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On the torsional dynamics of viscoelastic nanotubes on time-dependent boundary conditions using nonlocal elasticity theory

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Abstract. In this study, the torsional vibration of a viscoelastic nanotube is analyzed under viscoelastic boundary conditions. To incorporate both size effects and viscoelasticity into the model, the equations of motion are derived using the nonlocal theory of elasticity and the Kelvin-Voigt model, respectively. The problem is solved using Fourier series together with Stokes transforms and an eigenvalue problem is formulated in which the angular frequencies are determined. The results are compared with similar studies in the literature and presented in tables and figures. The findings of this study reveal some important results, such as that damping is more effective in more rigid boundary conditions and the effect of damping decreases as the size parameter increases.
Keywords: deformable boundary condition; nonlocal elasticity torsional vibration; stokes' transform; viscoelastic boundary condition; viscoelastic tube

1. Introduction

Many devices used today contain nano/micro-sized elements. These small-scale elements and materials, which have been discovered or produced and whose properties can be investigated thanks to nanotechnology, are remarkable. As a general acceptance, materials with at least one dimension smaller than 100 nm are called nanomaterials, and nanotechnology deals with them. These materials attract attention in the scientific community because they have much superior performance compared to their macro-scale equivalents. This good performance, which we encounter thanks to the high surface/volume ratio, allows nanomaterials to be used in various applications such as nanosensors, nano sorbents and fuel cells (Mazari *et al*. 2021). Quantum effects occur in these materials, which have versatile uses due to their size. Therefore, analyzing small-scale structures and elements formed from these materials should be carried out more precisely.

Experimental studies are the first approaches considered to investigate the mechanical response of small-scale materials, elements and structures. However, experimental studies have disadvantages such as high cost and long duration. Although molecular dynamics simulations are used to overcome these disadvantages, these simulations can also require both long durations and high computer capacities. Therefore, researchers have turned to theoretical studies to investigate the mechanical responses of small-scale structures and elements made of nano-sized materials. Studies carried out with classical elasticity theories have noticed that the responses of small-scale structures/ elements/materials could not be fully reflected. It has been understood that this is because the size effect could not be explored by classical theories. Therefore, in order to obtain better results, researchers have put forward various higher-order theories depending on the size effect.

Higher-order theories such as modified couple stress theory (MCST) (Yang *et al*. 2002), nonlocal (NL) elasticity theory (Eringen 1972), strain gradient theory (SGT) (Lam *et al*. 2003), simple strain gradient theory (Aifantis 1999), nonlocal strain gradient theory (NLSGT) (Lim *et al*. 2015) are frequently encountered in the investigation of the mechanical response of nano/microscale elements and structures. Researchers have sometimes modeled nano-materials and the elements in devices of nano-electro-mechanical-systems and micro-electro-mechanical systems as small-scale rod elements and performed their size effect-dependent analyses. Demir and Civalek (2013) studied microtubules' axial and torsional vibrations based on NL elasticity. Khosravi *et al*. (2020a) presented the torsional vibration of rectangular cross-section nanorods with warping function and NL effects for clamped-clamped (C-C), clamped-free (C-F) and clamped–torsional spring boundary conditions (BCs). In addition, in another study, Khosravi *et al*. (2020b) demonstrated the torsional vibration of elliptical nanorods with warping function and NL effects. NL torsional vibration of graphene oxide powder-strengthened nano-composite nanorods was studied by Ebrahimi *et al*. (2023). Yaylı (2018) investigated the torsional vibration frequencies of nonlocal rods with elastic boundary conditions. Numanoğlu and Civalek (2019) proposed the NL finite element method for the torsion dynamics of nanorods in an elastic medium. Loya *et al*.

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(2014) investigated the torsional vibration of cracked nanorods with NL theory. Abdullah *et al*. (2020) presented the NL torsional vibration of nanorods using the Rayleigh-Ritz method with elastic medium and temperature effects. Arda (2020) presented the NL torsional vibrations of carbon nanotubes (NTs) based on the Maxwell and Kelvin-Voigt type viscoelastic material models. Uzun *et al*. (2022) investigated the NL torsional vibration of functionally graded (FG) porous NTs with elastic medium and torsional spring effects. Li and Hu (2017) carried out the torsional vibration of NTs formed from FG composite material in two directions with NL elasticity theory and analytical solution. In addition to these studies carried out with the NL elasticity theory, some studies in the literature (Adeli *et al*. 2017, Barretta *et al*. 2020, El-Borgi *et al*. 2018, Gheshlaghi *et al*. 2010, Hassannejad *et al*. 2022, Jahangiri *et al*. 2020, Khosravi *et al*. 2020c, Uzun and Yaylı 2022, Yayli 2018) also performed torsional vibration analyses based on other higher-order theories. As can be seen from the literature summary, the torsional vibration response of the investigated structures has been studied mostly in the idealized rigid form of the boundary conditions, and the studies (Selim *et al*. 2023, Yaylı 2018, Uzun and Yaylı 2022) presenting the effect of deformable boundaries remain in the minority.

