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Size-dependent bending behavior of non-uniform beams resting on an elastic medium via modified couple stress theory

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Abstract. In this study, static analysis of variable cross-section embedded beams at micro scales is performed. It is considered that the microbeam is lying on Winkler foundation. Fundamental equations are obtained with the help of Euler-Bernoulli beam and modified couple stress theories. Four cases for the cross-section of cantilever microbeams as uniform, single tapered with variable width and constant height, single tapered with constant width and variable height, and double tapered with variable width and height are investigated. Displacements of embedded cantilever microbeams are obtained by Rayleigh-Ritz method for various taper ratios. The effects of small-scale parameter, Winkler parameter, tapering case, and taper ratio on displacements are examined in detail. It is found that the microstructure effect and tapering cases may play important roles in the displacements of microbeams.
Keywords: modified couple stress theory; Rayleigh-Ritz method; size effect; tapered microbeam; Winkler foundation model

1. Introduction

In the current century, thanks to developing technology, modern and efficient materials that provide the highest benefit to humanity are produced, and the materials currently used are revised accordingly. It is known that structural elements with regular cross-sections have the same functionality as structural elements with variable cross-sections. Structural elements with variable cross-sections are frequently used in construction, machinery, electronics and aviation engineering, and such structures are commonly encountered in daily life. Lighting and flagpoles, wind turbine wings, aircraft wings, inclined beams, gusseted beams, and the tip of the Atomic Force Microscope (Alibakhshi *et al*. 2022) can be given as examples of such structures (see Figs. 1-3). With the use of these types of structural elements, cost and dead load will be reduced due to the reduction in material consumption, and a more economical and safer combination will be created.

With the development of nanotechnology, remarkable features at micro and nano dimensions have become the subject of research. Micro and nanostructures have found many applications such as microelectronics, biomedical engineering, sensors and resonators (Sharmile *et al*. 2025, Yang *et al*. 2023, Fath *et al*. 2022, Alibakhshi *et al*. 2022). Many size-dependent continuum theories such as modified strain gradient, modified couple stress, nonlocal elasticity, nonlocal strain gradient, and reformulated strain gradient theories have been put forward for the analysis of micro and nano structures, and the interesting properties of these structures are still the subject of research today. Modified couple stress theory is a simple, useful, and popular higher-order continuum theory developed by Yang and his colleagues in 2002 (Yang *et al*. 2002). By applying this theory to micro beams by Park and Gao (2006), the bending analysis of cantilever micro beam under the effect of a single load was investigated according to the Bernoulli-Euler beam theory. Ma *et al*. (2008, 2010) conducted research on bending and free vibration of microbeams based on Timoshenko and Reddy-Levinson beam theories with the help of modified couple stress theory. The buckling analysis of microbeams was designed according to higher order continuum theories by Akgöz and Civalek (2011), and the vibration behavior of functionally graded variable cross-section microbeams was investigated according to Euler-Bernoulli and modified couple stress theories by Akgöz and Civalek (2013). Şimşek and Reddy (2013a, b) solved the bending, vibration and buckling problems of functionally graded microbeams based on various beam and modified couple stress theories. Kahrobaiyan *et al*. (2014) developed a finite element model based on the modified couple stress theory of Timoshenko microbeams. Shafiei *et al*. (2016) performed the vibration calculation of functionally graded variable cross-section rotating micro beams based on Euler-Bernoulli and Timoshenko beam theories and modified couple stress theory. Ebrahimi and Salari (2016) investigated the thermo-electro-mechanical behavior of functionally graded piezoelectric Timoshenko nanobeams. Tadi Beni (2016 a, b) conducted research on piezoelectric Timoshenko nanobeam and functionally graded piezoelectric material nanobeam. Vibration analysis of variable cross-section rotating micro beams was investigated by Shafiei *et al*. (2017) using the differential quadrature method according to the modified couple stress theory. Habibi *et al*. (2017) examined the free vibration response of magneto-electro-

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Fig. 1 Schematic representation of wind turbine blades with variable cross-section cantilever beams

Fig. 2 Schematic representation of lighting pole with variable cross-section cantilever beam

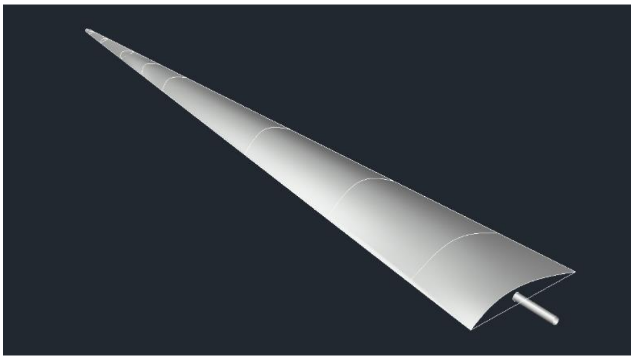


Fig. 3 Schematic representation of aircraft wing with variable cross-section cantilever beam

