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# Refined 6-DOF hyperbolic model for the free vibration analysis of multi-layered sandwich shallow shells

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**Abstract.** Sandwich shell panels are widely used in many engineering structures wherein those are subjected to dynamic forces, which leads to their failure. It is necessary to perform accurate dynamic analysis of multilayered sandwich shell panels to provide their safe design. To the best of the authors' knowledge, literature on the free vibration analysis of multi-layered sandwich shallow shells is limited and needs more attention. Therefore, the purpose of the present study is to find higher-order closed-form solutions for the free vibration problems of sandwich shallow shells with double curvature using refined computational model. In the present study, a new hyperbolic shape function is introduced in the refined shell theory to account for the effects of transverse shear and normal deformations. A theory involves six degrees of freedom and satisfies traction-free boundary conditions at the top and the bottom surfaces of the shell. The governing equations of motion and associated boundary conditions of the theory are produced by employing Hamilton's principle. Semi-analytical closed-form solutions for the free vibration problems are made by the Navier technique for simply supported boundary conditions of the shell. The non-dimensional natural frequencies of five layered symmetric and anti-symmetric sandwich shells are obtained for various parameters such as *a/h* ratio, *a/b* ratio, *tc/tf* ratio, radii of curvature, and modes of vibration. The present results are compared with results that have already been published to confirm the accuracy and efficiency of the current higher-order hyperbolic shell theory. It is concluded from the comparison of results that the present theory is in excellent agreement while predicting the natural frequencies of sandwich plates and shells. Also, this study presented new benchmarks in vibration analysis of sandwich shells. This study demonstrates results for a typical five-layer sandwich shell with composite faces and isotropic/foam cores. However, the proposed model can be readily extended to other material combinations and structural configurations.

**Keywords:** double-curvature; free vibration analysis; hyperbolic shell theory; sandwich shells; transverse normal strain

## 1. Introduction

Sandwich plate and shell structures are usually utilized in a variety of engineering fields where those are often subjected to dynamic loading conditions. For the safe design of sandwich structures

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under dynamic conditions, it is important to critically analyse the frequencies developed in the structures. Sandwich plates and shells can be analysed using classical and refined theories under dynamic conditions. The classical plate/shell theory (CST) which was developed by Kirchhoff (1850) is overestimating the natural frequencies due to neglecting transverse stresses. Mindlin (1951) overcame the limitations of the CST by developing the first-order shear deformation theory which accounts for the effects of shear deformation for the first time. The FSDT predicts the frequencies with a little more accuracy compared to the CST, however, this theory needs the shear correction factor $(\pi^2/12)$ during the solution of the Eigenvalue problem. These limitations of the CST and the FSDT forced the researchers to develop higher-order refined theories for the vibration analysis of sandwich structures. Below mentioned section covers the review of studies done on the vibration analysis of sandwich structures using classical and or higher-order theories.

Noor and Burton (1989) presented the development and classification of two-dimensional shear deformation theories based on displacement, stain and/or stress assumptions in the thickness direction for the free vibration analysis of multi-layered laminated structures. Noor (1973) analysed low-frequency free vibrations of multi-layered composite plates based on the classical plate theory. Pagano (1970) presented three-dimensional elasticity solutions for rectangular laminates with pinned edges which serves benchmark for many researchers. Whitney (1969) presented a transverse shear deformation bending theory for simply supported rectangular laminated plates. Closed-form solutions have been developed for buckling loads, flexural vibration frequencies, and bending deflections. Whitney and Pagano (1970) presented a vibration analysis of symmetric and non-symmetric anisotropic laminated plates.

Chalak *et al.* (2013) presented an FE model for the free vibration analysis of softcore sandwich plates with various boundary conditions and showed that the current FE model can accurately predict the free vibration behaviour of sandwich shells. Kant and Swaminathan (2001a, b, 2002) developed a higher-order refined plate theory considering the 12 degrees of freedom accounting for the effects of transverse shear and normal deformations for the static and free vibration analysis of sandwich plates. Rao *et al.* (2004) presented a semi-analytical solution based on a higher-order mixed approach and the propagator matrix technique to evaluate the natural frequencies of laminated plates based on equivalent single-layer theory. Reddy and Chao (1981) presented closed-form solutions for simply supported cross-ply and antisymmetric angle-ply plates under sinusoidal mechanical and thermal loadings along with free vibration problems. Reddy (1984) developed a third-order shear deformation theory in 1984 which satisfies the traction-free boundary conditions on the top and the bottom surfaces of the shell without using the shear correction factor. It is also known as parabolic shear deformation theory. Sayyad and Ghugal (2015) presented a review of various solution techniques that may be used for the free vibration analysis of sandwich and laminated plates highlighting the direction of future research. Several displacement fields of higher-order shear deformation theories have been documented in this paper. Sayyad and Ghugal (2016) presented the cylindrical bending of laminated composite and sandwich plates with simple supports using sinusoidal shear and normal deformation theory including the effects of transverse normal strain. Zhai *et al.* (2018) applied the first-order shear deformation theory for the vibration analysis of five-layered composite sandwich plates with viscoelastic cores. Keddouri (2019) presented the static response of functionally graded sandwich plates using a novel displacement-based high-order shear deformation theory considering the effects of porosity distributions.

Carrera (1999a, b) studied the effects of transverse normal stress on the static and vibration behaviour of laminated composite plates and shells. Azzara *et al.* (2024) analyzed the rotor dynamics of rotating variable angles to composite cylindrical structures using low and high-fidelity models. The authors have used Carrera's unified formulation (CUF) to develop shell finite element models. Carrera and Scano (2024) presented the Jacobi polynomial shape functions to permit the expression of displacement kinematics in a hierarchical form based on the CUF. Pagani *et al.* (2024) proposed Jacobi polynomials to approximate higher-order theories for beam, plate, and shell structures. Pagani *et al.* (2023) presented the CUF-based layerwise theories employed to characterize the complex phenomena in variable angle tow composite structures. Azzara *et al.* (2022) presented vibration and buckling analysis of composite shells subjected to combined internal pressure and axial compression by applying the finite element method and the principle of virtual work. Li and Carrera (2020) presented a class of refined hierarchical shell finite element models using Reissner's mixed variational theorem (RMVT) and layer-wise theories.

Chakrabarti and Sheikh (2004) presented a free vibration analysis of sandwich plates with stiff laminated face sheets based on refined plate theory. Chan and Cheung (1972) presented a finite strip approach for bending and vibration analysis of multi-layered sandwich rectangular plates. Ferreira *et al.* (2008) presented static and free vibration analysis of isotropic and laminated composite plates using a layerwise theory. Garg and Chalak (2020) presented the free vibration analysis of sandwich and laminated plates using a higher-order zigzag theory and the finite element approach. Ghugal and Sayyad (2011) presented orthotropic plate vibration based on a trigonometric shear deformation theory considering the effects of transverse normal strain. Raissi *et al.* (2019, 2022) presented stress distributions in rectangular and circular five-layer sandwich plates subjected to a uniform transverse load using layerwise theory along with the first-order and second-order shear deformation theories. Kant and Manjunatha (1994) presented an estimation of interlaminar transverse shear and normal stresses using the equations of three-dimensional equilibrium. Khare *et al.* (2005) presented higher-order shear deformation theories in closed-formed for the thermo-mechanical and free vibration analysis of simply supported, doubly curved, cross-ply laminated sandwich shells. Kulkarni and Kapuria (2008) presented a dynamic analysis of composite and sandwich plates with four-node discrete Kirchhoff quadrilateral elements based on the third-order zigzag theory. Kardooni *et al.* (2022) investigated the free vibration response of a five-layer sandwich composite plate resting on a Winkler elastic foundation in a thermal environment by using layer-wise theory with the help of Hamilton's principle. Navier solution is used to obtain the closed-form solutions. Moradi-Dastjerdi and Behdinan (2021) presented the impact of layer arrangement on the design of five-layer multifunctional smart sandwich plates under electromechanical loads. Pandya and Kant (1988) presented displacement analysis of laminated composite plates under transverse load using finite element formulation based on a higher-order theory. Rao and Meyer-Piening (1991) presented a bending analysis of simply supported and clamped sandwich plates subjected to a sinusoidal load based on hybrid-stress finite Element theory. Rao and Desai (2004) presented a semi-analytical approach using higher-order mixed theory to analyse the natural frequencies, displacement and stresses for simply supported, cross-ply laminated, and sandwich plates. Saeedi *et al.* (2013) presented the cylindrical bending of multi-layered plates by a two-dimensional layer-wise model based on the finite element method for various boundary conditions. Frosting and Thomsen (2004) presented free vibration analysis of sandwich panels with a flexible core based on the high-order sandwich panel theory. Sayyad and Ghugal (2019, 2022a) presented static bending and free vibration analysis of laminated and sandwich shells using a generalized higher-order shell theory. Analytical solutions for simply supported shells are obtained using Navier's technique by presenting non-dimensional natural frequencies for different modes. Senthilnathan *et al.* (1988) presented a frequency analysis of thick simply supported rectangular laminated plates using Reddy's higher-order plate theory. Shinde and Sayyad (2022) presented free vibration analysis of laminated composite and sandwich, cylindrical and spherical shells using fifth-order shear and normal deformation theory by considering the effects of both transverse shear and normal deformations. Shaikh and Sayyad (2024) presented static and free vibration analysis of three-layer sandwich shells by using a hyperbolic shell theory considering transverse normal strain effects based on the Navier technique. Jape and Sayyad (2023) presented static and free vibration analysis of simply supported laminated composite shells with double curvature using a hyperbolic shear deformation theory. Sayyad *et al.* (2023) presented the static and free vibration analysis of simply supported FGM sandwich shallow shells with double curvature using various equivalent single-layer shell theories based on Navier's technique. Sayyad *et al.* (2022) presented a dynamic analysis of laminated composite plates and various shells of double curvature by using Hamilton's principle and Navier's technique for the simply supported boundary conditions. Sayyad and Ghugal (2022b) presented the static and free vibration analysis of orthotropic laminated composite spherical shells using various refined shear deformation theories. Sayyad and Ghugal (2021) investigate the interlaminar stresses of laminated composite doubly curved shells on a rectangular planform under concentrated force using various equivalent single-layer shell theories recovered by generalized mathematical formulation. Sayyad and Ghugal (2020) presented stress analysis of laminated composite and sandwich cylindrical shells using equivalent single-layer higher-order shell theories. Zhen and Wanji (2006) presented the global-local higher-order theory for the free vibration analysis of sandwich and laminated plates. Zhen *et al.* (2010) presented analytical and numerical solutions for the free vibration analysis of laminated composite and sandwich plates using a higher-order theory and $C^0$ finite element formulation along with Navier's technique.