Furthermore, it should be noted that higher-order elasticity theories are also frequently used in the analysis of various nano/micro-scale structures such as nanobeams (Jena *et al*. 2019, 2020, Glabisz *et al*. 2019, Kianian *et al*. 2024, Devnath *et al*. 2022, Sheykhi *et al*. 2023, Yue *et al*. 2021), reinforced nanobeam (Madenci *et al*. 2024, Ebrahimi *et al*. 2021, Uzun and Yaylı 2024), curved imperfect nanobeams (Ebrahimi *et al*. 2019), piezoelectric nanobeams (Moradi *et al*. 2023), viscoelastic nanobeams (Attia and Abdel Rahman 2018, Eltaher *et al*. 2023, Gholipour *et al*. 2022, Kadıoğlu and Yaylı 2025, Rahmani *et al*. 2022, Sobhy and Zenkour 2020, Kadıoğlu *et al*. 2025), bi-directional FG nanobeams (Thang *et al*. 2021), FG sandwich nanobeams (Wu and Liu 2021), FG microtubes/ nanotubes (NTs) (Beni 2020a, b, Shakhlavi *et al*. 2020), single-walled carbon nanotubes (Jena *et al*. 2024), microbeams (Dehkordi and Beni, 2023, Ye *et al*. 2023), nanoplates (Chu *et al*. 2023, Khien *et al*. 2024), nanoshells (Yi *et al*. 2020).

This study shows the torsional vibration of viscoelastic nano-sized tubes restrained with both deformable torsion springs and viscous damper based on the small-size effect. The small-size effect is considered with Eringen's NL elasticity theory. The NL tube, which is thought to be made of viscoelastic material, has both torsion springs that allow rotation and a viscous damper attached to both boundaries. The aim of this work is to present an analytical solution that allows the calculation of the NL torsional vibration frequencies of viscoelastic NTs under general BCs. For this purpose, an approach based on the combination of Stokes' transform, and the Fourier sine series is used for the first time. End of the solution procedures, a 2x2 eigenvalue problem able to calculate the viscoelastic NT's free torsional vibration frequencies is obtained. Then, the analyses performed with this solution are presented. In the analyses,

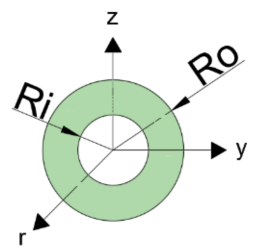

Fig. 1 Cross-section of a viscoelastic NT

the effects of the size parameter, viscous damping parameter, BCs and mode number on both imaginary and real frequencies of the viscoelastic NT are shown with a table and various graphs.

2. Definition of problem

NL theory will be used to obtain the equation of motion for torsional vibration. The general differential of NL theory is as follows (Eringen 1972).

$$(1-\gamma\nabla^2)\sigma_{kl}=\tau_{kl} \tag{1}$$

Here γ is the size parameter (SP) and ∇ is the Laplace operator. σ_{kl} and τ_{kl} are the NL stress tensor and the Cauchy stress tensor, respectively. This differential can be written in explicit form as follows.

$$\left(\sigma_{kl}-\gamma\frac{\partial^2\sigma_{kl}}{\partial x^2}\right)=\tau_{kl} \tag{2}$$

Thus, the equation of motion of a NL viscoelastic NT is found as follows (Arda 2020).

$$(1+\zeta\frac{\partial}{\partial t})GJ\frac{\partial^2\theta(x,t)}{\partial x^2}+\gamma\rho J\frac{\partial^4\theta(x,t)}{\partial x^2\partial t^2}-\rho J\frac{\partial^2\theta(x,t)}{\partial t^2}=0 \tag{3}$$

Here $\theta(x,t)=\theta$, G, J, ρ and ζ are rotation, shear modulus, polar moment of inertia, bulk density and viscous damping coefficient, respectively.

The polar moment of inertia for a NT presented in Fig. 1 is as follows.

$$J=2\pi\int_{R_i}^{R_o}r^3dr \tag{4}$$

Ro and Ri in Eq. 4 denote the outer and inner diameters of the NT respectively. Additionally, r is used to indicate the radius direction of the NT.

3. Solution of problem

In order to solve the problem, firstly, the equation of motion, which is bivariate depending on position and time, is converted to univariate by discretization using the following equation.

$$\theta(x,t)=\bar{\theta}(x)e^{i\omega t} \tag{5}$$

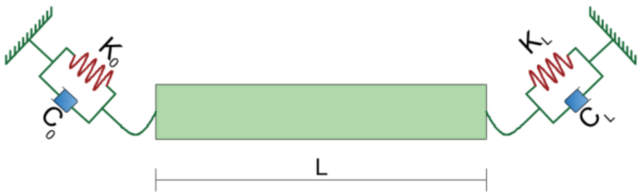

Fig. 2 BCs of a viscoelastic NT

Here ω is the angular frequency and $\bar{\theta}(x) = \bar{\theta}$ is the rotation function. Thus, the governing equation of a viscoelastic NL nanotube will be as follows.

$$(1 + i\omega\zeta)GJ\frac{d^2\bar{\theta}}{dx^2} - \omega^2\gamma\rho J\frac{d^2\bar{\theta}}{dx^2} + \omega^2\rho J\bar{\theta} = 0 \tag{6}$$

In order to express this equation in a simpler form, the following are presented.

$$A\frac{d^2\bar{\theta}}{dx^2} + B\bar{\theta} = 0 \tag{7}$$

$$A = (1 + i\omega\zeta)GJ - \omega^2\gamma\rho J \tag{8}$$

$$B = \omega^2\rho J \tag{9}$$

Fourier series are used together with Stokes transforms to solve this differential. Fourier sine series is chosen as the rotation function. The higher-order derivatives of this series with the help of Stokes' transform are as follows (Yaylı 2018).

$$\alpha_m = \frac{m\pi}{L} \tag{10}$$

$$\bar{\theta} = \sum_{m=1}^{N} A_m \mathrm{Sin}[\alpha_m x], \quad 0 < x < L \tag{11}$$

$$\frac{d\bar{\theta}}{dx} = \frac{\bar{\theta}_L - \bar{\theta}_0}{L} + \sum_{m=1}^{N}\left(\frac{2}{L}(\bar{\theta}_L(-1)^m - \bar{\theta}_0) + \alpha_m A_m\right)\mathrm{Cos}[\alpha_m x] \tag{12}$$

$$\frac{d^2\bar{\theta}}{dx^2} = -\sum_{m=1}^{N}\alpha_m\left(\frac{2}{L}(\bar{\theta}_L(-1)^m - \bar{\theta}_0) + \alpha_m A_m\right)\mathrm{Sin}[\alpha_m x] \tag{13}$$