elastic nanobeams in thermal environment based on modified couple stress theory. Ebrahimi and Salari (2018) studied the effects of various parameters such as temperature effect, aspect ratio, on the temperatures and natural frequencies of size-dependent functionally graded nanobeams in critical buckling. Khaniki *et al*. (2018) investigated the buckling behavior of variable cross-section microscale beam with nonlocal strain gradient theory. Keshmiri *et al*. (2018) The free vibration of a tapered cone beam with variable cross-section are studied. Saurin (2019) worked on the formulation of a boundary value problem to find the natural frequencies of a beam of variable cross-section. Ghannadiasl *et al*. (2020) studied the free vibration of variable cross-section beams. Noroozi *et al*. (2020) investigated the size effect on torsional vibration of functionally graded nonlinear nano cone. Kumar *et al*. (2021) studied the vibration behavior of nanoplate with functionally graded porous structure. Nazemizadeh *et al*. (2021) investigated the vibration equations of the variable cross-section nanobeam, modeling and frequency analysis of the nanobeam with the help of a mass sensor. Abouelregal *et al*. (2022) studied the thermoelastic vibration behavior of a nonlocal isotropic solid medium. Akbaş *et al*. (2022) analyzed the behavior of a porous microbeam composed of functionally graded materials under a moving load on the basis of modified couple stress theory. Cui *et al*. (2022) investigated the bending of laminates with variable cross-section. Azandariani *et al*. (2022) analyzed the nonlinear static analysis of Timoshenko nanobeams consisting of bidirectional functionally graded material with stationary ends. Sharma *et al*. (2022) studied exponential functionally graded beams and investigated the vibration analysis of beams whose width was varied along the axial direction according to an exponential function. Amir *et al*. (2022) analyzed the nonlinear bending and buckling behavior of functionally graded small-sized plates with the help of modified strain gradient theory. Šalinić *et al*. (2023) studied the axial bending vibration of an axially functionally graded irregular cross-section Timoshenko cantilever beam. Gohery *et al*. (2023) studied the three-dimensional free torsional-bending vibrations of Timoshenko beams with homogeneous and inhomogeneous properties and variable cross-section. Kumar and Ali (2023) analyzed the free transverse vibration and buckling analysis of nanobeam using Euler-Bernoulli beam theory and Eringen’s nonlocal elasticity theory together. Dangarwala and Gopal (2023) performed the free vibration analysis of rotating and irregular cantilever beams using the Ritz method. Alnujaie *et al*. (2024) analyzed coated functionally graded graphene reinforced composite nanoplates. Wrat *et al*. (2024) investigated the vibrations of regular and irregular cantilever beams in water. Burlayenko and Kouhia (2024) studied the free vibration behavior of functionally graded porous beams with non-uniform cross-sections. Bagheri *et al*. (2024) analyzed the free vibration and buckling of functionally graded porous beams with variable cross-sections. Burlayenko *et al*. (2024) studied to simulate the free vibrations of one-dimensional and three-dimensional modeled beams of axial functionally graded materials with irregular cross-sections. Malekzadeh and Moradi (2025) analyzed the large amplitude vibration case of thin beams with variable cross-section. Bazoune (2024) worked on a new method for the analysis of a Timoshenko-Ehrenfest beam under free bending vibration of variable cross section. Tadi Beni (2024) analyzed the behavior of flexoelectric microbeams. Yue *et al*. (2024) studied the nonlinear bending and vibration analysis of a variable cross-section piezoelectric nanoplate. Kai and Ling (2024) with the help of Euler beam theory and Hamilton principle, they tried to obtain the free vibration control equation for a variable cross-section beam. Burlayanko *et al*. (2025) analyzed the free vibration of variable cross-section carbon nanotube reinforced composite beams. Civalek and Akgöz (2025) investigated the static bending behavior of carbon nanotube reinforced composite strain gradient microbeams.

Civalek and Akgöz (2025) analyzed the elastic behavior of microbeams with double tapered based on Winkler elasticity. Guo and Alam (2025) studied the behavior of magneto-electro-elastic nanobeams in the framework of nonlocal strain gradient theory with surface effect. Li and Yao (2025) investigated the nonlinear vibration of a three-dimensional variable cross-section cantilever beam. Cai *et al*. (2025) analyzed the bending-torsional vibration response of modified Timoshenko thin-walled beams to moving harmonic loads. Zhang *et al*. (2025) studied the formulation for size-effective vibration of non-uniform Timoshenko beams.

As can be seen, remarkable features at micro and nano scales have taken their place in the literature as a research topic. In engineering structures, more importance is now given to geometric features and the effect of geometric effects on structural elements is investigated. Whether variable cross-section structural elements are different from homogeneous, uniform structural elements, mechanical properties such as displacement, vibration, and torsion under various loadings have been research topics. Variable cross-section structural elements are preferred both because they look more aesthetically pleasing in terms of appearance and because the cost burden is reduced. Of course, more than one criterion should be taken into consideration when choosing a structural element with this feature. In this study, we examined the displacements of a variable cross-section cantilever beam at the micro scale under varying cross-section dimensions.

2. Theory and formulation

The strain energy of a microbeam in the frame of modified couple stress theory (Yang *et al*. 2002) can be expressed as (Park and Gao 2006):

$$U=\frac{1}{2}\int_0^L\int_A(\sigma_{ij}\varepsilon_{ij}+m_{ij}^s x_{ij}^s)\,dAdx \tag{1}$$

where σ_{ij} and m_{ij}^s represent the components of classical stress and symmetric couple stress tensors, ε_{ij} and x_{ij}^s denote the components of strain and symmetric rotation gradient tensors, respectively. They can be written as follows (Park and Gao 2006):

$$\sigma_{ij}=\lambda\varepsilon_{mm}\delta_{ij}+2\mu\varepsilon_{ij} \tag{2}$$

$$m_{ij}^s=2\mu l^2 x_{ij}^s \tag{3}$$

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{4}$$

$$x_{ij}^s=\frac{1}{2}(\theta_{i,j}+\theta_{j,i}) \tag{5}$$

$$\theta_i=\frac{1}{2}e_{ijk}u_{k,j} \tag{6}$$

In these equations, u_i and θ_i represent the components of the displacement vector and the rotation vector, respectively. δ_{ij} and e_{ijk} denote Kronecker delta and permutation symbols. Additionally, l is the length scale parameter, λ and μ are Lamé constants.