Tornabene *et al.* (2024a, b) investigated the vibration response of laminated anisotropic doubly-curved shell structures reinforced with carbon nanotubes using higher-order theories. Tornabene *et al.* (2024c, 2023a, b) presented the dynamic analysis of doubly-curved shell laminated and porous FGM shells using higher-order theories and the generalized differential quadrature method. Tornabene *et al.* (2022, 2023c, d, 2024d, e) presented a dynamic analysis of doubly curved laminated shells based on the equivalent single-layer shell theories wherein governing equations of motion are derived from Hamilton's principle. Sobhani *et al.* (2023) presented a dynamic analysis of a porous nano-composite assembled paraboloid cylindrical shell wherein Hamilton's principle is used to determine the governing equations of motion. Mahmure *et al.* (2021) presented the free vibration behaviour of carbon nanotube-reinforced thin-walled composite shell structures resting on a Winkler-Pasternak foundation using the Galerkin and Grigolyuk methods. Mantari *et al.* (2021, 2012) presented static and dynamic analysis of laminated composite and sandwich plates and shells by using a new higher-order shear deformation theory. The governing equations are derived by using the principle of virtual work and these equations are solved using the Navier-type, solution technique. Monge *et al.* (2022, 2019) presented a three-dimensional bending solution of doubly-curved shells subjected to mechanical, thermal and hygro-thermal loadings using the differential quadrature method and Navier's technique for simply supported boundary conditions. Monge *et al.* (2018) present a static analysis of laminated composite doubly curved shells using a refined kinematic model based on Navier's closed-form solution. Monge *et al.* (2024) presented free vibration analysis of laminated and sandwich shells using non-polynomial type theories based on axiomatic/asymptotic methods. Hamilton's principle is used to derive governing equations of motion and semi-analytical solutions for free vibration problems are obtained using the Navier technique. Santos *et al.* (2005), Correia *et al.* (2005), Soares *et al.*

(1989) presented bending, vibration and buckling analysis of laminated axisymmetric shells using semi-analytical finite element model and higher-order theory. Asadi and Qatu (2012, 2024), and Asadi *et al.* (2012) presented static and free vibration analysis of laminated cylindrical shells using the general differential quadrature method based on higher-order shear deformation theories. Qatu and Asadi (2012) presented a vibration analysis of doubly curved shallow shells subjected to 21 possible boundary conditions using the thin shallow shell theory. Wang and Qatu (2012) presented a vibration analysis of thick cylindrical shells using the three-dimensional elasticity theory. Qatu (1995) presented a static and dynamic analysis of laminated shells using a triangular conforming element. Qatu (1993), Qatu and Leissa (1991, 1993) presented vibration analysis of cantilever laminated shallow shells using the Ritz method. Recently, Jape and Sayyad (2025) presented a free vibration analysis of anti-symmetric angle-ply laminated shells using a modified hyperbolic shear deformation theory based on Navier's solution technique. Brischetto (2014) presented an exact 3D free vibration analysis of multilayered composite and sandwich plates and shells for several vibration modes, thickness ratios, and imposed wave numbers.

Bennai *et al.* (2013) developed a refined higher-order shear and normal deformation theory for functionally graded sandwich beams, while Bennai *et al.* (2022) studied the stability and free vibration of porous FGM beams. Mellal *et al.* (2023) investigated vibration and buckling of porous FG beams on variable elastic foundations, and Atmane *et al.* (2021) analyzed their dynamic response using a novel higher-order shear deformation theory. Kehli *et al.* (2024) examined cracked FG beams on viscoelastic foundations with a quasi-3D model, whereas Nebab *et al.* (2024) focused on frequency characteristics of cracked porous FG beams on elastic supports.

Hadji *et al.* (2011) proposed a four-variable refined theory for free vibration of FG sandwich plates, further extending to porous multidirectional FG sandwich plates (Hadji *et al.* 2023) and thermal buckling of FG plates with temperature-dependent properties (Hadji *et al.* 2024). Zouatnia and Hadji (2019a, b) introduced refined shear deformation theories for FG beams and sandwich plates, respectively. Saad *et al.* (2025) presented a sinusoidal shear deformation theory for dynamic analysis of FG plates, while Plevris *et al.* (2025) developed an nth-order refined plate theory for graphene-reinforced FG composites. Ould Larbi *et al.* (2025) analyzed bi-directional porous FG plates under buckling loads, and Pham *et al.* (2024) applied an improved finite element model for in-plane behavior of FG plates.

Other contributions include Djebbour *et al.* (2025) on porous FG CNT beams with orthotropic viscoelastic foundations, Nassah *et al.* (2025) on static response of FG cantilever beams under polynomial loading, Meski *et al.* (2024) on FG sandwich beams with ceramic cores, and Hadji *et al.* (2024, 2025) on buckling, free vibration, and dynamic responses of multidirectional FG sandwich plates and porous FG beams with imperfections.

In this work a detailed study on the free vibration behavior of multilayered sandwich shells is presented. The non-dimensional natural frequencies of five-layered symmetric and anti-symmetric sandwich shells are determined for various structural and geometric parameters. In the present study, a new hyperbolic shear deformation shell theory to account for the effects of transverse shear and normal deformations is developed considering six degrees of freedom. The equations of motion are produced by employing Hamilton's principle. Semi-analytical closed-form solutions for the free vibration problems are made by the Navier technique for simply supported boundary conditions of the shell. The non-dimensional natural frequencies of five-layer symmetric (0°/90°/core/90°/0°) and anti-symmetric (0°/90°/core/0°/90°) sandwich shells are obtained for various geometric parameters. The present results are compared with those presented by Kant and Swaminathan (2001), Sayyad and Ghugal (2015), Reddy (1984), Rao and Desai (2004), Pandya and Kant (1988), Senthilnathan *et al.* (1988), Whitney-Pagano (1970).

## 2. Research gap in the existing literature and the present objectives

1. It has been proved in the literature (Carrera 1999a, b) that the analysis of laminated composite plates and shells is meaningless if the effects of transverse normal strains are discarded from the mathematical modelling and the development of the theory. Based on the aforementioned literature review, it is found that despite of the fact that several higher-order shear deformation and layer-wise theories have been developed in the literature considering the effects of transverse normal strain on the free vibration analysis of laminated plates and shells such as Pandya and Kant (1988), Verijenko (1993), Kant and Manjunatha (1994), Kant and Swaminathan (2001a, b, 2002), Khare *et al.* (2005), Bosia *et al.* (2004), Ghugal and Sayyad (2011), Asadi *et al.* (2012), Brischetto (2014), Sayyad and Ghugal (2014, 2015, 2016), Shah and Batra (2018), Sayyad *et al.* (2023), Jape and Sayyad (2023, 2025), Shaikh and Sayyad (2024), Shinde and Sayyad (2022), Tornabene *et al.* (2022, 2023a, b, 2024e), Kolapkar and Sayyad (2024, 2025a, b), Tamnar and Sayyad (2025), etc. But, use of transverse normal strain to address the free vibration response of multi-layered sandwich shells remains limited in the literature and needs more attention. Thus, the novelty lies in extending the refined hyperbolic function to capture transverse normal effects in five-layered symmetric and antisymmetric sandwich shells, providing new benchmark frequency solutions where the literature is sparse. Hence, the first objective of this study is to formulate and apply a modified hyperbolic shell theory for the free vibration analysis of multi-layered laminated sandwich shells considering the effects of transverse normal strain.
2. Based on the above literature survey it is observed that plenty of research studies have been reported in the literature on the free vibration analysis of antisymmetric sandwich plates and few studies on antisymmetric sandwich shells. However, literature on the free vibration analysis of multi-layered symmetric sandwich shells is rare. Therefore, the second objective of the present study is to perform a free vibration analysis of five-layer symmetric (0°/90°/core/90°/0°) and anti-symmetric (0°/90°/core/0°/90°) sandwich shells.
3. The numerous studies have been addressed on the free vibration analysis of laminated shells and sandwich plates by Carrera and co-workers (Carrera and Scano 2024, Pagani *et al.* 2024, Pagani *et al.* 2023, Li and Carrera 2020, Carrera *et al.* 2020, 2025, Foroutan *et al.* 2022) considering the effects of transverse normal strain. However, the problems presented in this paper on the free vibration analysis of multi-layered sandwich shallow shells incorporating transverse normal strain effects are not addressed in these studies. These problems have been addressed for the first time in the present study.
4. The third objective of the present study is to analyse the effects of a wide range of parameters such as $a/b$, $a/h$, $R/a$, $t_c/t_f$ and modes of vibration on the natural frequencies of multi-layered sandwich shells to understand their influence on the vibration behaviour of sandwich shells.

Based on the aforementioned research gaps and the objectives of the present study, a modified hyperbolic shell theory to account for the effects of transverse shear and normal deformations is formulated here. A theory involves six unknowns and satisfies traction-free boundary conditions at the top and the bottom surfaces of the shell. The equations of motion are produced by employing Hamilton's principle. Semi-analytical solutions for the free vibration problems are obtained by the

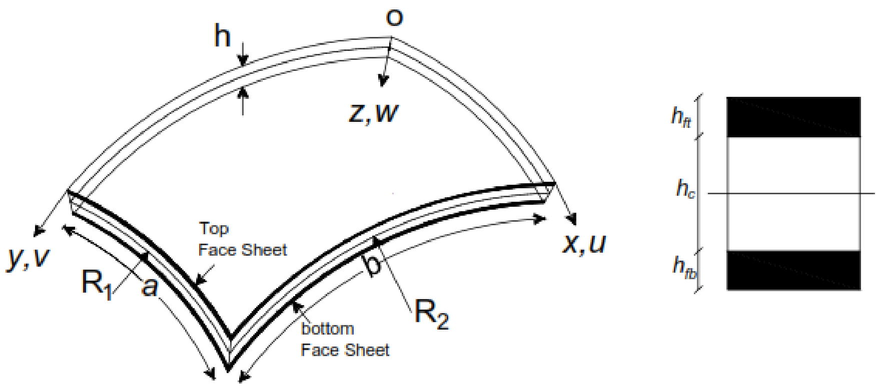

Fig. 1 Geometry and coordinates of the sandwich shell under consideration

Navier technique. The non-dimensional natural frequencies of five layered symmetric and anti-symmetric sandwich shells are obtained for various parameters. The present results are compared with results that have already been published to confirm the accuracy and efficiency of the current hyperbolic shell theory.

## 3. Kinematic assumption and formulation

A sandwich shell element under consideration is shown in Fig. 1 A geometry of the shell element is considered in the (*x*, *y*, z) coordinate system on a rectangular platform. The *x* and *y* curves show the in-plane directions where *z* represents the transverse direction which is taken as positive in downward. $R_1$ and $R_2$ present the principal radii of curvature of the mid-plane along the *x* and *y* axes. A mathematical formulation of the present theory is based on a few kinematic assumptions.

1. In-plane displacement polynomials of the theory consist of three components (extension, bending, and shear). The extension components ($u_0$, $v_0$) are middle surface displacements in *x*- and *y*- directions; the bending components ($w_0$) are analogous to displacement in classical shell theory; and shear components are assumed to be hyperbolic functions in terms of thickness coordinates.
2. The Transverse displacement polynomial of the theory is a function of thickness coordinate (*z*) which accounts for the effects of transverse normal strain.
3. The face sheets of the sandwich shell are made up of fibrous composite material and the middle core is made up of isotropic or foam material.
4. All the layers of the sandwich shell are assumed to be perfectly bonded together.