The rotation functions for the points $x = 0$ and $x = L$ are defined by $\bar{\theta}_0$ and $\bar{\theta}_L$, respectively. N is defined as the number of terms, where N is infinite for a Fourier series. However, to solve the problem, it must be mathematically interrupted at some point. A_m is the Fourier coefficient. Substituting the above series in Eq. (8), the Fourier coefficient is found as follows.

$$A_m = -\frac{2A\alpha_m(\bar{\theta}_0 + (-1)^{1+m}\bar{\theta}_L)}{L(B - A\alpha_m^2)} \tag{14}$$

Fig. 2 illustrates the BCs under which the solution of the problem will be realized. As can be noticed, a viscoelastic model is utilized in the BCs. The solution is performed with the following equations.

$$x = 0 \rightarrow A\frac{\partial\theta}{\partial x} = K\theta + C\frac{\partial\theta}{\partial t} \tag{15}$$

$$x = L \rightarrow A\frac{\partial\theta}{\partial x} = -K\theta - C\frac{\partial\theta}{\partial t} \tag{16}$$

$$A\frac{d\bar{\theta}}{dx}|_{x=0} = K_0\bar{\theta}_0 + i\omega C_0\bar{\theta}_0 \tag{17}$$

$$A\frac{d\bar{\theta}}{dx}|_{x=L} = -K_L\bar{\theta}_L - i\omega C_L\bar{\theta}_L \tag{18}$$

Here K_0 and K_L are the torsion spring constants and C_0 and C_L are the viscosity. These equations can be written in matrix form as follows.

$$\begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix}\begin{bmatrix} \bar{\theta}_0 \\ \bar{\theta}_L \end{bmatrix} = 0 \tag{19}$$

when the determinant of this matrix is taken, angular frequencies can be calculated.

$$\begin{vmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{vmatrix} = 0 \tag{20}$$

where, K_{11}, K_{12}, K_{21} and K_{22} are defined as follows.

$$\mathrm{K}_{11} = (-\mathrm{K}_0 - i\omega\mathrm{C}_0 - \frac{\mathrm{A}}{L}) + \sum_{m=1}^{N}\frac{2LAB}{m^2\pi^2\mathrm{A} - L^2\mathrm{B}} \tag{21}$$

$$\mathrm{K}_{12} = \frac{A}{L} + \sum_{m=1}^{N}\frac{2(-1)^m LAB}{-m^2\pi^2\mathrm{A} + L^2\mathrm{B}} \tag{22}$$

$$\mathrm{K}_{21} = \frac{A}{L} + \sum_{m=1}^{N}\frac{2(-1)^m LAB}{-m^2\pi^2\mathrm{A} + L^2\mathrm{B}} \tag{23}$$

$$\mathrm{K}_{22} = (-\mathrm{K}_L - i\omega\mathrm{C}_L - \frac{\mathrm{A}}{L}) + \sum_{m=1}^{N}\frac{2LAB}{m^2\pi^2\mathrm{A} - L^2\mathrm{B}} \tag{24}$$

The above equations are the most general solution for torsional vibration of a NL viscoelastic tube under viscoelastic BCs. Thanks to the solution method used, if the torsion spring constants are chosen to be zero, the analysis can be performed for viscous BCs, while if the viscosity is chosen to be zero, calculations can be performed for elastic BCs. Most of the studies in the literature are for rigid BCs. Since this study is carried out for non-rigid BCs, it will fill the gap in the literature in this field.

4. Result and discussion

In this part, results and deductions on the torsional vibration of a NL viscoelastic tube are presented. The rubber material is selected to represent the viscoelastic

Table 1 Comparison of nanotube DFs

B.C.	γ	Eq. (27)					In this work N=20				
		λ1	λ2	λ3	λ4	λ5	λ1	λ2	λ3	λ4	λ5
C-C	0	3.142	6.283	9.425	12.566	15.708	3.142	6.283	9.425	12.566	15.708
	1	3.104	5.994	8.526	10.640	12.353	3.104	5.994	8.526	10.640	12.353
	2	3.067	5.742	7.843	9.394	10.510	3.067	5.742	7.843	9.394	10.510
	3	3.031	5.519	7.301	8.503	9.304	3.031	5.519	7.301	8.503	9.304
	4	2.997	5.320	6.859	7.825	8.436	2.997	5.320	6.859	7.825	8.436
	5	2.964	5.141	6.488	7.287	7.773	2.964	5.141	6.488	7.287	7.773
C-F	0	1.571	4.712	7.854	10.996	14.137	1.587	4.759	7.933	11.106	14.280
	1	1.566	4.587	7.310	9.635	11.544	1.582	4.630	7.374	9.709	11.622
	2	1.561	4.471	6.866	8.681	9.998	1.577	4.511	6.919	8.735	10.048
	3	1.556	4.363	6.494	7.963	8.943	1.572	4.400	6.538	8.005	8.979
	4	1.552	4.263	6.177	7.398	8.164	1.567	4.298	6.215	7.431	8.191
	5	1.547	4.169	5.902	6.939	7.559	1.562	4.202	5.935	6.966	7.580