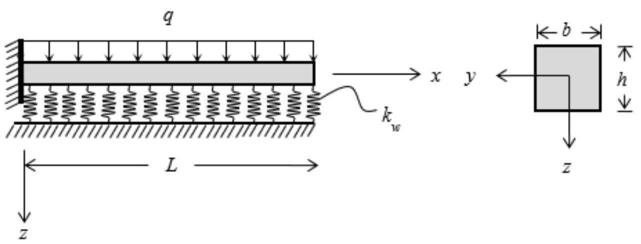


Fig. 4 A schematic representation of a rectangular cantilever microbeam resting on an elastic medium under uniformly distributed load

The coordinate system, loading case, and cross-section of a prismatic cantilever microbeam are illustrated in Fig. 4. L, b, and h represent the length, width, and thickness of microbeam. q denotes the uniformly distributed load and k_w is Winkler foundation parameter.

The nonzero displacement components of a microbeam based on simple beam theory can be written as:

$$u_1(x,z)=-z\frac{dw(x)}{dx} \tag{7}$$

$$u_3(x,z)=w(x) \tag{8}$$

where u_1 and u_3 are the longitudinal and lateral components of displacement vector and w is the transverse displacement. The nonzero component of strain tensor is obtained by using Eqs. (7) and (8) in Eq. (4) as follows:

$$\varepsilon_{11}=-z\frac{d^2w(x)}{dx^2} \tag{9}$$

Similarly, the nonzero components of rotation vector and symmetric rotation gradient tensor can be expressed as:

$$\theta_2=-\frac{dw}{dx},\qquad x_{12}^s=x_{21}^s=-\frac{1}{2}\frac{d^2w}{dx^2} \tag{10}$$

Substituting the above strain components in Eqs. (2) and (3) yields the nonzero strain components as (neglecting the Poisson effect (Reddy 2011))

$$\sigma_{11}=-Ez\frac{d^2w}{dx^2} \tag{11}$$

$$m_{12}^s=m_{21}^s=-\mu l^2\frac{d^2w}{dx^2} \tag{12}$$

Subsequently the strain energy in Eq. (1) can be rewritten for microbeams with constant cross section as follows:

$$U=\frac{1}{2}\int_0^L(EI_0+\mu A_0 l^2)\,(\frac{d^2w}{dx^2})^2dx \tag{13}$$

where A_0 and I_0 represent the area of cross section and second moment of area of microbeam at the left end $x=0$, respectively and they are defined as

$$A_0=\int_A d\,A, I_0=\int_A z^2dA \tag{14}$$

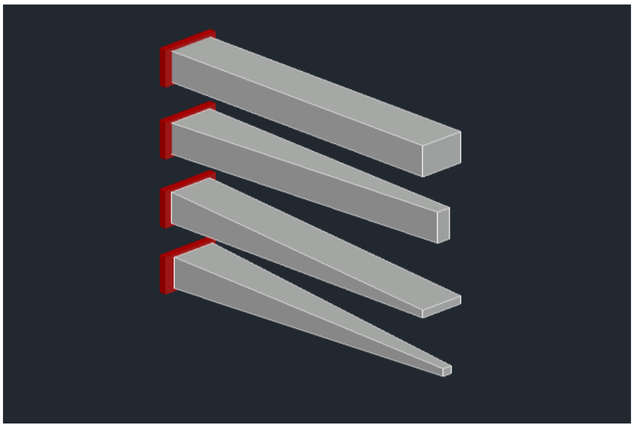


Fig. 5 Schematic representation of cantilever beams with constant and various variable cross sections

Fig. 5 depicts various cantilever microbeams having uniform cross section, variable width and constant height, constant width and variable height, and variable width and height (double tapered), respectively.

The properties of cross sections given in Fig. 5 can be formulated for four different cases as:

• *Case 1: Prismatic microbeam* ($\alpha = \beta = 0$)

$$A(x) = A_0 = b_0 h_0, \quad I(x) = I_0 = \frac{b_0 h_0^3}{12} \tag{15}$$

where b_0 and h_0 denote respectively the width and thickness of microbeam at $x = 0$.

• *Case 2: Constant thickness* ($\beta = 0$) *and variable width* ($\alpha \neq 0$)

$$\begin{aligned} b(x) &= b_0\left(1 - \alpha\frac{x}{L}\right), h(x) = h_0, \\ A(x) &= A_0\left(1 - \alpha\frac{x}{L}\right), I(x) = I_0\left(1 - \alpha\frac{x}{L}\right) \end{aligned} \tag{16}$$

• *Case 3: Constant width* ($\alpha = 0$)*and variable thickness* ($\beta \neq 0$)

$$\begin{aligned} b(x) &= b_0, \quad h(x) = h_0\left(1 - \beta\frac{x}{L}\right), \\ A(x) &= A_0\left(1 - \beta\frac{x}{L}\right), I(x) = I_0\left(1 - \beta\frac{x}{L}\right)^3 \end{aligned} \tag{17}$$

• *Case 4: Variable width and thickness* ($\alpha, \beta \neq 0$)

$$\begin{aligned} b(x) &= b_0\left(1 - \alpha\frac{x}{L}\right), \ h(x) = h_0\left(1 - \beta\frac{x}{L}\right), \\ A(x) &= A_0\left(1 - \alpha\frac{x}{L}\right)\left(1 - \beta\frac{x}{L}\right), \\ I(x) &= I_0\left(1 - \alpha\frac{x}{L}\right)\left(1 - \beta\frac{x}{L}\right)^3 \end{aligned} \tag{18}$$

where α and β represent taper ratios throughout width and thickness, respectively. It is notable that they can take values between 0 and 1, and if they are equal to zero, the cross section will be constant along the length of the microbeam.