The present theory is developed by introducing a new hyperbolic shape function in the displacement polynomial to represent the expansions of in-plane and transverse displacements. Following is the displacement polynomial assumed for the present hyperbolic shell theory considering the effects of transverse shear and normal deformations. The theory involves six unknown variables in the polynomial which are functions of (*x*, *y*, *t*) variables.

$$u(x,y,z,t) = \left(1+\frac{z}{R_1}\right)u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + f(z)\theta_x(x,y,t)$$

$$v(x,y,z,t) = \left(1+\frac{z}{R_2}\right)v_0(x,y,t) - z\frac{\partial w_0}{\partial y} + f(z)\theta_y(x,y,t)$$
$$w(x,y,z,t) = w_0(x,y,t) + f'(z)\theta_z(x,y,t) \tag{1}$$

where $u$, $v$, $w$ are the displacements at any point of the shell surface; $u_0$, $v_0$, $w_0$ is the displacement of any point of the middle surface of the shell; and $\theta_x, \theta_y, \theta_z$ are the unknown rotations to be determined. Here, $f(z)$ represents the shape function assumed to represent the transverse shear strain distributions through the thickness of the shell in the realistic form. Non-zero strain components from the displacement polynomial stated in Eq. (1) are obtained using the linear theory of elasticity (14). According to Niordson (1985) and Donnell-Mushtari-Vlasov kinematics, $\left(1+\frac{z}{R}\right)$ factors are the typical shallow-shell scaling in principal-curvature directions. It is standard shallow shell assumption according to Donnell (1933), Mushtari (1938), Vlasov (1949). The following strain components apply to shallow shells.

$$\varepsilon_x = \left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_1}\right) - z\frac{\partial^2 w_0}{\partial x^2} + f(z)\frac{\partial \theta_x}{\partial x} + \frac{f'(z)}{R_1}\theta_z$$
$$\varepsilon_y = \left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_2}\right) - z\frac{\partial^2 w_0}{\partial y^2} + f(z)\frac{\partial \theta_y}{\partial y} + \frac{f'(z)}{R_2}\theta_z$$
$$\varepsilon_z = f''(z)\theta_z$$
$$\gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2z\frac{\partial^2 w_0}{\partial x \partial y} + f(z)\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \tag{2}$$
$$\gamma_{xz} = f'(z)\theta_x + f'(z)\frac{\partial \theta_x}{\partial x}$$
$$\gamma_{yz} = f'(z)\theta_y + f'(z)\frac{\partial \theta_z}{\partial y}$$

Where

$$f(z) = \left[z\cosh\left(\frac{\xi}{2}\right)\right] - \left[\left(\frac{h}{\xi}\right)\sinh\left(\frac{\xi z}{h}\right)\right]$$
$$f'(z) = \left[\cosh\left(\frac{\xi}{2}\right)\right] - \left[\cosh\left(\frac{\xi z}{h}\right)\right] \tag{3}$$
$$\xi = 2.634$$

The value of the shear correction parameter $\xi$ =2.634 has been used by Jape and Sayyad (2023), where this constant was derived to ensure that the transverse shear stresses disappear at the top and bottom surfaces, while keeping close agreement with three-dimensional elasticity solutions. Jape and Sayyad (2023) also presented convergence of this constant in their study. In-plane and transverse stresses of a $k^{th}$ layer of multilayered sandwich shells are also calculated using the linear theory of elasticity (14).

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \tag{4}$$

where $(\sigma,\tau)^k$ are the stress components; $(\varepsilon,\gamma)^k$ are the strain components, and $(\mathrm{Q_{ij}})^k$ are the reduced stiffness coefficients at any point of the shell domain.

$$\begin{gathered}Q_{11}=\frac{E_1(1-\mu_{23}\mu_{32})}{\Delta};Q_{12}=\frac{E_1(\mu_{21}+\mu_{31}\mu_{23})}{\Delta};Q_{13}=\frac{E_1(\mu_{31}+\mu_{21}\mu_{32})}{\Delta};\\ Q_{22}=\frac{E_2(1-\mu_{13}\mu_{31})}{\Delta};Q_{23}=\frac{E_2(\mu_{32}+\mu_{12}\mu_{31})}{\Delta};Q_{33}=\frac{E_3(1-\mu_{12}\mu_{21})}{\Delta};\\ Q_{44}=G_{23};Q_{55}=G_{13};Q_{66}=G_{12};\\ \Delta=1-\mu_{12}\mu_{21}-\mu_{23}\mu_{32}-\mu_{13}\mu_{31}-2\mu_{21}\mu_{32}\mu_{13}\end{gathered} \tag{5}$$

The equations of motion associated with the present theory are formulated using Hamilton's principle stated in Eq. (6).

$$\int_{t1}^{t2}(\delta U-\delta V+\delta K) \tag{6}$$

where $\delta$ is the variational operator, $t_1$ and $t_2$ are the initial and final time; $\delta U$ represents strain energy, $\delta V$ represents potential energy, and $\delta K$ represents kinetic energy. After the substitution of expressions of these energies into Eq. (6), it takes the following form where $\rho$ is the density of the material.

$$\begin{gathered}\textstyle\sum_{k=1}^{N}\int_0^a\int_0^b\int_{h_k}^{h_{k+1}}(\sigma_x\delta\mathcal{E}_x+\sigma_y\delta\mathcal{E}_y+\sigma_z\delta\mathcal{E}_z+\tau_{xy}\delta\Upsilon_{xy}+\tau_{xz}\delta\Upsilon_{xz}+\tau_{yz}\delta\Upsilon_{yz})\,dzdydx-\\ \int_0^a\int_0^b\int_{h_k}^{h_{k+1}}q\ dzdydx+\textstyle\sum_{k=1}^{N}\int_0^a\int_0^b\int_{h_k}^{h_{k+1}}\rho^k\left(\frac{\partial^2u}{\partial^2t}\delta u+\frac{\partial^2v}{\partial^2t}\delta v+\frac{\partial^2w}{\partial^2t}\delta w\right)dzdydx=0\end{gathered} \tag{7}$$

To obtain the six equations of motion, Eq. (7) needs to be integrated by parts after substituting the strains into it. Therefore, after performing integration by parts; collecting the coefficients of unknown variables $(u_0,v_0,w_0,\theta_x,\theta_y,\theta_z)$ and equating them with zero, the following equations of motion are obtained.

$$\begin{gathered}\delta u_0\!:\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=\left(I_1+2\frac{I_2}{R_1}+\frac{I_3}{R_1^2}\right)\frac{\partial^2u_0}{\partial t^2}-\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3w_0}{\partial x\partial t^2}+\left(I_4+\frac{I_6}{R_1}\right)\frac{\partial^2\theta_x}{\partial t^2}\\ \delta v_0\!:\frac{\partial N_{yy}}{\partial y}+\frac{\partial N_{xy}}{\partial x}=\left(I_1+2\frac{I_2}{R_2}+\frac{I_3}{R_2^2}\right)\frac{\partial^2v_0}{\partial t^2}-\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3w_0}{\partial y\partial t^2}+\left(I_4+\frac{I_6}{R_2}\right)\frac{\partial^2\theta_y}{\partial t^2}\\ \delta w_0\!:\frac{\partial^2M_{xx}^b}{\partial x^2}+\frac{\partial^2M_{yy}^b}{\partial y^2}+2\frac{\partial^2M_{xy}^b}{\partial x\partial y}-\frac{N_{xx}}{R_1}-\frac{N_{yy}}{R_2}+q=\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3u_0}{\partial x\partial^2t}-I_3\frac{\partial^4w_0}{\partial x^2\partial t^2}\\ +I_6\frac{\partial^3\theta_x}{\partial x\partial t^2}+\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3v_0}{\partial y\partial t^2}-I_3\frac{\partial^4w_0}{\partial y^2\partial t^2}+I_6\frac{\partial^3\theta_y}{\partial y\partial t^2}+I_1\frac{\partial^2w_0}{\partial t^2}+I_7\frac{\partial^2\theta_z}{\partial t^2}\\ \delta\theta_x\!:\ \frac{\partial M_{xx}^S}{\partial x}+\frac{\partial M_{xy}^S}{\partial y}-Q_{xz}=\left(I_4+\frac{I_6}{R_1}\right)\frac{\partial^2u_0}{\partial t^2}-I_6\frac{\partial^3w_0}{\partial x\partial t^2}+I_5\frac{\partial^2\theta_x}{\partial t^2}\\ \delta\theta_y\!:\ \frac{\partial M_{yy}^S}{\partial y}+\frac{\partial M_{xy}^S}{\partial x}-Q_{yz}=\left(I_4+\frac{I_6}{R_2}\right)\frac{\partial^2v_0}{\partial t^2}-I_6\frac{\partial^3w_0}{\partial y\partial t^2}+I_5\frac{\partial^2\theta_y}{\partial t^2}\\ \delta\theta_z\!:\ \frac{\partial \mathrm{Q}_{xz}^s}{\partial x}+\frac{\partial \mathrm{Q}_{yz}^s}{\partial y}-\frac{\mathrm{V}_{xx}^s}{R_1}-\frac{\mathrm{V}_{yy}^s}{R_2}-\mathrm{V}_{zz}^s=I_7\frac{\partial^2w_0}{\partial t^2}+I_8\frac{\partial^2\theta_z}{\partial t^2}\end{gathered} \tag{8}$$

where

$$\left(N_{xx},N_{yy},N_{xy},M_{xx}^{b},M_{yy}^{b},M_{xy}^{b}\right)=\int_{-h/2}^{h/2}\left(\sigma_{x},\sigma_{y},\tau_{xy},z\sigma_{x},z\sigma_{y},z\tau_{xy}\right)dz;$$
$$\left(M_{xx}^{S},M_{yy}^{S},M_{xy}^{S}\right)=\int_{-h/2}^{h/2}\left[f(z)\left(\sigma_{x},\sigma_{y},\tau_{xy}\right)\right]dz;$$
$$\left(Q_{xz},Q_{yz}\right)=\int_{-h/2}^{h/2}\left[f'(z)\left(\tau_{xz},\tau_{yz}\right)\right]dz; \tag{9}$$
$$\left(\mathrm{V}_{xx}^{s},\mathrm{V}_{yy}^{s},\mathrm{Q}_{xz}^{s},\mathrm{Q}_{yz}^{s}\right)=\int_{-h/2}^{h/2}\left[f'(z)\left(\sigma_{x},\sigma_{y},\tau_{xz},\tau_{yz}\right)\right]dz;$$
$$\mathrm{V}_{zz}^{s}=\int_{-h/2}^{h/2}\left(\sigma_{z}f''(z)\right)dz$$

and

$$(I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8)=\int_{-h/2}^{h/2}\rho[1,z,z^2,f(z),[f(z)]^2,zf(z),f'(z),[f'(z)]^2]dz \tag{10}$$

One can find the expressions of stress resultants by substituting stress expression into Eq. (9) and then integrating concerning thickness coordinate $z$. Once the expressions of stress resultants are derived, those can be used to determine the equations of motion in terms of unknown displacement variables $\left(u_0,v_0,w_0,\theta_x,\theta_y,\theta_z\right)$. These equations of motion are expanded into a system of six coupled PDEs in terms of the displacements as shown in the Appendix A.