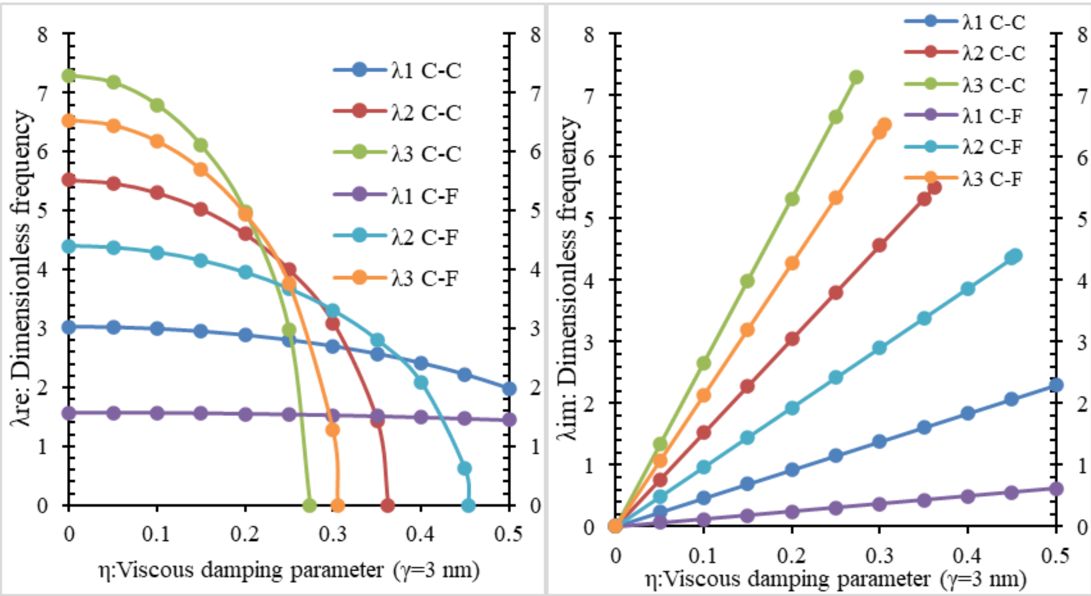

Fig. 3 Variation of DFs depending on the VDP

properties in a more realistic manner. The mechanical properties of this material E=4 MPa, ρ=1200 kg/m^3 and μ=0.45 (Possion's ratio) have been used for the analysis. Geometric properties are chosen as Ro=2 nm, Ri=1 nm and L/Ro=10 unless otherwise stated. Torsion spring constants 10^9 and viscosity 0 have been selected in order to obtain the results with the rigid BCs in the literature. Solutions have been calculated using 20 terms. In the evaluation of the results, a dimensionlessization process is applied. For this process, λ the dimensionless frequency (DF) and η viscous damping parameter (VDP) are defined as follows.

$$\lambda = \omega L\sqrt{\frac{\rho}{\mathrm{G}}} \tag{25}$$

$$\eta = \frac{\zeta}{L\sqrt{\frac{\rho}{\mathrm{G}}}} \tag{26}$$

In their study, Demir and Civalek (2013) presented the following equation to calculate the angular frequencies in torsional vibration of a NL tube with clamped-clamped BCs. In this study, the accuracy of the results obtained by using this equation is confirmed as can be clearly visible in Table 1.

$$\omega = \beta\sqrt{\frac{\mathrm{G}}{\rho(\beta^2\gamma + 1)}} \tag{27}$$

$\beta = \frac{n\pi}{L}$ for C-C BC and $\beta = \frac{(2n-1)\pi}{2L}$ for C-F BC. n is mode number.

Table 2 presents the variation of the DFs depending on the VDP at various SP values for NTs with two different BCs. As can be seen, the DFs appear as complex numbers for positive values of the VDP. For this reason, these complex numbers are evaluated separately as real and imaginary, as well as significant results that reveal the