The strain energy for uniform microbeams in Eq. (13) can be rewritten for nonuniform microbeams as follows:

$$U = \frac{1}{2}\int_0^L \left(EI_{(x)} + \mu A_{(x)} l^2\right) \left(\frac{d^2 w}{dx^2}\right)^2 dx \tag{19}$$

The external work done by the uniformly distributed load and Winkler foundation can be written as

$$W = \int_0^L q\, w dx - \frac{1}{2}\int_0^L k_w w^2 dx \tag{20}$$

3. Solution procedure

The total potential energy Π for microbeams can be given as

$$\Pi = U - W \tag{21}$$

Use of Eqs. (19) and (20) in Eq. (21) yields the total potential energy of nonuniform embedded microbeams subjected to uniformly distributed load as

$$\begin{aligned} \Pi = \frac{1}{2}\int_0^L &\left(\left(EI_{(x)} + \mu A_{(x)} l^2\right)\left(\frac{d^2 w}{dx^2}\right)^2 + k_w w^2\right) dx \\ &- \int_0^L q\, w dx \end{aligned} \tag{22}$$

The displacement can be approximated according to Rayleigh-Ritz method as:

$$w(x) = \sum_{i=1}^{N} c_i \varphi_i(x) \tag{23}$$

where c_i is unknown coefficient, $\varphi_i(x)$ is the trial function, and N is the number of polynomials. It is notable that in Rayleigh-Ritz method, the trial function must satisfy only the geometric (essential) boundary conditions. Also, it is well-known that when the number of polynomials increases, the obtained results approach the exact results. $\varphi_i(x)$ is selected to satisfy the essential boundary conditions of cantilever microbeam as

$$\varphi_i(x) = x^{(i+1)} \qquad i = 1,2,3, \ldots, N \tag{24}$$

On the other hand, the trial function can be respectively given for simply supported, both ends clamped, and propped cantilever microbeams as

$$\varphi_i(x) = x^i (L - x) \qquad i = 1,2,3, \ldots, N \tag{25a}$$

$$\varphi_i(x) = x^{(i+1)} (L - x)^2 \qquad i = 1,2,3, \ldots, N \tag{25b}$$

$$\varphi_i(x) = x^{(i+1)} (L - x) \qquad i = 1,2,3, \ldots, N \tag{25c}$$

To obtain a solution according to Rayleigh-Ritz method, the value of the partial derivatives of the total potential energy in Eq. (22) with respect to each arbitrary coefficient must be equal to zero.

$$\frac{\partial \Pi}{\partial c_i} = 0 \tag{26}$$

By solving this expression, the value of N arbitrary coefficients is obtained. By substituting them in Eq. (23), the displacement expression for the variable cross-section microbeams is obtained.

Table 1 Convergence of dimensionless classical maximum deflections ($\bar{w} = 10w\frac{EI_0}{qL^4}$) of cantilever beams with various tapering cases

N	$\alpha = \beta = 0$	$\alpha = 0.9, \beta = 0$	$\alpha = 0, \beta = 0.9$	$\alpha = \beta = 0.9$
2	1.2500	1.5957	3.2831	5.2207
3	1.2500	1.5947	3.2864	5.6859
4	1.2500	1.5944	3.2755	5.7707
5	1.2500	1.5943	3.2731	5.7599
6	1.2500	1.5943	3.2738	5.7473
7	1.2500	1.5943	3.2746	5.7473
8	1.2500	1.5943	3.2749	5.7475
9	1.2500	1.5942	3.2749	5.7496
10	1.2500	1.5942	3.2749	5.7506
11	1.2500	1.5942	3.2749	5.7509
12	1.2500	1.5942	3.2749	5.7508
13	1.2500	1.5942	3.2748	5.7507
14	1.2500	1.5942	3.2748	5.7506
15	1.2500	1.5942	3.2748	5.7506
16	1.2500	1.5942	3.2748	5.7506
17	1.2500	1.5942	3.2748	5.7505
18	1.2500	1.5942	3.2748	5.7505
19	1.2500	1.5942	3.2748	5.7505
20	1.2500	1.5942	3.2748	5.7505

Table 2 Maximum displacements μm of uniform/ nonuniform embedded microbeams for various taper ratios based on classical theory ($h_0 = 2\ \mu m, b_0 = h_0, L = 30h_0, l = 0, q = 1\ \mu N/\mu m,\ Kw = 10$)

Taper ratio	$\alpha = \beta = 0$	$\beta = 0$	$\alpha = 0$	$\alpha = \beta$
0.15	3.2114	3.2659	3.3787	3.4370
0.30	3.2114	3.3250	3.5688	3.6989
0.45	3.2114	3.3896	3.7878	4.0059
0.60	3.2114	3.4614	4.0446	4.3681
0.75	3.2114	3.5427	4.3530	4.7853
0.90	3.2114	3.6381	4.7315	5.1394

4. Numerical results and discussion

In this section, the bending behaviors of nickel microbeams having various variable cross sections under uniformly distributed load are investigated on the basis of modified couple stress theory. The material properties for nickel microbeam are used as (Lei *et al*. 2016): $E = 207GPa$, $v = 0.31$ and $l = 1.553\ \mu m$. Also, Winkler foundation parameter is nondimensionalized as $Kw = k_w L^4/EI_0$.