## 4. Free vibration analysis

According to the literature, Navier's technique is the simplest technique available for the analysis of simply supported beams, plates and shells. Therefore, a semi-analytical solution for the free vibration problem of five-layered sandwich shells is obtained using the Navier method. For the free vibration analysis, a transverse load acting on the structure is assumed to be zero i.e., $q$ ($x$, $y$)=0. The unknown variables involved in the equations of motion are assumed in the following double trigonometric series form.

$$\begin{Bmatrix}u_0\\v_0\\w_0\\\theta_x\\\theta_y\\\theta_z\end{Bmatrix}=\begin{Bmatrix}u_{mn}\cos\alpha x\sin\beta y\\v_{mn}\sin\alpha x\cos\beta y\\w_{mn}\sin\alpha x\sin\beta y\\\theta_{xmn}\cos\alpha x\sin\beta y\\\theta_{ymn}\sin\alpha x\cos\beta y\\\theta_{zmn}\sin\alpha x\sin\beta y\end{Bmatrix}e^{i\omega t} \tag{11}$$

where $(u_{mn}, v_{mn}, w_{mn}, \theta_{xmn}, \theta_{ymn}, \theta_{zmn})$ are the unknown coefficients to be found; $\alpha = m\pi/a$, $\beta = n\pi/b$; $m$ and $n$ are the positive integers defining the modes of vibration; and $\omega$ is the natural frequency of the shell. When Eq. (11) is substituted into equations of motion as shown in Appendix A, the Eigenvalue problem mentioned in Eq. (12) is obtained.

$$([K] - \omega^2[M])\{\Delta\} = 0 \tag{12}$$

The Nontrivial solution of Eigen value Eq. (12) is obtained when $|K - \omega^2 M\ | = 0$. Here matrix $[K]$ represents the stiffness matrix and matrix ($M$) represents the mass matrix. Elements of these matrices are given in the Appendix B. Solution of Eq. (12) gives the natural frequencies for various cases of multi-layered sandwich shells.

*Numerical Result and Discussion*

In this study, non-dimensional natural frequencies are obtained for five layered symmetric (0°/90°/core/90°/0°) and anti-symmetric (0°/90°/core/0°/90°) sandwich plates and shells. The following material properties are used to generate the numerical results of natural frequencies.

Face Sheet (Graphite Epoxy)

$$\begin{gathered} E_1 = 19 \times 10^6 \text{ psi (131 Gpa)}, E_2 = 1.5 \times 10^6 \text{ psi (10.34 Gpa)}, E_2 = E_3 \\ G_{12} = 1 \times 10^6 \text{ (6.895 Gpa)}, G_{13} = 0.90 \times 10^6 \text{(6.205 Gpa)}, G_{23} = 1 \times 10^6 \text{ (6.895 Gpa)} \\ \mu_{12} = 0.22, \mu_{13} = 0.22, \mu_{23} = 0.49, \rho = 0.057 \text{ lb/inch}^3 \text{(1637 kg/m}^3\text{)} \end{gathered} \tag{13}$$

Core Properties (isotropic)

$$\begin{gathered} E_2 = E_2 = E_3 = 2\text{G} = 100 \text{ psi } (6.89 \times 10^{-3} \text{ Gpa}) \\ G_{12} = G_{13} = G_{23} = 500 \text{ psi } (3.45 \times 10^{-3} \text{ Gpa}) \\ \mu_{12} = \mu_{13} = \mu_{23} = 0 \\ \rho = 0.3403 \times 10^{-2} \text{ lb/inch}^3 (97 \text{ kg/m}^3) \end{gathered} \tag{14}$$

For comparison purposes, the numerical results are presented in the following non-dimensional form.

$$\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2}) \tag{15}$$

where $E_2$ is the modulus of elasticity of the core of cylindrical and spherical shells. A sandwich shell consists of five layers. The overall thickness of the sandwich shell is $h$ and $t_c/t_f$=10 unless and until mentioned. Here, $t_c$ represents the thickness of the core and $t_f$ represents the thickness of the face sheets.

## *5.1 Free vibration analysis of multilayered sandwich plates*

For the verification of the present theory and the validation of the present formulation, the free vibration analysis of symmetric and anti-symmetric sandwich plates is performed. The non-dimensional fundamental frequencies of an anti-symmetric (0°/90°/core/0°/90°) sandwich plates by varying side-to-thickness ratio (*a/h*) are reported in Table 1, for the varying aspect ratio (*a/b*) are shown in Table 2 and for the varying thickness of the core to thickness of the face sheet ($t_c/t_f$), those are presented in Table 3. The obtained results are compared with those presented by Kant and Swaminathan (2001), Sayyad and Ghugal (2015), Reddy (1984), Rao and Desai (2004), Pandya and Kant (1988), Senthilnathan *et al.* (1988), Whitney-Pagano (1970). The close

Table 1 Non-dimensional fundamental frequencies $\left(\bar{\omega} = (\omega b^2/h)\sqrt{\rho/E_2}\right)$ of an antisymmetric (0°/90°/core/0°/90°) sandwich plates (*a/b*=1 and $t_c/t_f$=10)

| *a/h* | Present | Kant and Swaminathan (2001) | Sayyad and Ghugal (2015) | Reddy (1984) | Rao and Desai (2004) | Pandya and Kant (1988) | Senthilnathan *et al.* (1988) | Whitney-Pagano (1970) |
|---|---|---|---|---|---|---|---|---|
| 2 | 1.0509 | 1.1734 | 0.8209 | 1.6252 | 0.7141 | 1.1734 | 1.6252 | 5.2017 |
| 4 | 2.0339 | 2.0913 | 1.6439 | 3.1013 | 0.9363 | 2.0913 | 3.1013 | 9.0312 |
| 10 | 4.8271 | 4.8519 | 3.9964 | 7.0473 | 1.848 | 4.8519 | 7.0473 | 13.869 |
| 20 | 8.4476 | 8.5838 | 7.2820 | 11.266 | 3.4791 | 8.5838 | 11.266 | 15.530 |
| 50 | 13.029 | 13.6577 | 12.300 | 15.032 | 7.7355 | 13.658 | 15.032 | 16.126 |
| 100 | 14.544 | 15.4647 | 14.347 | 15.952 | 11.94 | 15.465 | 15.952 | 16.218 |

Table 2 Non-dimensional fundamental frequencies $\left(\bar{\omega} = (\omega b^2/h)\sqrt{\rho/E_2}\right)$ of an anti-symmetric (0°/90°/core/0°/90°) sandwich plates (*a/h*=10 and $t_c/t_f$=10)

| *a/b* | Present | Kant and Swaminathan (2001) | Reddy (1984) | Senthilnathan *et al.* (1988) | Whitney-Pagano (1970) |
|---|---|---|---|---|---|
| 0.5 | 15.045 | 15.032 | 21.45 | 21.667 | 39.484 |
| 1.0 | 4.8496 | 4.8594 | 7.0473 | 7.0473 | 13.869 |
| 1.5 | 2.8050 | 2.8188 | 4.1725 | 4.1725 | 9.4910 |
| 2.0 | 2.2527 | 2.4560 | 3.6582 | 3.6582 | 10.166 |
| 2.5 | 1.5413 | 1.5719 | 2.3413 | 2.3413 | 6.5059 |
| 3.0 | 1.2651 | 1.3040 | 1.9216 | 1.9216 | 5.6588 |
| 5.0 | 0.7502 | 0.8187 | 1.1550 | 1.1550 | 3.6841 |

Table 3 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2})$ of an antisymmetric (0°/90°/core/0°/90°) sandwich plates (*a/b*=1 and *a/h*=10)

| $t_c/t_f$ | Present | Kant and Swaminathan (2001) | Reddy (1984) | Senthilnathan *et al.* (1988) | Whitney- Pagano (1970) |
|---|---|---|---|---|---|
| 4 | 8.7855 | 8.9948 | 10.7409 | 10.7409 | 13.9190 |
| 10 | 4.8271 | 4.8594 | 7.0473 | 7.0473 | 13.8694 |
| 20 | 3.3124 | 3.1435 | 4.3734 | 4.3734 | 12.8946 |
| 30 | 3.0690 | 2.8481 | 3.4815 | 3.4815 | 11.9760 |
| 40 | 3.0387 | 2.8266 | 3.1664 | 3.1664 | 11.2036 |
| 50 | 3.0472 | 2.8625 | 3.0561 | 3.0561 | 10.5557 |
| 100 | 3.0678 | 3.0290 | 3.0500 | 3.0500 | 8.4349 |

agreement of the present results with Kant and Swaminathan (2001) arises because both the models account for transverse shear and normal deformation effects, unlike classical or first-order theories. The model of Kant and Swaminathan (2001) expanded upto third-order based on Tylor series expansion and involves 12 DOF whereas the present model involves 6 DOF. Models such as Whitney-Pagano (1970) overestimate frequencies due to their first-order variation through the thickness, whereas Reddy's (1984) parabolic theory, though accurate, does not fully capture transverse normal strain effects. The models of Whitney-Pagano (1970), Reddy (1984) involves 5 DOF in the displacement field. As a side-to-thickness ratio (*a/h*) increases the fundamental

Table 4 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2})$ of an antisymmetric (0°/90°/core/0°/90°) sandwich plates (*a/b*=1 and $t_c/t_f$=10)

| *a/h* | Source | Modes of vibration | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 |
| 10 | Present | 4.8270 | 7.9083 | 10.086 | 11.365 | 12.983 | 14.989 |
| | Sayyad and Ghugal (2015) | 4.1312 | 6.7339 | 8.6150 | 9.6638 | 11.088 | 13.123 |
| | Reddy (1984) | 7.0473 | 11.908 | 15.289 | 17.321 | 19.812 | 23.506 |
| | Rao and Desai (2004) | 4.9624 | 8.1934 | 10.517 | 11.986 | 13.748 | 16.451 |
| | Kant and Manjunatha (1994) | 4.8594 | 8.0187 | 10.296 | 11.738 | 13.470 | 16.132 |
| | Pandya and Kant (1988) | 4.8519 | 7.9965 | 10.255 | 11.680 | 13.388 | 16.003 |
| | Senthilnathan *et al.* (1988) | 7.0473 | 11.962 | 15.289 | 17.369 | 19.832 | 23.506 |
| | Whitney-Pagano (1970) | 13.869 | 30.643 | 41.557 | 50.938 | 58.363 | 71.372 |
| 100 | Present | 14.544 | 37.128 | 52.117 | 70.195 | 80.277 | 101.73 |
| | Sayyad and Ghugal (2015) | 15.597 | 38.377 | 53.516 | 69.802 | 80.072 | 100.39 |
| | Reddy (1984) | 15.952 | 42.227 | 60.127 | 83.998 | 96.313 | 124.20 |
| | Rao and Desai (2004) | 15.548 | 39.265 | 73.495 | 55.151 | 84.291 | 106.58 |
| | Kant and Manjunatha (1994) | 15.509 | 39.029 | 54.761 | 72.757 | 83.441 | 105.37 |
| | Pandya and Kant (1988) | 15.464 | 38.923 | 54.633 | 72.592 | 83.269 | 105.18 |
| | Senthilnathan *et al.* (1988) | 15.952 | 42.370 | 60.127 | 84.421 | 96.725 | 124.20 |
| | Whitney-Pagano (1970) | 16.217 | 44.707 | 64.504 | 94.909 | 108.90 | 143.79 |