Table 2 Variation of DFs depending on the VDP

	γ	η	λ1	λ2	λ3	λ4	λ5
C-C	1	0	3.10354	5.99434	8.52558	10.6404	12.3534
		0.05	3.09418+0.240799i	5.92664+0.898302i	8.32967+1.81714i	10.257+2.83043i	11.7495+3.81514i
		0.10	3.06594+0.481597i	5.71877+1.7966i	7.71217+3.63427i	9.00954+5.66086i	9.71516+7.63027i
		0.15	3.01829+0.722397i	5.3544+2.6949i	6.55497+5.45142i	6.41211+8.49129i	4.64845+11.44542i
		0.20	2.95029+0.963194i	4.79801+3.59321i	4.45575+7.26855i	-	-
		0.25	2.86048+1.20399i	3.96969+4.49151i	-	-	-
C-F		0	1.58151	4.63018	7.37379	9.70944	11.6215
		0.05	1.58028+0.0625293i	4.59905+0.535963i	7.24741+1.35932i	9.41905+2.35683i	11.1202+3.3765i
		0.10	1.57656+0.125059i	4.50439+1.07193i	6.85432+2.71864i	8.4885+4.71366i	9.45817+6.75299i
		0.15	1.57035+0.187588i	4.34203+1.60789i	6.14353+4.07795i	6.65443+7.07049i	5.69678+10.1295i
		0.20	1.56161+0.250117i	4.10395+2.14385i	4.98084+5.43727i	2.32354+9.42732i	-
		0.25	1.5503+0.312647i	3.77586+2.67982i	2.85991+6.79659i	-	-
C-C	3	0	3.0314	5.51903	7.30143	8.50254	9.3037
		0.05	3.02268+0.229735i	5.46624+0.761492i	7.17876+1.33277i	8.30823+1.80733i	9.04854+2.16397i
		0.10	2.99638+0.459469i	5.30473+1.52298i	6.79748+2.66554i	7.69593+3.61466i	8.23576+4.32794i
		0.15	2.95201+0.689204i	5.02403+2.28448i	6.10936+3.99831i	6.54944+5.42198i	6.66438+6.49191i
		0.20	2.88876+0.918939i	4.60236+3.04597i	4.98903+5.33108i	4.47551+7.22931i	3.41095+8.65587i
		0.25	2.80534+1.14867i	3.99536+3.80746i	2.98394+6.66386i	-	-
C-F		0	1.57172	4.40034	6.53838	8.00449	8.97877
		0.05	1.5705+0.061757i	4.37363+0.484074i	6.45044+1.06876i	7.84258+1.6018i	8.74964+2.01546i
		0.10	1.56686+0.123514i	4.29251+0.968148i	6.17911+2.13752i	7.33545+3.20359i	8.02309+4.03092i
		0.15	1.56076+0.185271i	4.1538+1.45222i	5.69826+3.20628i	6.40157+4.80539i	6.63775+6.04637i
		0.20	1.55218+0.247028i	3.95142+1.9363i	4.94717+4.27504i	4.7979+6.40719i	3.95287+8.06183i
		0.25	1.54109+0.308785i	3.67489+2.42037i	3.76752+5.3438i	-	-
C-C	5	0	2.96407	5.14138	6.48778	7.28694	7.77255
		0.05	2.95592+0.219643i	5.09874+0.660846i	6.40187+1.05228i	7.165+1.32749i	7.6244+1.51031i
		0.10	2.93134+0.439285i	4.9686+1.32169i	6.13694+2.10456i	6.78606+2.65497i	7.16159+3.02063i
		0.15	2.8899+0.658928i	4.74377+1.98254i	5.66794+3.15684i	6.10242+3.98246i	6.31531+4.53094i
		0.20	2.83087+0.878571i	4.4098+2.64338i	4.93706+4.20912i	4.99038+5.30995i	4.89038+6.04126i
		0.25	2.75311+1.09821i	3.93902+3.30423i	3.7959+5.26141i	3.00731+6.63744i	1.8402+7.55157i
C-F		0	1.5621	4.20165	5.93479	6.96607	7.58008
		0.05	1.56091+0.0610035i	4.1784+0.441345i	5.8691+0.880543i	6.85962+1.21315i	7.44273+1.43644i
		0.10	1.55733+0.122007i	4.10788+0.882691i	5.66748+1.76109i	6.52987+2.4263i	7.01457+2.87288i
		0.15	1.55134+0.183011i	3.98757+1.32404i	5.31446+2.64163i	5.93973+3.63946i	6.23597+4.30932i
		0.20	1.54292+0.244014i	3.81277+1.76538i	4.77661+3.52217i	4.99783+4.85261i	4.94407+5.74576i
		0.25	1.53203+0.305018i	3.5755+2.20673i	3.97968+4.40271i	3.42529+6.06576i	2.42355+7.1822i

relationships between each other are obtained by converting the results in this table into graphs.

Fig. 3 displays the variation of real and imaginary values of the DFs of the first three modes for C-C and C-F BCs depending on the VDP. When these graphs are analyzed, it is observed that the real values of the DFs in larger modes are zero for smaller values of the VDP. This is thought to be due to the fact that the Kelvin Voigt model is more suitable for single degree of freedom systems. However, if the VDP is limited to a certain value, it can be preferred for large modes. Eltaher *et al*. (2023) found similar results when they performed free vibration analysis of perforated NL beams. It is also evident from the graph that the effect of damping will be greater for more rigid BCs. Because the vibration of a more flexible structure is thought to be more difficult to damp. For this reason, it is predicted that the effect of damping will be higher in more rigid BCs. The imaginary values increase linearly as the VDP grows. The more vertical these lines are, the stronger the effect of the VDP on the real values of the DFs. These curves are terminated at the point where the real values are zero.

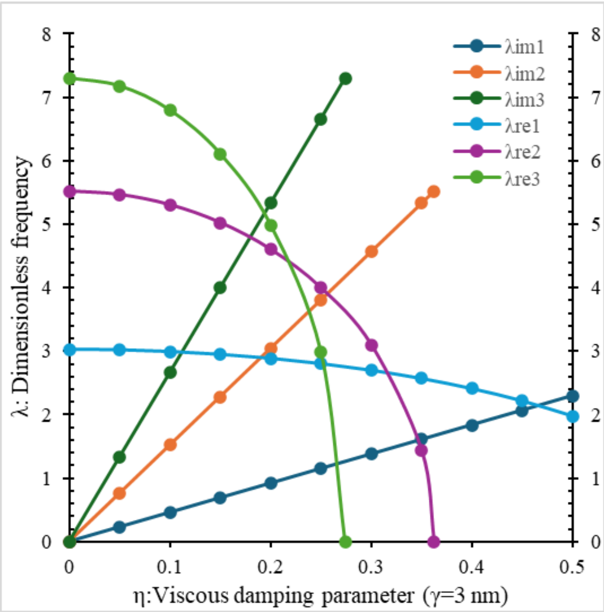

Fig. 4 Variation of real and imaginary values of DFs depending on the VDP for C-C BC