First of all, a convergence study is addressed to determine the minimum number of terms (N) required to ensure sufficient sensitivity in the present results. The dimensionless maximum deflections of cantilever beams are tabulated in Table 1 for various tapering cases to find the suitable number of terms. It is observed from this table that more terms are required for nonuniform beams with variable width and/or height while $N = 2$ is enough for uniform beams ($\alpha = \beta = 0$). The expected sensitivities are reached at $N = 9$, $N = 13$, and $N = 17$ for single tapering in width ($\alpha = 0.9, \beta = 0$), single tapering in height ($\alpha = 0, \beta = 0.9$), and double tapering ($\alpha = \beta = 0.9$), respectively. So, the number of terms (N) is selected 17 in the current work.

Table 3 Maximum displacements μm of uniform/ nonuniform embedded microbeams for various taper ratios based on modified couple stress theory ($h_0 = 2\ \mu m, b_0 = h_0, L = 30h_0, l = 1.553\ \mu m, q = 1\ \mu N/\mu m,\ Kw = 10$)

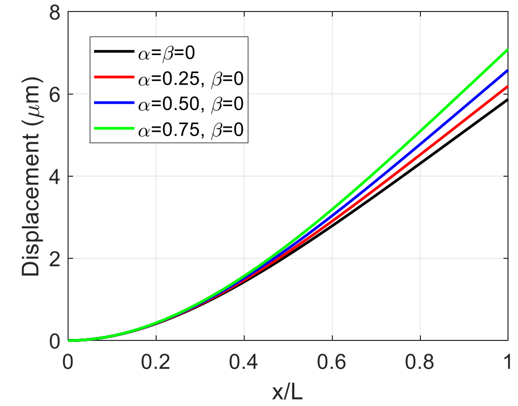
Taper ratio	$\alpha = \beta = 0$	$\beta = 0$	$\alpha = 0$	$\alpha = \beta$
0.15	1.2806	1.3138	1.3312	1.3665
0.30	1.2806	1.3503	1.3867	1.4661
0.45	1.2806	1.3912	1.4481	1.5838
0.60	1.2806	1.4377	1.5170	1.7266
0.75	1.2806	1.4919	1.5960	1.9074
0.90	1.2806	1.5579	1.6899	2.1559

The displacements at the free end of uniform/ nonuniform embedded microbeams for various taper ratios are tabulated in Table 2 based on classical theory. Here, the constant height and width indicate a prismatic microbeam and therefore the change in the taper ratio has no effect on the displacement values of this case. While the maximum displacement value is the least in the prismatic microbeam, it is the most in the double tapered microbeam. It is also clearly seen that the maximum displacement values gradually increase as the taper ratio increases.

Table 3 lists the maximum displacements of uniform/ nonuniform embedded microbeams for various taper ratios based on modified couple stress theory. Similar observations in Table 2 can be found here. It can be emphasized that the displacements obtained by modified couple stress theory are smaller than those evaluated by classical theory about 60%.

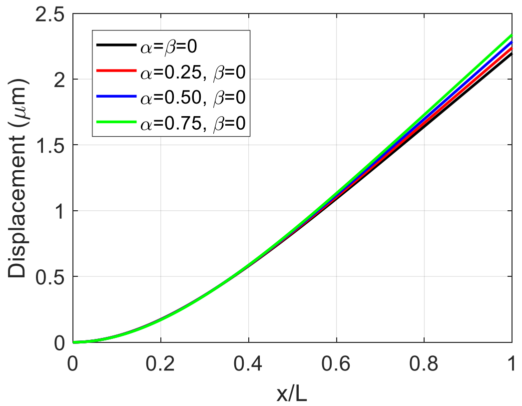
Variations of transverse displacements of tapered microbeams with respect to various taper ratios are depicted in Figs. 6-17 corresponding to variable width and constant height, constant width and variable height, and variable width and height (double tapered), respectively. It is notable that the classical displacements are seen in Figs. 6, 7, 10, 11, 14 and 15 while the size-dependent displacements are illustrated in Figs. 8, 9, 12, 13, 16 and 17. It can be observed from these figures that while the displacement is zero at the fixed end, it gradually increases along the length of the microbeam and reaches the maximum value at the free end. Also, it can be stated that the displacements increase depending on the increase in the taper ratio. In addition, this increase is most evident in double tapered microbeams. Moreover, the displacements become lower with the Winkler foundation effect.

The effects of the initial thickness-to-length scale parameter ratio on the maximum displacements of the tapered microbeams are plotted in Figs. 18-23 corresponding to variable width and constant height, constant width and variable height, and variable width and height (double tapered), respectively. In these figures, $\alpha = \beta = 0$ case



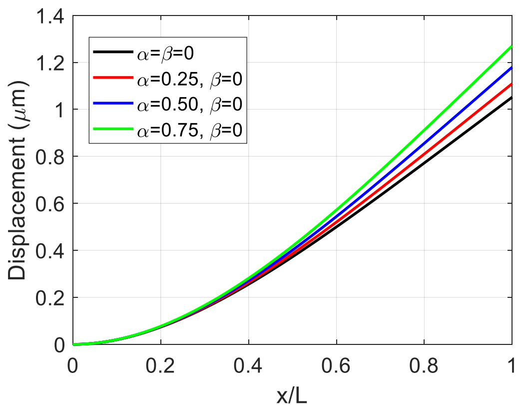
$Kw = 0$

Fig. 6 Classical transverse displacement distribution of a tapered microbeam with variable width and constant height according to various taper ratios $(h_0 = 1.553\ \mu m, l = 0)$



