/10.12989/anr.2024.17.3.275.

Nebab, M., Atmane, H.A. and Bennai, R. (2024), "Investigating wave propagation in sigmoid-FGM imperfect plates with accurate Quasi-3D HSDTs", *Steel Compos. Struct.*, **51**(2), 185-202.

https://doi.org/10.12989/scs.2024.51.2.185.
Nie, G.J., Zhong, Z. and Chen, S. (2013), "Analytical solution for a functionally graded beam with arbitrary graded material properties", *Compos. Part B: Eng.*, **44**(1), 274-282. http://doi.org/10.1016/j.compositesb.2012.05.029.
Reddy, J.N., Ruocco, E., Loya, J.A. and Neves, A.M. (2021), "Theories and analysis of functionally graded beams", *Appl. Sci.*, **11**(15), 7159. https://doi.org/10.3390/app11157159.
Saidi, H., Tounsi, A., Bourada, F., Bousahla, A.A., Tounsi, A. and Al-Juboori, F.I.S. (2024), "Vibrational behavior of porous composite laminated plates using four unknown integral shear deformation theory", *Steel Compos. Struct.*, **52**(3), 249-271. https://doi.org/10.12989/scs.2024.52.3.249.
Songsuwan, W., Wattanasakulpong, N. and Vo, T.P. (2022), "Nonlinear vibration of third-order shear deformable FG-GPLRC beams under time-dependent forces: Gram–Schmidt–Ritz method", *Thin Wall. Struct.*, **176**, 109343. https://doi.org/10.1016/j.tws.2022.109343.
Sun, S.L., Zhang, X.Y. and Li, X.F. (2024), "A consistent shear beam theory for free vibration of functionally graded beams based on physical neutral plane", *Mech. Adv. Mater. Struct.*, **31**(16), 3844-3854. https://doi.org/10.1080/15376494.2023.2185709.
Sun, S.L., Zhang, X.Y. and Li, X.F. (2024), "A consistent shear beam theory for free vibration of functionally graded beams based on physical neutral plane", *Mech. Adv. Mater. Struct.*, **31**(16), 3844-3854. https://doi.org/10.1080/15376494.2023.2185709.
Tang, C., Dui, G. and Fu, Y. (2025), "Elasticity solutions for functionally graded beams with arbitrary distributed loads", *Compos. Struct.*, **351**, 118578. https://doi.org/10.1016/j.compstruct.2024.118578.
Tlidji, Y., Zidour, M., Zerrouki, R., Benahmed, A., Serbah, B., Draiche, K. and Bouakkaz, K. (2024), "Nonlinear FG-CNT effect on the critical buckling load of nanocomposite beams with different boundary conditions", *Adv. Nano Res.*, **17**(4), 323-334. https://doi.org/10.12989/anr.2024.17.4.323.
Wang, M. and Liu, Y. (2010), "Analytical solution for bi-material beam with graded intermediate layer", *Compos. Struct.*, **92**(10), 2358-2368. https://doi.org/10.1016/j.compstruct.2010.03.013.
Wang, M. and Liu, Y. (2013), "Elasticity solutions for orthotropic functionally graded curved beams", *Eur. J. Mech.-A/Solid.*, **37**, 8-16. https://doi.org/10.1016/j.euromechsol.2012.04.005.
Xu, Y., Yu, T. and Zhou, D. (2014), "Two-dimensional elasticity solution for bending of functionally graded beams with variable thickness", *Meccanica*, **49**, 2479-2489. https://doi.org/10.1007/s11012-014-9958-1.
Yang, Q., Zheng, B.L., Zhang, K. and Li, J. (2014), "Elastic solutions of a functionally graded cantilever beam with different modulus in tension and compression under bending loads", *Appl. Math. Model.*, **38**(4), 1403-1416. http://doi.org/10.1016/j.apm.2013.08.021.
Yang, Q., Zheng, B.L., Zhang, K. and Zhu, J.X. (2013), "An elastic solution of a functionally graded cantilever beam with different modulus in tension and compression under uniform pressure", *Adv. Mater. Res.*, **717**, 396-402. http://doi.org/10.4028/www.scientific.net/AMR.717.396.
Yang, Q., Zheng, B., Zhang, K. and Zhu, J. (2013), "Analytical solution of a bilayer functionally graded cantilever beam with concentrated loads", *Arch. Appl. Mech.*, **83**, 455-466. https://doi.org/10.1007/s00419-012-0693-6.
Yu, T. and Zhong, Z. (2007), "Bending analysis of a functionally graded piezoelectric cantilever beam", *Sci. China Ser. G: Phys., Mech. Astron.*, **50**(1), 97-108. https://doi.org/10.1007/s11433-007-2006-6.
Zahedinejad, P., Zhang, C., Zhang, H. and Ju, S. (2020), "A comprehensive review on vibration analysis of functionally graded beams", *Int. J. Struct. Stab. Dyn.*, **20**(04), 2030002. https://doi.org/10.1142/S0219455420300025.
Zhong, Z. and Yu, T. (2007), "Analytical solution of a cantilever functionally graded beam", *Compos. Sci. Technol.*, **67**(3-4), 481-488. https://doi.org/10.1016/j.compscitech.2006.08.023.