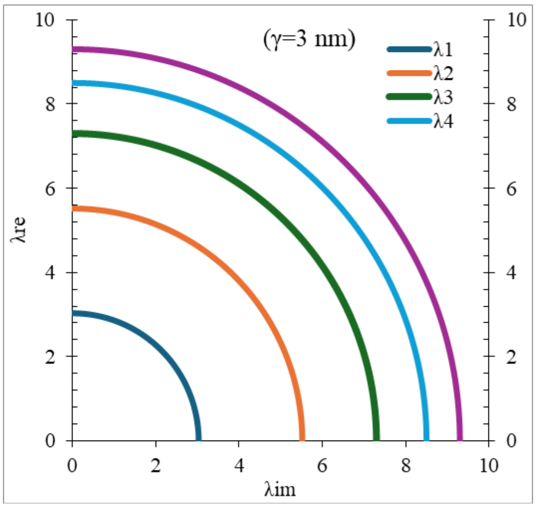

Fig. 5 Variation of real and imaginary values of the DFs of the viscoelastic NT for C-C BC

The DFs are expressed in terms of the sum of imaginary (λ_{im}) and real (λ_{re}) values depending on the variation of the VDP as follows.

$$\lambda = \lambda_{re} + \lambda_{im} \tag{28}$$

As indicated in Figure 3, the imaginary values change linearly. This change can be defined by a line equation as follows.

$$\lambda_{im} = k\eta \tag{29}$$

here k is the slope of the line.

Fig. 4 demonstrates the real and imaginary values of the DFs on a single graph. There is a noticeable point here. When the real value reaches zero, the imaginary value reaches the undamped DF (λ_{ud}), i.e., the DF at which the VDP is zero. From this it can be deduced at which value of the VDP the DF is equal to zero, as in the equation below.

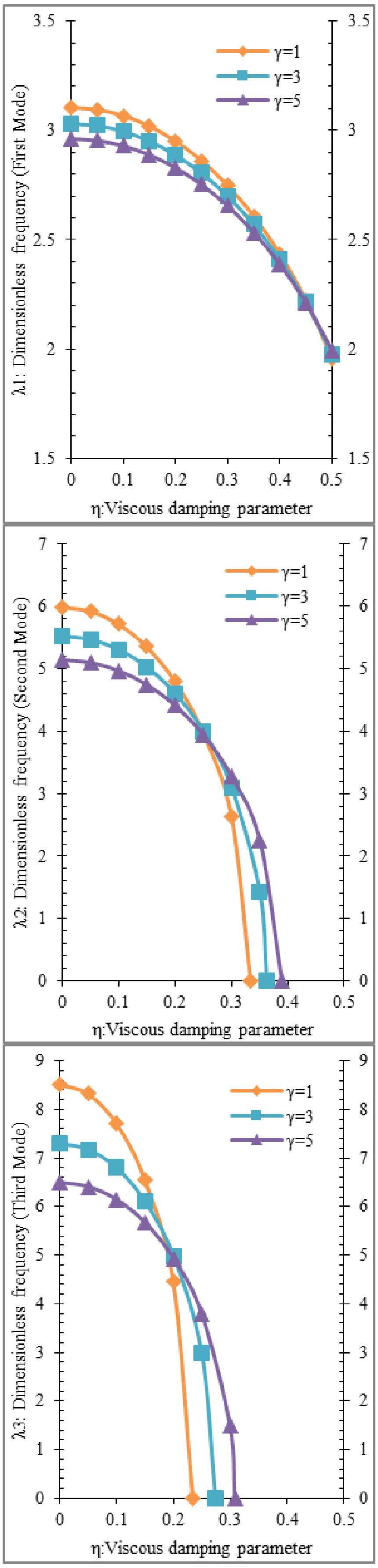

Fig. 6 Variation of real values of DFs at different SPs depending on the VDP for C-C BC

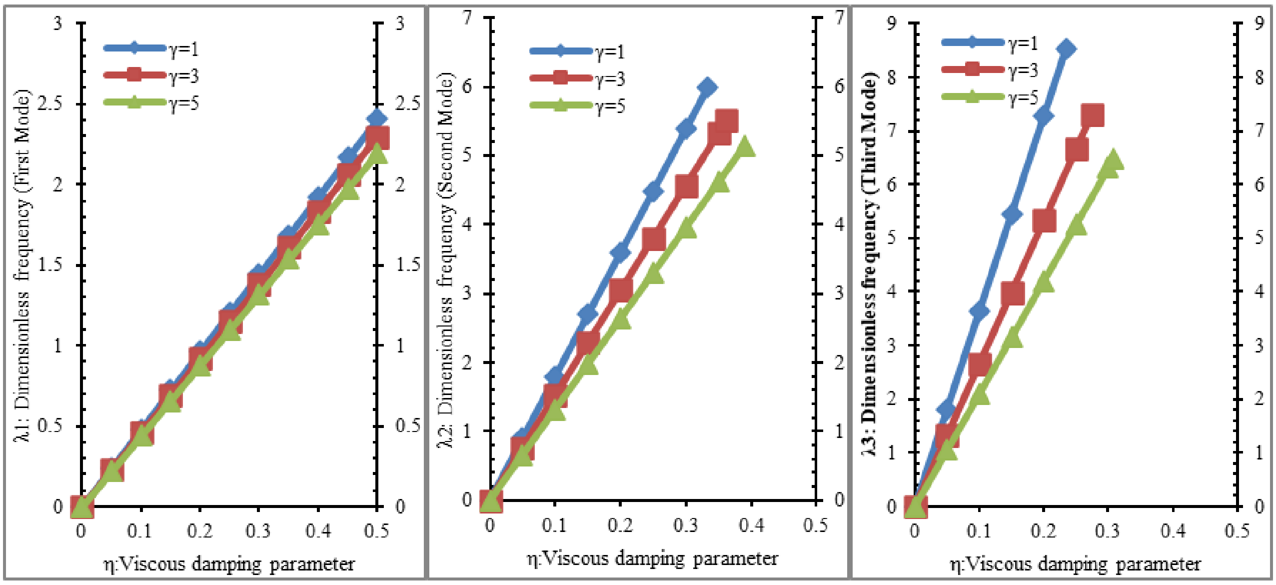

Fig. 7 Variation of imaginary values of DFs at different SPs depending on the VDP for C-C BC