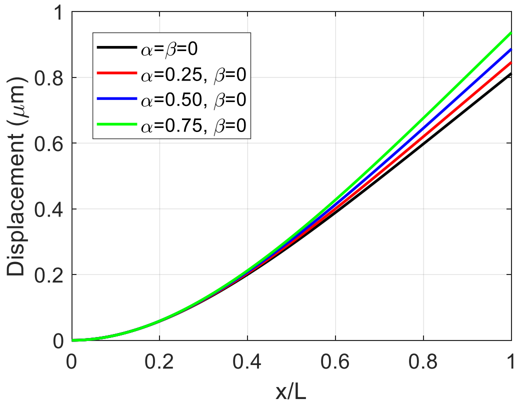
$Kw = 20$

Fig. 7 Classical transverse displacement distribution of a tapered microbeam with variable width and constant height according to various taper ratios $(h_0 = 1.553\ \mu m, l = 0)$



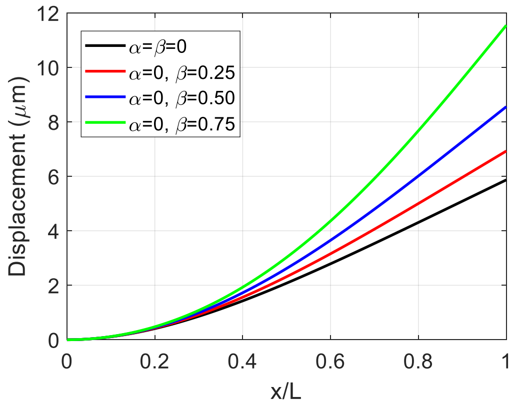
$Kw = 0$

Fig. 8 Size-dependent transverse displacement distribution of a tapered microbeam with variable width and constant height according to various taper ratios $(h_0 = l = 1.553\ \mu m)$



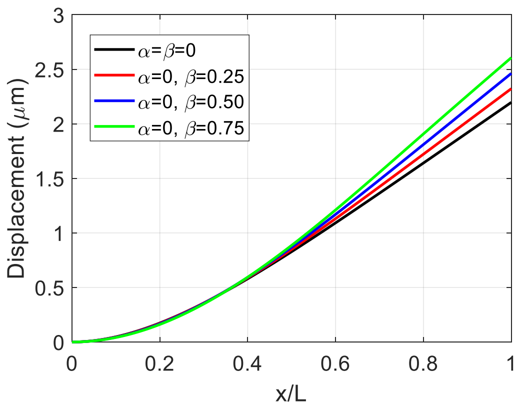
$Kw = 20$

Fig. 9 Size-dependent transverse displacement distribution of a tapered microbeam with variable width and constant height according to various taper ratios $(h_0 = l = 1.553\ \mu m)$



$Kw = 0$

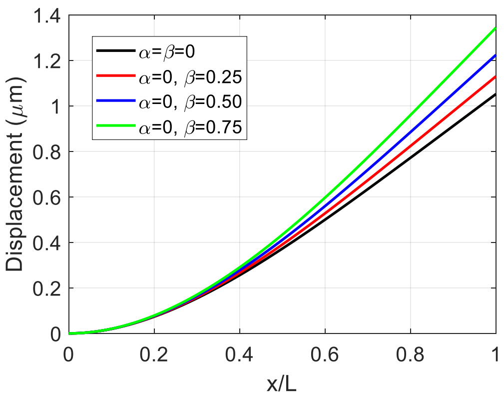
Fig. 10 Classical transverse displacement distribution of a tapered microbeam with constant width and variable height according to various taper ratios $(h_0 = 1.553\ \mu m, l = 0)$



$Kw = 20$

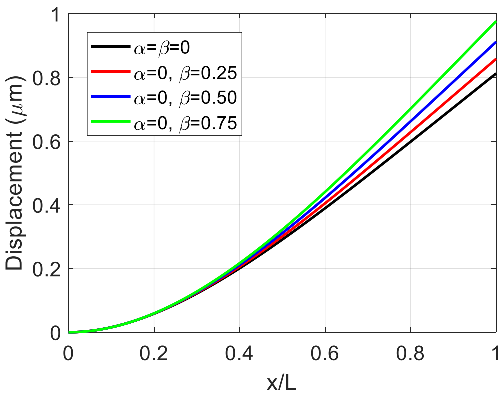
Fig. 11 Classical transverse displacement distribution of a tapered microbeam with constant width and variable height according to various taper ratios $(h_0 = 1.553\ \mu m, l = 0)$

represents the uniform microbeam and the smallest displacements occur in this case. It is evident that an increase in the initial thickness-to-length scale parameter ratio gives rise to increase the displacements due to the size



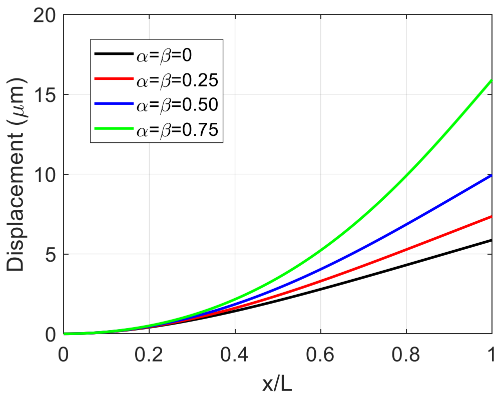
$Kw = 0$

Fig. 12 Size-dependent transverse displacement distribution of a tapered microbeam with constant width and variable height according to various taper ratios ($h_0 = l = 1.553\ \mu m$)