Table 5 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2}$ of symmetric (0°/90°/core/90°/0°) sandwich rectangular plates ($t_c/t_f$=10)

| *a/b* | Source | Modes | *a/h* | | | |
|---|---|---|---|---|---|---|
| | | | 100 | 50 | 10 | 5 |
| 1 | Present | I (Flexure) | 16.164 | 15.824 | 10.413 | 6.2994 |
| | | II (Thickness-shear) | 772.36 | 386.18 | 77.236 | 38.618 |
| | | III (Thickness-twist) | 889.68 | 444.84 | 88.968 | 42.988 |
| | Sayyad and Ghugal (2015) | I (Flexure) | 16.129 | 15.702 | 9.6657 | 5.7107 |
| | | II (Thickness-shear) | 771.60 | 385.80 | 77.161 | 38.580 |
| | | III (Thickness-twist) | 861.48 | 430.73 | 86.148 | 43.074 |
| | Exact 3D (2014) | I (Flexure) | 15.754 | 14.440 | 5.9275 | 3.2639 |
| | | II (Thickness-shear) | 771.58 | 385.73 | 71.631 | 17.398 |
| | | III (Thickness-twist) | 861.39 | 430.56 | 76.817 | 37.351 |
| 3 | Present | I (Flexure) | 11.487 | 11.295 | 7.9034 | 4.9441 |
| | | II (Thickness-shear) | 345.35 | 172.67 | 34.535 | 17.267 |
| | | III (Thickness-twist) | 807.39 | 403.69 | 76.372 | 24.199 |
| | Sayyad and Ghugal (2015) | I (Flexure) | 11.448 | 11.169 | 7.0265 | 4.1626 |
| | | II (Thickness-shear) | 342.45 | 171.22 | 34.245 | 17.122 |
| | | III (Thickness-twist) | 791.08 | 395.54 | 79.108 | 39.554 |
| | Exact 3D (Brischetto 2014) | I (Flexure) | 11.231 | 10.421 | 4.5385 | 2.4968 |
| | | II (Thickness-shear) | 342.45 | 171.22 | 34.219 | 19.965 |
| | | III (Thickness-twist) | 791.01 | 395.40 | 48.460 | 17.876 |

Table 6 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2})$ of an antisymmetric (0°/90°/core/0°/90°) sandwich shells (*a/h*=10, *a/b*=1 and $t_c/t_f$=10)

| Shell Panel | Reference | *R/a* | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 10 | 20 |
| Cylindrical | Present | 7.15669 | 5.54208 | 5.16004 | 5.01734 | 4.94958 | 4.85763 | 4.83452 |
| | Khare *et al.* (2005) | 7.64969 | 5.80102 | 5.35586 | 5.18814 | 5.10809 | 4.99873 | 4.97087 |
| Spherical | Present | 11.5345 | 7.33638 | 6.09499 | 5.58098 | 5.3235 | 4.95642 | 4.85977 |
| | Khare *et al.* (2005) | 12.9654 | 8.02198 | 6.52916 | 5.90091 | 5.58291 | 5.12468 | 5.00286 |

Table 7 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2})$ of an antisymmetric (0°/90°/core/0°/90°) sandwich shells (*a/b*=1 and $t_c/t_f$=10)

| Shell Panel | *R/a* | *a/h* | | | | | |
|---|---|---|---|---|---|---|---|
| | | 2 | 4 | 10 | 20 | 50 | 100 |
| Cylindrical Shell | 1 | 1.47806 | 2.92341 | 7.15669 | 13.5999 | 30.05925 | 56.45014 |
| | 2 | 1.18164 | 2.30491 | 5.54208 | 10.0773 | 19.10814 | 31.68311 |
| | 3 | 1.11216 | 2.16007 | 5.16004 | 9.21497 | 16.05046 | 23.83129 |
| | 4 | 1.08624 | 2.10613 | 5.01734 | 8.88837 | 14.81138 | 20.31888 |
| | 5 | 1.07389 | 2.08053 | 4.94958 | 8.73225 | 14.19751 | 18.45622 |
| | 10 | 1.05695 | 2.04572 | 4.85763 | 8.51914 | 13.33131 | 15.61698 |
| | 20 | 1.05253 | 2.03689 | 4.83452 | 8.46522 | 13.10529 | 14.81981 |
| Spherical Shell | 1 | 2.32346 | 4.64552 | 11.5345 | 22.6562 | 54.49307 | 107.1349 |
| | 2 | 1.51966 | 2.99967 | 7.33638 | 13.9364 | 30.78814 | 57.81647 |
| | 3 | 1.28518 | 2.51853 | 6.09499 | 11.2740 | 22.86444 | 40.39864 |
| | 4 | 1.18936 | 2.32098 | 5.58098 | 10.1462 | 19.22974 | 31.88092 |
| | 5 | 1.14175 | 2.22253 | 5.32350 | 9.57311 | 17.26870 | 26.98857 |
| | 10 | 1.07443 | 2.08289 | 4.95642 | 8.74391 | 14.21333 | 18.47512 |
| | 20 | 1.05683 | 2.04629 | 4.85977 | 8.52276 | 13.33550 | 15.62113 |

frequency is increased due to a decrease in the thickness of the sandwich plate and flexural rigidity. Table 2 reveals that the non-dimensional frequencies decrease with an increase in the aspect ratio i.e., square to rectangular sandwich plates. The examination of Table 3 reveals that the increase in the ($t_c/t_f$) ratio decreases the values of non-dimensional fundamental frequencies. This is in fact due to an increase in the $t_c/t_f$ ratio increases the thickness of the core (soft material) and reduces the thickness of face sheets (stiff material) which ultimately reduces the stiffness of the sandwich shell. The present theory is also validated for predicting the frequencies of higher modes of vibration after comparing the present results with those presented by Sayyad and Ghugal (2015), Reddy (1984), Rao and Desai (2004), Kant and Manjunatha (1994), Pandya and Kant (1988), Senthilnathan *et al.* (1988), Whitney-Pagano (1970), etc. The comparison is shown in Table 4 which demonstrates the strong agreement between present findings and previous literature. Table 4 also reveals that the non-dimensional frequencies are always more for thin plates and less for thick plates. At the end, the present model is applied for the free vibration analysis of rectangular sandwich plates and the obtained frequencies are presented in Table 5. The non-dimensional frequencies are obtained for different values of *a/h* ratio and *a/b*. An important contribution of this investigation is it consider the effect of different deformation modes such as

Table 8 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2}$ of an antisymmetric (0°/90°/core/0°/90°) sandwich shells (*a*/*b*=1 and *R*/*a*=1)

| Shell Panel | $t_c/t_f$ | *a/h* | | | | | |
|---|---|---|---|---|---|---|---|
| | | 2 | 4 | 10 | 20 | 50 | 100 |
| Cylindrical Shell | 4 | 2.39233 | 4.61414 | 10.43198 | 17.43177 | 34.38273 | 63.89932 |
| | 10 | 1.55695 | 3.04572 | 4.85763 | 11.51914 | 25.33131 | 45.61698 |
| | 20 | 1.14273 | 2.28061 | 4.28061 | 11.01709 | 25.45060 | 48.26348 |
| | 30 | 1.02836 | 2.05611 | 4.09125 | 9.89389 | 22.69266 | 42.92614 |
| | 40 | 0.96688 | 1.93331 | 4.77242 | 9.19778 | 20.75639 | 39.08619 |
| | 50 | 0.92687 | 1.85231 | 4.55398 | 8.68971 | 19.27456 | 36.14804 |
| | 100 | 0.83293 | 1.65654 | 3.97284 | 7.19574 | 14.92782 | 27.68367 |
| Spherical Shell | 4 | 3.11438 | 6.12801 | 14.54775 | 26.85557 | 61.92034 | 121.39230 |
| | 10 | 2.57443 | 3.08289 | 10.95642 | 19.74391 | 46.51333 | 48.47512 |
| | 20 | 1.91221 | 3.84357 | 9.59604 | 19.03804 | 46.48228 | 91.63595 |
| | 30 | 1.70452 | 3.43145 | 8.56716 | 16.97759 | 41.35443 | 81.46911 |
| | 40 | 1.56750 | 3.15728 | 7.87424 | 15.55582 | 37.68528 | 74.14533 |
| | 50 | 1.46752 | 2.95612 | 7.36075 | 14.48415 | 34.88079 | 68.54753 |
| | 100 | 1.19873 | 2.40970 | 5.92901 | 11.39963 | 26.78667 | 52.47091 |

Table 9 Non-dimensional fundamental frequencies $\bar{\omega} = (\omega b^2/h)(\sqrt{\rho/E_2})$ of a symmetric (0°/90°/core/90°/0°) sandwich shells (*a*/*b*=1 and $t_c/t_f$ =10)

| Shell Panel | *R/a* | *a/h* | | | | | |
|---|---|---|---|---|---|---|---|
| | | 2 | 4 | 10 | 20 | 50 | 100 |
| Cylindrical Shell | 1 | 2.95009 | 5.59420 | 11.82299 | 18.41223 | 35.18310 | 65.38892 |
| | 2 | 2.83694 | 5.31568 | 10.81260 | 15.25865 | 22.66196 | 36.48549 |
| | 3 | 2.81284 | 5.25772 | 10.59881 | 14.54639 | 19.20396 | 27.25145 |
| | 4 | 2.80414 | 5.23691 | 10.52166 | 14.28435 | 17.81220 | 23.09125 |
| | 5 | 2.80007 | 5.22718 | 10.48553 | 14.16064 | 17.12536 | 20.87228 |
| | 10 | 2.79460 | 5.21411 | 10.43693 | 13.99319 | 16.15959 | 17.46338 |
| | 20 | 2.79323 | 5.21082 | 10.42470 | 13.95088 | 15.90814 | 16.49817 |
| Spherical Shell | 1 | 3.56655 | 6.90382 | 15.65668 | 27.81212 | 63.34344 | 124.1834 |
| | 2 | 3.03168 | 5.73438 | 12.11045 | 18.85872 | 36.03647 | 66.97466 |
| | 3 | 2.90387 | 5.45429 | 11.21937 | 16.35557 | 26.94808 | 46.67733 |
| | 4 | 2.85630 | 5.34973 | 10.88051 | 15.35438 | 22.80421 | 36.71425 |
| | 5 | 2.83374 | 5.30007 | 10.71824 | 14.86272 | 20.58048 | 30.97007 |
| | 10 | 2.80312 | 5.23257 | 10.49617 | 14.17500 | 17.14276 | 20.89347 |
| | 20 | 2.79536 | 5.21545 | 10.43958 | 13.99675 | 16.16370 | 17.46783 |

flexure, thickness-shear and thickness-twist modes, to provide a comparative understanding of the vibrational behavior of multilayered sandwich plates. Present results are compared with Sayyad and Ghugal (2015) and Exact 3D (Brischetto 2014) which shows good agreement with them.