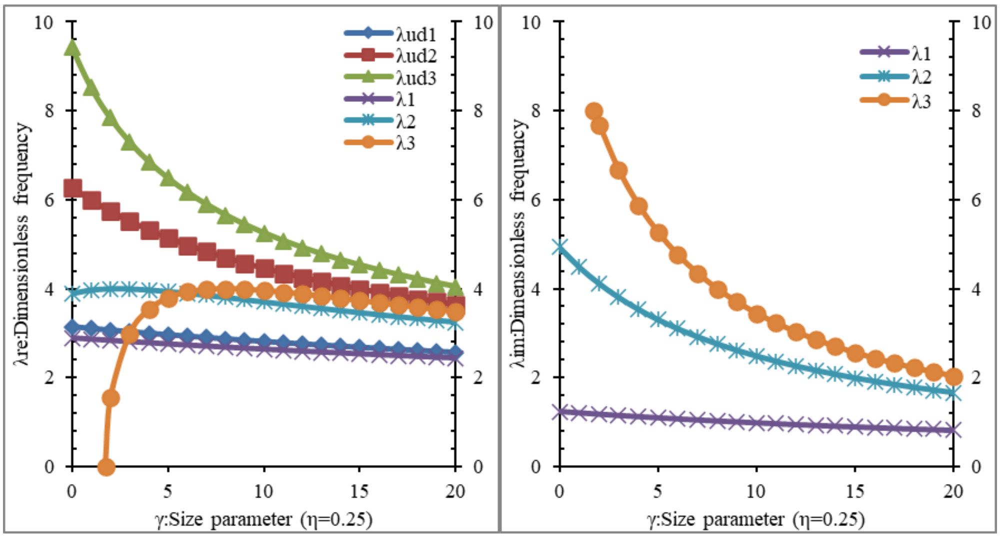

Fig. 8 Variation of real and imaginary values of the DFs of the viscoelastic NT for the first 3 modes depending on the SP for C-C BC

$$\lambda_{re} = 0 \rightarrow \eta = \frac{\lambda_{ud}}{k} \tag{30}$$

There is a relationship between real and imaginary values of DFs as in the equation below. A similar relationship was also found by Kadioglu *et al*. (2025).

$$\lambda_{re}{}^2 + \lambda_{im}{}^2 = \lambda_{ud}{}^2 \tag{31}$$

In the above equation, λ_{re} and λ_{im} are variables and λ_{ud} is a constant. This gives the formula for a circle of radius λ_{ud} with its center at the origin. This relationship is visualized and presented in Fig. 5.

Fig. 6 displays the variation of the real values of the DFs at different SPs as a function of the VDP. It is noticed that the effect of damping reduces as the value of the SP gets larger. This phenomenon is attributed to the increased flexibility of the beam due to the influence of the SP. According to NL theory, deformation at a given point is not solely dependent on local strain but also on the collective interaction with neighboring points. As a result, a more flexible structure exhibits vibrations that are more resistant to damping, leading to a reduced damping effect as the SP increases.

Fig. 7 illustrates the variation of the imaginary values. As mentioned before, the imaginary values vary linearly. Attia and Abdel Rahman (2018) proposed in their study that the variation of the VDP changes the imaginary values linearly.

Fig. 8 demonstrates the variations of the real and imaginary values of the DFs of the viscoelastic NT depending on the SP. The figure reveals that as the SP rises, the real values of the damped DFs and the real values of the undamped DFs converge. This convergence is more pronounced in larger modes. The imaginary values exhibit a decreasing trend as the SP increases.

Fig. 9 displays the variation of the real values of the DFs

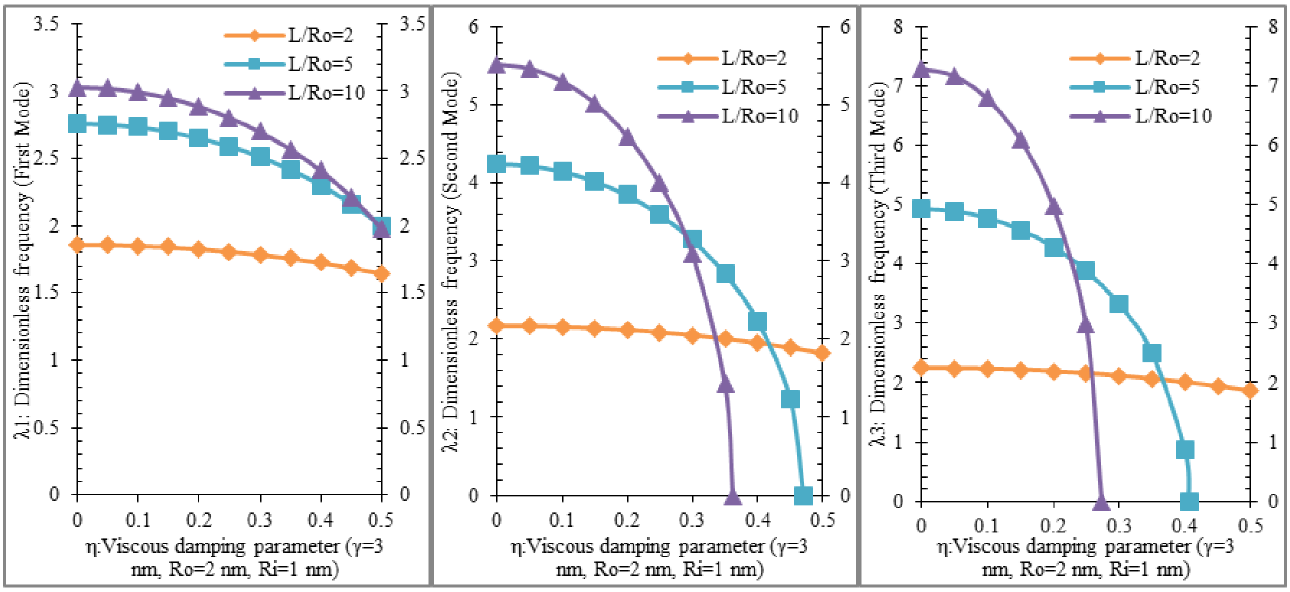

Fig. 9 Variation of real values of DFs for different L/Ro values depending on the VDP for C-C BC