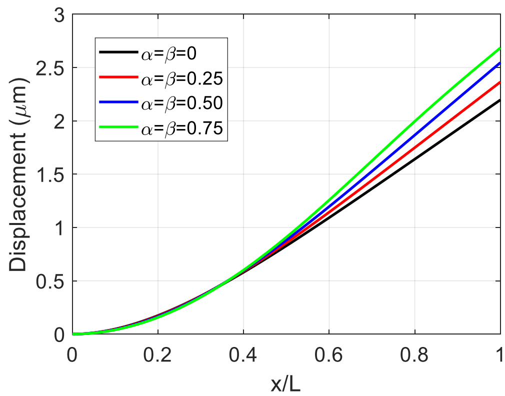
$Kw = 20$

Fig. 13 Size-dependent transverse displacement distribution of a tapered microbeam with constant width and variable height according to various taper ratios ($h_0 = l = 1.553\ \mu m$)



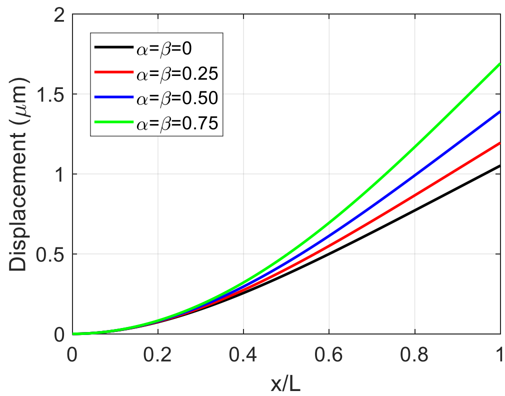
$Kw = 0$

Fig. 14 Classical transverse displacement distribution of a double tapered microbeam with variable width and height according to various taper ratios ($h_0 = 1.553\ \mu m, l = 0$)



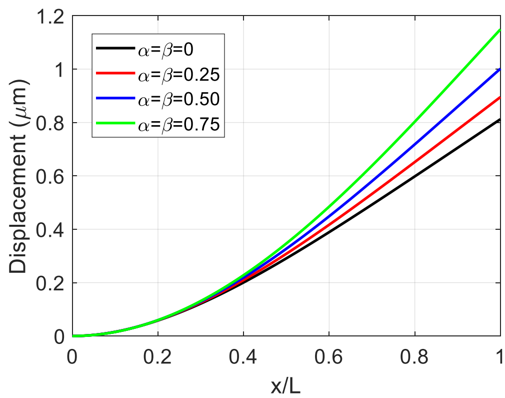
$Kw = 20$

Fig. 15 Classical transverse displacement distribution of a double tapered microbeam with variable width and height according to various taper ratios ($h_0 = 1.553\ \mu m, l = 0$)



$Kw = 0$

Fig. 16 Size-dependent transverse displacement distribution of a double tapered microbeam with variable width and height according to various taper ratios ($h_0 = l = 1.553\ \mu m$

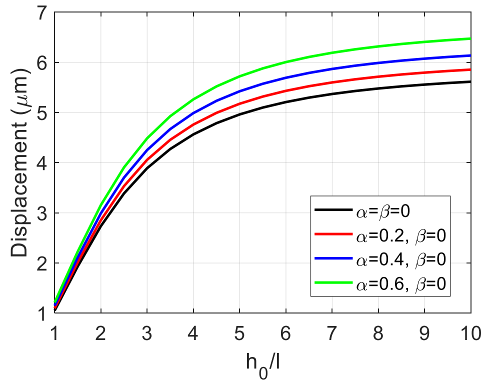


$Kw = 20$

Fig. 17 Size-dependent transverse displacement distribution of a double tapered microbeam with variable width and height according to various taper ratios ($h_0 = l = 1.553\ \mu m$

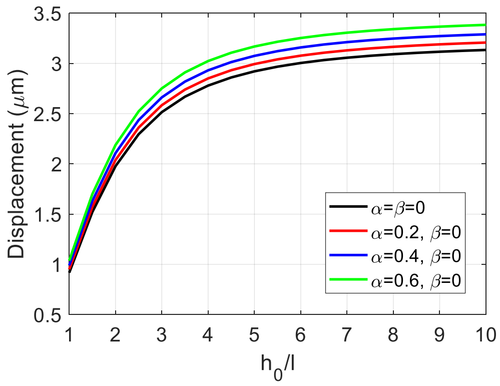
effect diminishes. Also, it can be found that the differences between the curves are more prominent for isolated microbeams without foundation effect.

The effect of Winkler foundation parameter on the normalized displacements $w_{tapered}/w_{prismatic}$ of the tapered microbeams is illustrated in Figs. 24-25 based on classical



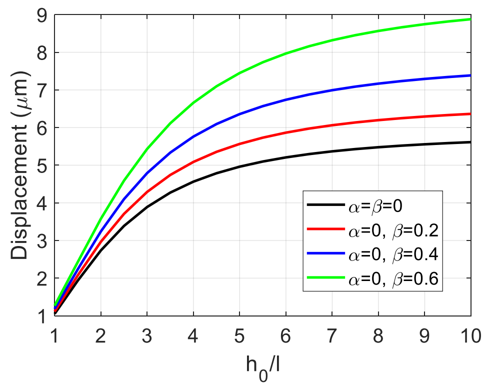
$Kw = 0$

Fig. 18 Variation of maximum displacements of a tapered microbeam with variable width and constant height against the initial thickness-to-length scale parameter ratio