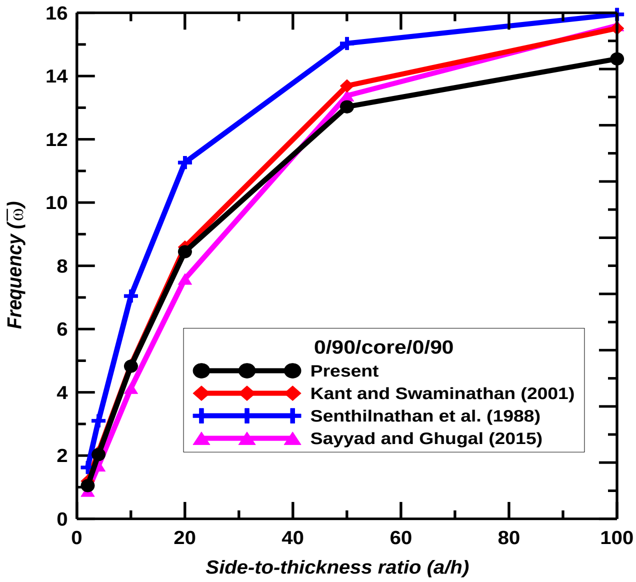


Fig. 2 Plot of non-dimensional frequencies ($\bar{\omega}$) versus side-to-thickness ratio ($a/h$) of a simply supported five-layer anti-symmetric sandwich plates ($a/b$=1 and $t_c/t_f$=10)

### *5.2 Free vibration analysis of multilayered sandwich shells*

Table 6 presented a non-dimensional fundamental frequency for anti-symmetric (0°/90°/core/0°/90°) sandwich shells on rectangular planform for varying radius-to-side ratio ($R/a$). The present results are compared with those reported by Khare *et al.* (2005) using higher-order shear deformation theory. Fundamental frequencies are presented for cylindrical and spherical shells. Table 6 demonstrates a strong agreement of the present results with Khare *et al.* (2005). Examination of Table 6 reveals that the fundamental frequencies are inversely proportional to the radii of curvature of the shell i.e., increase in the radii of curvature of the shell decreases their fundamental frequencies. Due to being singly-curved, the fundamental frequencies of cylindrical shells are lower than the spherical shells which have double curvature.

Reference solutions for the effects of various parameters ($a/h$, $t_c/t_f$) on the fundamental frequencies of multilayered sandwich shells are not available in the literature. Therefore, the numerical results presented in Tables 7-9 can be considered as benchmark solutions for future studies. The effects of varying $a/h$ ratio and $R/a$ ratio on the fundamental frequencies of anti-symmetric (0°/90°/core/0°/90°) sandwich shells are examined in Table 7. It is pointed out from Table 7 that the fundamental frequencies are maximum for thin shells and minimum for thick shells, also, fundamental frequencies decrease with an increase in the radii of curvature**.** Table 8 presents the non-dimensional fundamental frequency of anti-symmetric shells at $a/b$=1 $R/a$=1 for varying $t_c/t_f$ ratio $a/h$ ratio ranging from 2 to 100. Table 8 reveals that the $t_c/t_f$ ratio $a/h$ ratio provides a significant change in fundamental frequencies due to variations in the stiffness and rigidity of the shell. Spherical shells predict the higher frequencies due to double curvature effects compared to cylindrical shells. Table 9 presents the non-dimensional fundamental frequencies of symmetric sandwich shells at $a/b$=1, $t_c/t_f$=10 for varying $R/a$ and $a/h$ ratios. Table 9 reveals that curvature and geometric parameters significantly impact the fundamental frequencies of sandwich

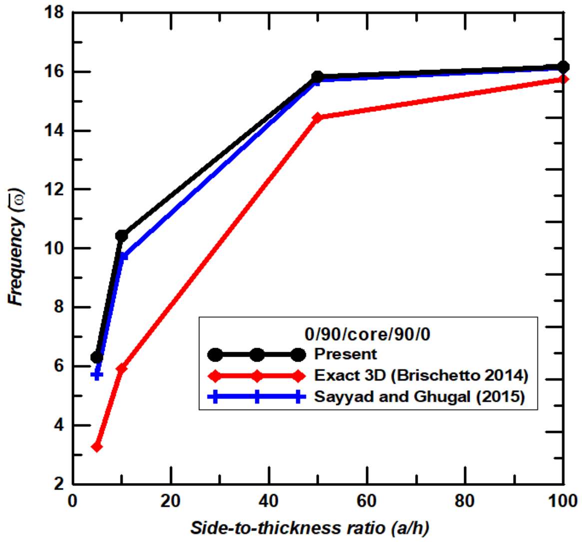


Fig. 3 Plot of non-dimensional frequencies ($\bar{\omega}$) versus side-to-thickness ratio (*a/h*) of a simply supported five-layer symmetric sandwich plates (*a/b*=1 and $t_c/t_f$=10)

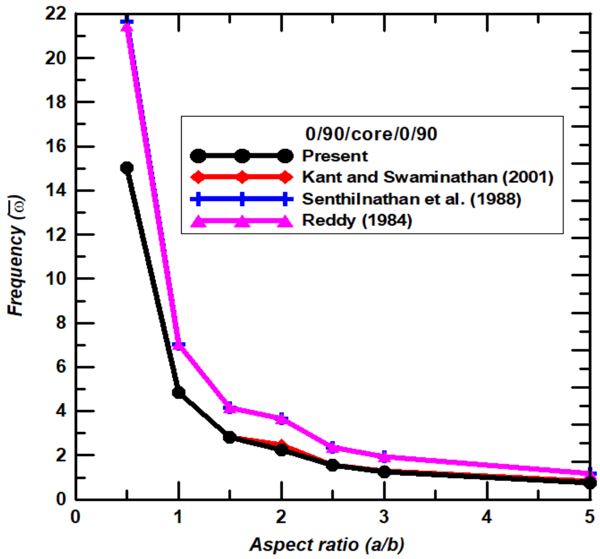


Fig. 4 Plot of non-dimensional frequencies ($\bar{\omega}$) versus aspect ratio (a/b) of a simply supported five-layer anti-symmetric sandwich plates (*a/h*=10 and $t_c/t_f$=10)

shells. The spherical shells exhibit higher fundamental frequencies than cylindrical shells due to structural rigidity.

Figs. 2 and 3 show the effects of side-to-thickness ratio on the natural frequencies of anti-symmetric and symmetric sandwich plates. These figures reveal that the non-dimensional natural frequencies are more for the thin plates. Figs. 4 and 5 show the influence of the a/b ratio and $t_c/t_f$ ratio on the frequencies of anti-symmetric sandwich plates. Fig. 5 reveals that the increase in the core thickness reduces the non-dimensional value of natural frequencies for five-layer anti-

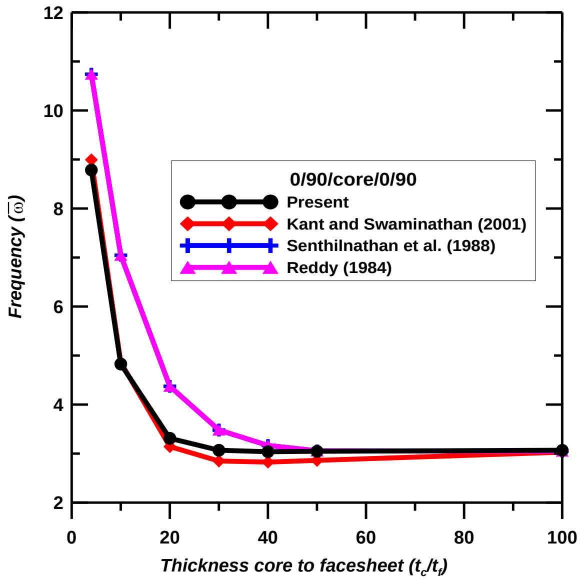


Fig. 5 Plot of non-dimensional frequencies ($\bar{\omega}$) versus thickness of core to thickness of face sheet ratio (*tc/tf*) of a simply supported five-layer anti-symmetric sandwich plates (*a/b*=1 and *a/h*=10)

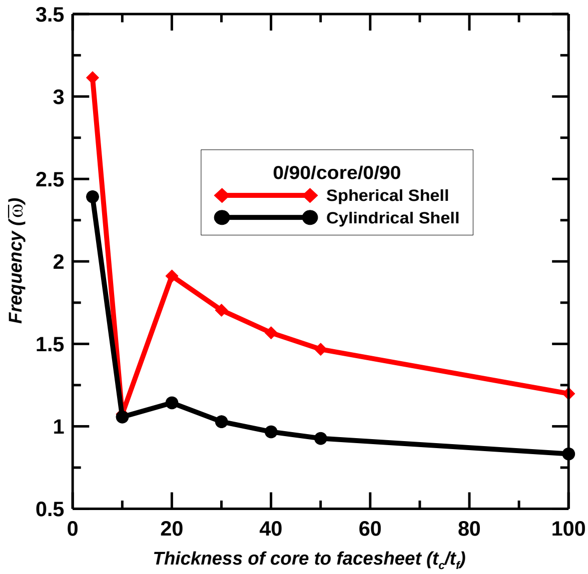


Fig. 6 Plot of non-dimensional frequencies ($\bar{\omega}$) versus thickness of core to thickness of face sheet ratio ($t_c/t_f$) of a simply supported five-layer anti-symmetric sandwich shell (*a/b*=1 and *a/h*=10)

symmetric sandwich plates. Similar results are presented for the five-layered symmetric sandwich shell in Fig. 6. The influence of radii of curvature as well as $t_c/t_f$ ratio for predicting the non-dimensional frequencies of symmetric and anti-symmetric sandwich shells are presented in Figs. 7 and 8. Also, the effects of *a/h* ratio on the non-dimensional values of natural frequencies of a five-layer anti-symmetric sandwich shell are plotted in Figs. 9 and 10.

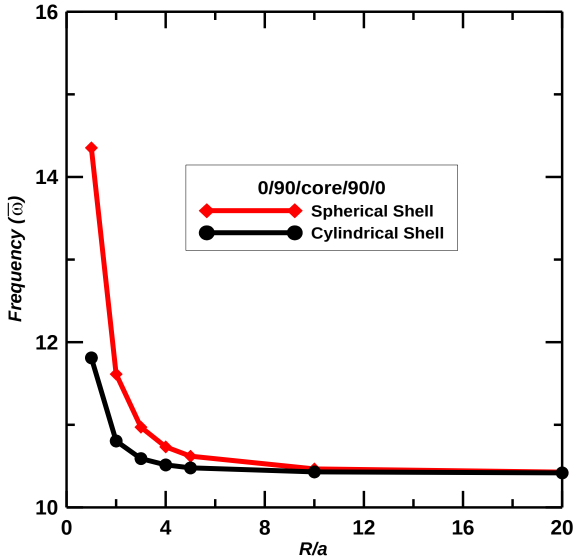


Fig. 7 Plot of non-dimensional frequencies ($\overline{\omega}$) versus ($R/a$) of a simply supported five-layer symmetric sandwich shell ($a/h$=10 and $t_c/t_f$=10)

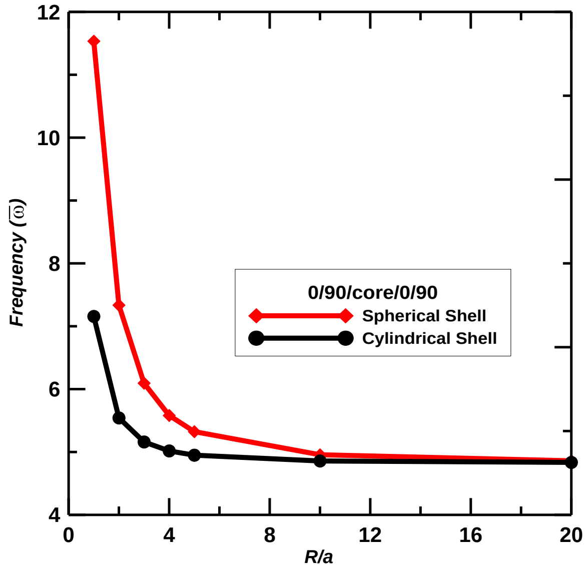


Fig. 8 Plot of non-dimensional frequencies ($\overline{\omega}$) versus ($R/a$) of a simply supported five-layer anti-symmetric sandwich shell ($a/h$=10 and $t_c/t_f$=10)