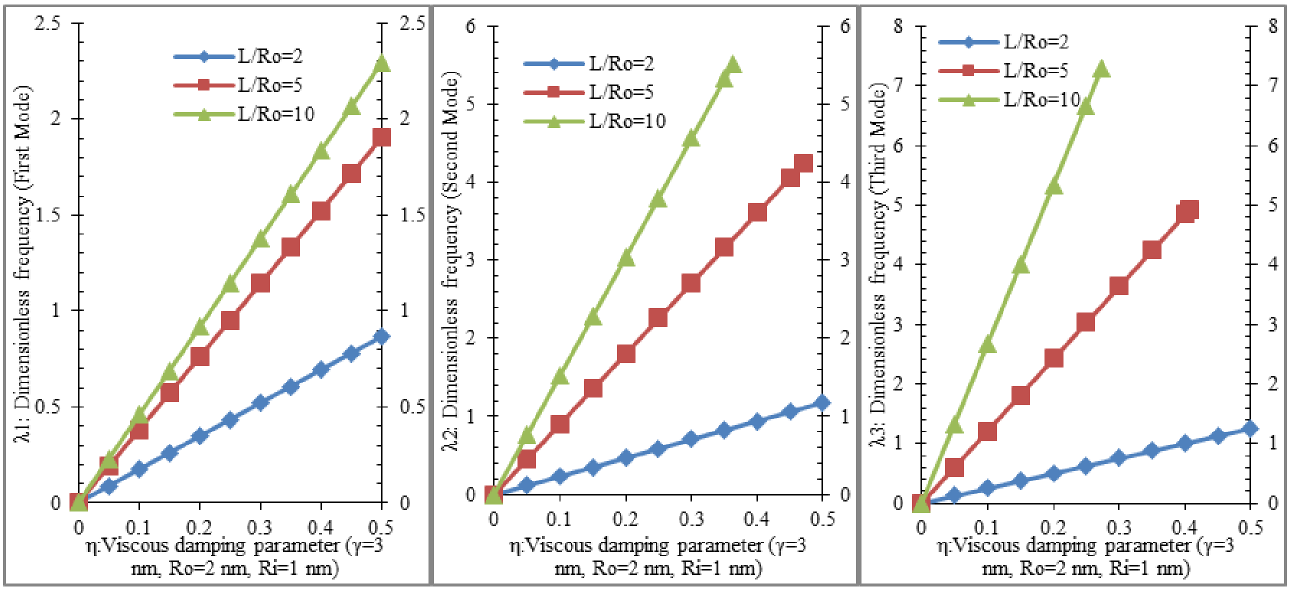

Fig. 10 Variation of imaginary values of DFs for different L/Ro values depending on the VDP for C-C BC

of viscoelastic NTs at three different L/Ro values dependent on the VDP. When the figures are examined, it may be thought that the effect of damping on the DFs with larger L/Ro values is greater. However, this may not the case. Since the increase in L/Ro reduces the influence of the SP, graphs like the figure may be formed. It should also be noted that this behavior is also a result of the dimensionlessization process performed. Because the frequency will change as a consequence of this, as the stiffness will decrease as L/Ro increases. Fig. 10 displays the variation of the imaginary values.

5. Conclusions

In this study, the torsional vibration of a viscoelastic NT is carried out for viscoelastic BCs using NL elasticity theory. Mathematical formulations are obtained using NL elasticity theory and the Kelvin Voight viscoelastic model. Fourier series and Stokes transforms, a popular solution method, are employed to solve the problem and the results are verified by checking the results with a closed solution available in the literature for rigid BCs. It should be emphasized that the most important advantage of this solution method is that it is possible to calculate for non-rigid BCs. Despite the fact that there are many studies for rigid BCs in the literature, there are very few studies for non-rigid BCs.

The results achieved in this study are briefly presented below.

• As the VDP rises, the real values of the DFs decrease in a curved shape while the imaginary values grow linearly.

• For more rigid BCs, the variation of the VDP has a more influence on the variation of the DFs.

• There is a relationship between the real and imaginary values of the DF that gives the geometric formula for a circle whose center is at the origin and whose radius is the undamped DF value. For this reason, at the point where the real values are zero, the imaginary values reach the undamped DF value.

• As the SP increases, the effect of damping on the real values of the DFs decreases.

• In classical theory, the increase of L/Ro has no impact on the DF. However, it reduces the effect of the SP on the DF.

This study presents a feasible solution method to analyze the torsional vibration of viscoelastic NTs under viscoelastic BCs. The results obtained reveal in detail the influence of the VDP on the frequency behavior, the role of the SP and its relationship with the stiffness of the BCs. Moreover, the method presented in this study is expected to provide a new perspective to the literature for viscoelastic BCs. In this manner, the proposed approach is expected to make a significant contribution to the dynamic analysis of NTs with non-rigid BCs and provide a good basis for future research.

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