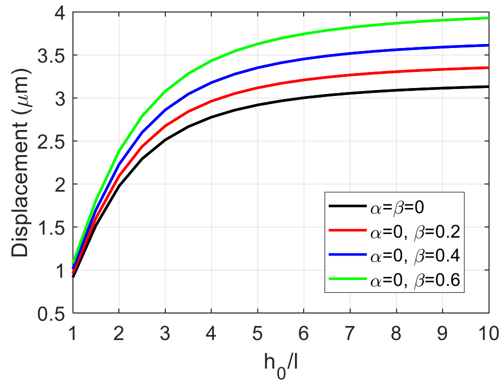
$Kw = 10$

Fig. 19 Variation of maximum displacements of a tapered microbeam with variable width and constant height against the initial thickness-to-length scale parameter ratio



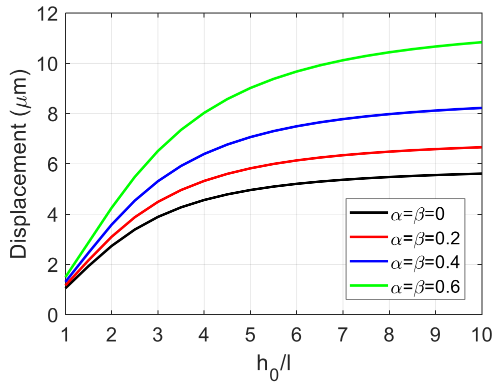
$Kw = 0$

Fig. 20 Variation of maximum displacements of a tapered microbeam with constant width and variable height against the initial thickness-to-length scale parameter ratio



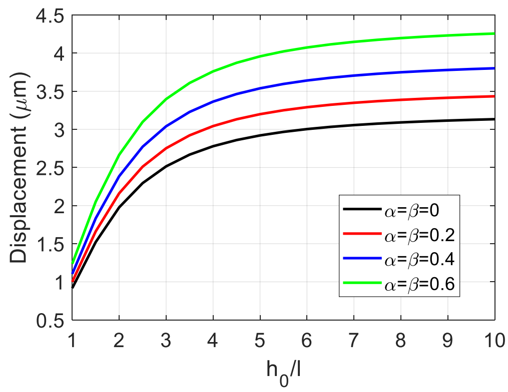
$Kw = 10$

Fig. 21 Variation of maximum displacements of a tapered microbeam with constant width and variable height against the initial thickness-to-length scale parameter ratio



$Kw = 0$

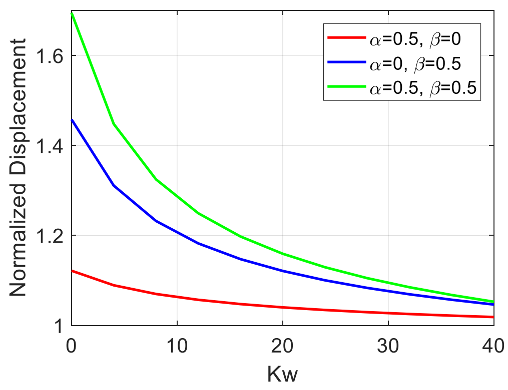
Fig. 22 Variation of maximum displacements of a double tapered microbeam with variable width and height against the initial thickness-to-length scale parameter ratio



$Kw = 10$

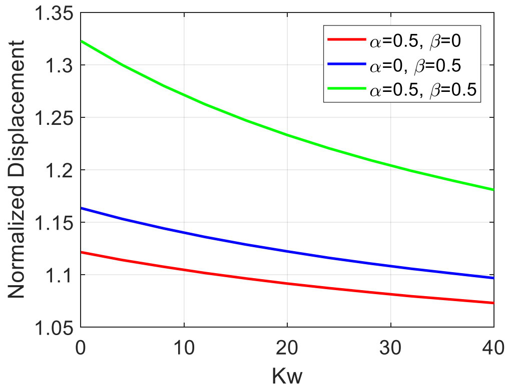
Fig. 23 Variation of maximum displacements of a double tapered microbeam with variable width and height against the initial thickness-to-length scale parameter ratio

and modified couple stress theories. It is found that the greatest normalized displacements occur in double tapered case $(\alpha = \beta = 0.5)$. It is also observed that an increase in Winkler foundation parameter leads to decrease in the normalized displacements. In addition, it can be found that the differences between the curves are more evident for size-dependent normalized displacements than classical ones.



$l = 0$

Fig. 24 Variation of normalized displacements of tapered microbeams with respect to Winkler foundation parameter ($h_0 = 1.553\ \mu m,\ L = 30h_0$)



$l = 1.553\ \mu m$

Fig. 25 Variation of normalized displacements of tapered microbeams with respect to Winkler foundation parameter ($h_0 = 1.553\ \mu m,\ L = 30h_0$)

4. Conclusions

In the present work, the microstructure-dependent deflection analysis of cantilever microbeams having variable cross section resting on an elastic medium is performed. Winkler foundation model is employed to consider the effect of elastic medium. The basic formulations are based on Bernoulli-Euler beam and modified couple stress theories that include an additional material length scale parameter. Future studies can be expected to utilize theories that involve another nonclassical elasticity theories such as nonlocal strain gradient theory and reformulated strain gradient theory, which include more additional length scale parameters. In terms of cross section, four cases are examined: uniform, single tapered with variable width and constant height, single tapered with constant width and variable height, double tapered with variable width and height. Rayleigh-Ritz method is applied with enough number of terms to obtain the displacements for cantilever microbeams. The influences of microstructure, taper ratio, and Winkler parameter on the deflections are perused. The main observations from the present results can be outlined as follows:

- While the displacements of uniform microbeams become the smallest, those of double tapered ones are the largest related to the variation of bending rigidity.
- The beam model based on classical theory overestimates the deflections.
- The displacements increase by an increment in taper ratio.
- The elastic foundation parameter has a decreasing effect on the deflections.
- The size effect becomes more prominent when the initial thickness of the microbeam approaches the length scale parameter.

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