## 5. Conclusions

In this study, a refined hyperbolic shear deformation shell theory considering transverse shear and normal deformation effects has been developed for the free vibration analysis of multi-layered sandwich shallow shells. The equations of motion are derived using Hamilton's principle and solved semi-analytically by Navier's method for simply supported boundary conditions of the

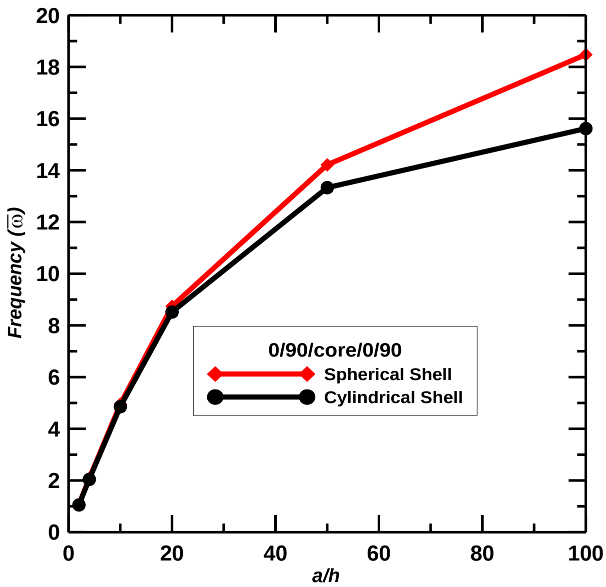


Fig. 9 Plot of non-dimensional frequencies ($\bar{\omega}$) versus ($a/h$) of a simply supported five-layer anti-symmetric sandwich shell ($a/b$=1 and $t_c/t_f$=10)

shell. The numerical results demonstrate the accuracy and efficiency of the present theory by comparison with several benchmark solutions available in the literature. A close agreement with existing higher-order shell theories is observed, which can be attributed to the inclusion of transverse normal strain effects. It was also shown that the proposed theory performs better than classical and first-order theories, which tend to overestimate the natural frequencies due to ignoring the effects of transverse normal strain. The parametric study revealed that the natural frequencies of sandwich shells are strongly influenced by geometric parameters such as side-to-thickness ratio, aspect ratio, core-to-face thickness ratio, and radii of curvature. In general, frequencies increase with decreasing thickness, decrease with increasing core thickness, and are inversely proportional to the radii of curvature. Furthermore, spherical shells exhibit higher natural frequencies than cylindrical shells due to their double curvature and greater stiffness. A significant contribution of this work is the provision of new benchmark frequency results for symmetric and anti-symmetric five-layer sandwich shells, which are not widely available in the literature. These results may serve as reference solutions for future analytical and numerical studies.

Future perspectives of this study include extending the present theory for the analysis of functionally graded, porous, and nano-reinforced sandwich shells; incorporating various boundary conditions and loading environments; and integrating the formulation with finite element methods for the analysis of complex shell problems. The developed model also holds potential for practical applications in aerospace, defense, marine, automotive, and civil engineering structures, particularly in vibration control, optimization, and damage detection. Some of the practical applications of the present model and the approach are aircraft fuselages and wings, space structures, helicopter rotor blades, fan casings, engine nacelles, armored vehicles and naval vessels, protective shelters, Ship hulls and decks, offshore platforms, roof and dome structures, bridges and walkways, vehicle body panels, train carriages and buses, space telescopes and large reflectors, robotic arms and manipulators, acoustic and vibration-sensitive structures, etc.

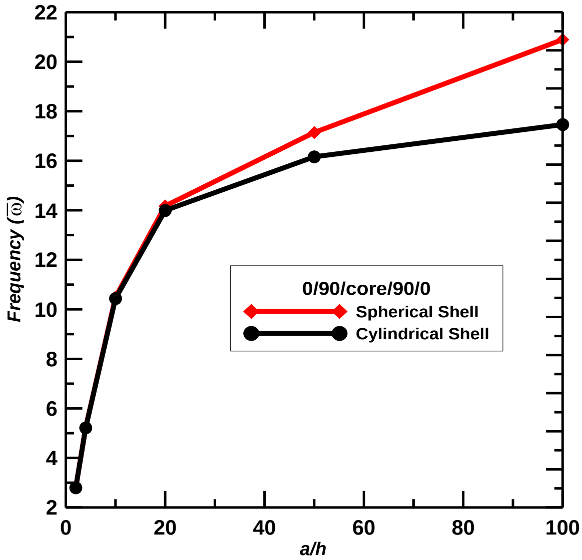


Fig. 10 Plot of non-dimensional frequencies ($\bar{\omega}$) versus (*a/h*) of a simply supported five-layer symmetric sandwich shell (*a/b*=1 and $t_c/t_f$=10)

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*AP*

## Appendix A

Governing equations of motion in terms of unknown displacement variables.

$$\delta u_0 : A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{R_1 \partial x}\right)-B_{11}\frac{\partial^3 w_0}{\partial x^3}+C_{11}\frac{\partial^2 \theta_x}{\partial x^2}+\frac{F_{11}}{R_1}\frac{\partial \theta_z}{\partial x}+A_{12}\left(\frac{\partial^2 v_0}{\partial x \partial y}+\frac{\partial w_0}{R_2 \partial x}\right)$$
$$-B_{12}\frac{\partial^3 w_0}{\partial x \partial y^2}+C_{12}\frac{\partial^2 \theta_y}{\partial x \partial y}+\frac{F_{12}}{R_2}\frac{\partial \theta_z}{\partial x}+\mathrm{D}_{13}\frac{\partial \theta_z}{\partial x}+A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x \partial y}\right)-2B_{66}\frac{\partial^3 w_0}{\partial x \partial y^2} \quad \text{(A.1)}$$
$$+C_{66}\left(\frac{\partial^2 \theta_x}{\partial y^2}+\frac{\partial^2 \theta_y}{\partial x \partial y}\right)=\left(I_1+\frac{2I_2}{R_1}+\frac{I_3}{R_1^2}\right)\frac{\partial^2 u_0}{\partial t^2}-\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3 w_0}{\partial x \partial t^2}+\left(I_4+\frac{I_6}{R_1}\right)\frac{\partial^2 \theta_x}{\partial t^2}$$

$$\delta v_0 : A_{21}\left(\frac{\partial^2 u_0}{\partial x \partial y}+\frac{\partial w_0}{R_1 \partial y}\right)-B_{21}\frac{\partial^3 w_0}{\partial x^2 \partial y}+C_{21}\frac{\partial^2 \theta_x}{\partial x \partial y}+\frac{F_{21}}{R_1}\frac{\partial \theta_z}{\partial y}+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{R_2 \partial y}\right)$$
$$-B_{22}\frac{\partial^3 w_0}{\partial y^3}+C_{22}\frac{\partial^2 \theta_y}{\partial y^2}+\frac{F_{22}}{R_2}\frac{\partial \theta_z}{\partial y}+\mathrm{D}_{23}\frac{\partial \theta_z}{\partial y}+A_{66}\left(\frac{\partial^2 u_0}{\partial x \partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)-2B_{66}\frac{\partial^3 w_0}{\partial x^2 \partial y} \quad \text{(A.2)}$$
$$+C_{66}\left(\frac{\partial^2 \theta_x}{\partial x \partial y}+\frac{\partial^2 \theta_y}{\partial x^2}\right)=\left(I_1+\frac{2I_2}{R_2}+\frac{I_3}{R_2^2}\right)\frac{\partial^2 v_0}{\partial t^2}-\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3 w_0}{\partial t^2 \partial y}+\left(I_4+\frac{I_6}{R_2}\right)\frac{\partial^2 \theta_y}{\partial t^2}$$

$$\delta w_0 : B_{11}\left(\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial^2 w_0}{R_1 \partial x^2}\right)-\mathrm{H}_{11}\frac{\partial^4 w_0}{\partial x^4}+I_{11}\frac{\partial^3 \theta_x}{\partial x^3}+\frac{J_{11}}{R_1}\frac{\partial^2 \theta_z}{\partial x^2}+B_{12}\left(\frac{\partial^2 v_0}{\partial x^2 \partial y}+\frac{\partial^2 w_0}{R_2 \partial x^2}\right)$$
$$-H_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2}+I_{12}\frac{\partial^3 \theta_y}{\partial x^2 \partial y}+\frac{J_{12}}{R_2}\frac{\partial^2 \theta_z}{\partial x^2}+\mathrm{K}_{13}^{S}\frac{\partial^2 \theta_z}{\partial x^2}+B_{12}\left(\frac{\partial^3 u_0}{\partial x \partial y^2}+\frac{\partial^2 w_0}{R_1 \partial y^2}\right)-H_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2}$$
$$+I_{12}\frac{\partial^3 \theta_x}{\partial x \partial y^2}+\frac{J_{12}}{R_1}\frac{\partial^2 \theta_z}{\partial y^2}+B_{22}\left(\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial^2 w_0}{R_2 \partial y^2}\right)-H_{22}\frac{\partial^4 w_0}{\partial y^4}+I_{22}\frac{\partial^3 \theta_y}{\partial y^3}+\frac{\mathrm{J}_{22}}{R_2}\frac{\partial^2 \theta_z}{\partial y^2}$$
$$+\mathrm{K}_{23}^{S}\frac{\partial^2 \theta_z}{\partial y^2}+2B_{66}\left(\frac{\partial^3 u_0}{\partial x \partial y^2}+\frac{\partial^3 v_0}{\partial x^2 \partial y}\right)-4H_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2}+2I_{66}\left(\frac{\partial^3 \theta_x}{\partial x \partial y^2}+\frac{\partial^3 \theta_y}{\partial x^2 \partial y}\right)$$
$$-\left[\frac{A_{11}}{R_1}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)-\frac{B_{11}}{R_1}\frac{\partial^2 w_0}{\partial x^2}+\frac{C_{11}}{R_1}\frac{\partial \theta_x}{\partial x}+\frac{F_{11}}{R_1^2}\theta_z\right]-\frac{A_{12}}{R_1}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)+\frac{B_{12}}{R_1}\frac{\partial^2 w_0}{\partial y^2} \quad \text{(A.3)}$$
$$-\frac{C_{12}}{R_1}\frac{\partial \theta_y}{\partial y}-\frac{F_{12}}{R_1 R_2}\theta_z-\frac{\mathrm{D}_{13}}{R_1}\theta_z-\frac{A_{12}}{R_2}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)+\frac{B_{12}}{R_2}\frac{\partial^2 w_0}{\partial x^2}-\frac{C_{12}}{R_2}\frac{\partial \theta_x}{\partial x}-\frac{F_{12}}{R_1 R_2}\theta_z$$
$$-\frac{A_{22}}{R_2}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)+\frac{B_{22}}{R_2}\frac{\partial^2 w_0}{\partial y^2}-\frac{C_{22}}{R_2}\frac{\partial \theta_y}{\partial y}-\frac{F_{22}}{R_2^2}\theta_z-\frac{\mathrm{D}_{23}}{R_2}\theta_z+q=\left(I_2+\frac{I_3}{R_1}\right)\frac{\partial^3 u_0}{\partial x \partial t^2}$$
$$-I_3\frac{\partial^4 w_0}{\partial x^2 \partial t^2}+I_6\frac{\partial^3 \theta_x}{\partial x \partial t^2}+\left(I_2+\frac{I_3}{R_2}\right)\frac{\partial^3 v_0}{\partial t^2 \partial y}-I_3\frac{\partial^4 w_0}{\partial y^2 \partial t^2}+I_6\frac{\partial^3 \theta_y}{\partial y \partial t^2}+I_1\frac{\partial^2 w_0}{\partial t^2}+I_7\frac{\partial^2 \theta_z}{\partial t^2}$$

$$\delta \theta_x : C_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{R_1 \partial x}\right)-I_{11}\frac{\partial^3 w_0}{\partial x^3}+L_{11}\frac{\partial^2 \theta_x}{\partial x^2}+\frac{M_{11}}{R_1}\frac{\partial \theta_z}{\partial x}+C_{12}\left(\frac{\partial^2 v_0}{\partial x \partial y}+\frac{\partial w_0}{R_2 \partial x}\right)$$
$$-I_{12}\frac{\partial^3 w_0}{\partial x \partial y^2}+L_{12}\frac{\partial^2 \theta_y}{\partial x \partial y}+\frac{M_{12}}{R_2}\frac{\partial \theta_z}{\partial x}+N_{13}\frac{\partial \theta_z}{\partial x}+C_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x \partial y}\right)-2I_{66}\frac{\partial^3 w_0}{\partial x \partial y^2}$$

$$+L_{66}\left(\frac{\partial^2\theta_x}{\partial y^2}+\frac{\partial^2\theta_y}{\partial x\partial y}\right)-O_{55}\left(\theta_x+\frac{\partial\theta_z}{\partial x}\right)=\left(I_4+\frac{I_6}{R_1}\right)\frac{\partial^2 u_0}{\partial t^2}-I_6\frac{\partial^3 w_0}{\partial x\partial t^2}+I_5\frac{\partial^2\theta_x}{\partial t^2} \quad (A.4)$$

$$\begin{aligned}\delta\theta_y: & C_{21}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{R_1\partial y}\right)-I_{21}\frac{\partial^3 w_0}{\partial x^2\partial y}+L_{21}\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{M_{21}}{R_1}\frac{\partial\theta_z}{\partial y}+C_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{R_2\partial y}\right)\\ & -I_{22}\frac{\partial^3 w_0}{\partial y^3}+L_{22}\frac{\partial^2\theta_y}{\partial y^2}+\frac{M_{22}}{R_2}\frac{\partial\theta_z}{\partial y}+N_{23}\frac{\partial\theta_z}{\partial y}+C_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)-2I_{66}\frac{\partial^3 w_0}{\partial x^2\partial y}\\ & +L_{66}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)-O_{44}\left(\theta_y+\frac{\partial\theta_z}{\partial y}\right)=\left(I_4+\frac{I_6}{R_2}\right)\frac{\partial^2 v_0}{\partial t^2}-I_6\frac{\partial^3 w_0}{\partial t^2\partial y}+I_5\frac{\partial^2\theta_y}{\partial t^2}\end{aligned} \quad (A.5)$$

$$\begin{aligned}\delta\theta_z: & -O_{55}\left(\frac{\partial\theta_x}{\partial x}+\frac{\partial^2\theta_z}{\partial x^2}\right)+O_{44}\left(\frac{\partial\theta_y}{\partial y}+\frac{\partial^2\theta_z}{\partial y^2}\right)-\frac{F_{11}}{R_1}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)+\frac{J_{11}}{R_1}\frac{\partial^2 w_0}{\partial x^2}-\frac{M_{11}}{R_1}\frac{\partial\theta_x}{\partial x}\\ & -\frac{O_{11}}{R_1^2}\theta_z-\frac{F_{12}}{R_1}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)+\frac{J_{12}}{R_1}\frac{\partial^2 w_0}{\partial y^2}-\frac{M_{12}}{R_1}\frac{\partial\theta_y}{\partial y}-\frac{O_{12}}{R_1R_2}\theta_z-2\frac{P_{13}}{R_1}\theta_z\\ & -\frac{F_{12}}{R_2}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)+\frac{J_{12}}{R_2}\frac{\partial^2 w_0}{\partial x^2}-\frac{M_{12}}{R_2}\frac{\partial\theta_x}{\partial x}-\frac{O_{12}}{R_1R_2}\theta_z-\frac{F_{22}}{R_2}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)+\frac{J_{22}}{R_2}\frac{\partial^2 w_0}{\partial y^2}\\ & -\frac{M_{22}}{R_2}\frac{\partial\theta_y}{\partial y}-\frac{O_{22}}{R_2^2}\theta_z-2\frac{P_{23}}{R_2}\theta_z-D_{13}\left(\frac{\partial u_0}{\partial x}+\frac{w_0}{R_1}\right)+K_{13}^S\frac{\partial^2 w_0}{\partial x^2}-N_{13}\frac{\partial\theta_x}{\partial x}\\ & -D_{23}\left(\frac{\partial v_0}{\partial y}+\frac{w_0}{R_2}\right)+K_{23}^S\frac{\partial^2 w_0}{\partial y^2}-N_{23}\frac{\partial\theta_y}{\partial y}-S_{33}\theta_z=I_7\frac{\partial^2 w_0}{\partial t^2}+I_8\frac{\partial^2\theta_z}{\partial t^2}\end{aligned} \quad (A.6)$$

Where an extension, bending, bending-extension coupling laminate stiffness coefficients appeared in the governing equations of motion are defined in terms of stiffness coefficients as follows.

$$\begin{aligned}&\left(A_{ij},B_{ij},H_{ij},C_{ij},I_{ij},F_{ij},D_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}\left[1.0,z,z^2,f(z),zf(z),f'(z),f''(z)\right]dz;\\ &\left(L_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}[f(z)]^2\,dz;\quad \left(O_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}[f'(z)]^2\,dz;\\ &\left(J_{ij},M_{ij},P_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}f'(z)\left[z,f(z),f''(z)\right]dz;\\ &\left(K_{ij}^s,N_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}f''(z)[z,f(z)]\,dz;\ S_{ij}=Q_{ij}\int_{-h/2}^{h/2}\left[f''(z)\right]^2dz\end{aligned} \quad (A.7)$$

## Appendix B

Elements of stiffness matrices and mass matrix appeared in the Eq. (12) are as follows.

$$
\begin{aligned}
&K_{11} = -A_{11}\alpha^2 - A_{66}\beta^2, \quad K_{12} = -A_{12}\alpha\beta - A_{66}\alpha\beta,\\
&K_{13} = \frac{A_{11}}{R_1}\alpha + \frac{A_{12}}{R_2}\alpha + B_{11}\alpha^3 + B_{12}\alpha\beta^2 + 2B_{66}\alpha\beta^2,\\
&K_{14} = -C_{11}\alpha^2 - C_{66}\beta^2,\ K_{15} = -C_{12}\alpha\beta - C_{66}\alpha\beta, K_{16} = \left[\frac{F_{11}}{R_1}\alpha + \frac{F_{12}}{R_2}\alpha + D_{13}\alpha\right],\\
&K_{22} = -A_{22}\beta^2 - A_{66}\alpha^2,\\
&K_{23} = B_{22}\beta^3 + [B_{12} + 2B_{66}]\alpha^2\beta + \left[\frac{A_{12}}{R_1} + \frac{A_{22}}{R_2}\right]\beta\\
&K_{24} = -C_{21}\alpha\beta - C_{66}\alpha\beta, K_{25} = -C_{22}\beta^2 - C_{66}\alpha^2, K_{26} = \left(D_{23} + \frac{F_{21}}{R_1} + \frac{F_{22}}{R_2}\right)\beta,\\
&K_{33} = -(H_{11}\alpha^4 + H_{22}\beta^4) - 2\alpha^2\beta^2(H_{12} + 2H_{66}) - 2\alpha^2\left(\frac{B_{11}}{R_1} + \frac{B_{12}}{R_2}\right)\\
&\qquad - 2\beta^2\left(\frac{B_{12}}{R_1} + \frac{B_{22}}{R_2}\right) - \left(\frac{2A_{12}}{R_1R_2} + \frac{A_{11}}{R_1^2} + \frac{A_{22}}{R_2^2}\right),\\
&K_{34} = I_{11}\alpha^3 + I_{21}\alpha\beta^2 + 2I_{66}\alpha\beta^2 + \frac{C_{11}}{R_1}\alpha + \frac{C_{21}}{R_2}\alpha,\\
&K_{35} = I_{12}\alpha^2\beta + I_{22}\beta^3 + 2I_{66}\alpha^2\beta + \frac{C_{12}}{R_1}\beta + \frac{C_{22}}{R_2}\beta,\\
&K_{36} = -K_{13}\alpha^2 - K_{23}\beta^2 - \frac{D_{13}}{R_1} - \frac{D_{23}}{R_2} - \left(\frac{J_{11}}{R_1} + \frac{J_{12}}{R_2}\right)\alpha^2 - \left(\frac{J_{21}}{R_1} + \frac{J_{22}}{R_2}\right)\beta^2\\
&\qquad - \left(2\frac{F_{12}}{R_1R_2} + \frac{F_{22}}{R_2^2} + \frac{F_{11}}{R_1^2}\right)\\
&K_{44} = -L_{11}\alpha^2 - L_{66}\beta^2 - O_{55}, K_{45} = -(L_{12} + L_{66})\alpha\beta,\\
&K_{46} = \left(N_{13} - O_{55} + \frac{M_{11}}{R_1} + \frac{M_{12}}{R_2}\right)\alpha, K_{55} = -L_{66}\alpha^2 - L_{22}\beta^2 - O_{44},\\
&K_{56} = \left(-O_{44} + N_{23} + \frac{M_{21}}{R_1} + \frac{M_{22}}{R_2}\right)\beta,\\
&K_{66} = \left(-O_{55}\alpha^2 - O_{44}\beta^2 - S_{33} + 2\frac{P_{23}}{R_2} - 2\frac{P_{13}}{R_1} - \frac{O_{11}}{R_1^2} - 2\frac{O_{12}}{R_1R_2} - \frac{O_{22}}{R_2^2}\right)
\end{aligned}
\tag{B.1}
$$

and

$$
\begin{aligned}
&M_{11} = \left(I_1 + \frac{2I_2}{R_1} + \frac{I_3}{R_1^2}\right);\ M_{12} = M_{21} = 0;\ M_{13} = M_{31} = -\left(I_2 + \frac{I_3}{R_1}\right)\alpha;\\
&M_{14} = M_{41} = \left(I_4 + \frac{I_6}{R_1}\right);\ M_{15} = M_{51} = 0;\ M_{16} = M_{61} = 0;\\
&M_{22} = \left(I_1 + \frac{2I_2}{R_2} + \frac{I_3}{R_2^2}\right); M_{23} = M_{32} = -\left(I_2 + \frac{I_3}{R_2}\right)\beta; M_{24} = M_{42} = 0;
\end{aligned}
$$

$$M_{26}=M_{62}=0; M_{25}=M_{52}=\left(I_4+\frac{I_6}{R_2}\right); M_{34}=M_{43}=-I_6\alpha;$$
$$M_{33}=(I_3\alpha^2+I_3\beta^2+I_1);\ M_{35}=M_{53}=-I_6\beta;\quad M_{36}=M_{63}=I_7; M_{44}=I_5; \tag{B.2}$$
$$M_{55}=I_5;\ M_{45}=M_{54}=0;\ M_{46}=M_{64}=0;\ M_{56}=M_{65}=0;\ M_{66}=I_8$$