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Dynamic analysis of piezoelectric smart sport balls based on analytical solution

Zhiwei Lu*[1], Li-Cai Zhao[2] and Yasser Alashker[3,4]

[1]*Department of Physical Education, Luoyang Institute of Science and Technology, Luoyang, Henan, 471023, China*
[2]*The Third Engineering Co., Ltd., China Railway 19th Bureau Group Co., Ltd., Shenyang 110136, Liaoning Province, P.R. China*
[3]*Department of Civil Engineering, College of Engineering, King Khalid University, PO Box 394, Abha 61411 Kingdom of Saudi Arabia*
[4]*Center for Engineering and Technology Innovations, King Khalid University, Abha 61421, Saudi Arabia*

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Abstract. A spherical shell panel with two nanocomposite piezoelectric (NCP) facesheets and a functionally graded porous (FGP) core is the subject of the present investigation. Not only are the layers firmly linked to one another, but their qualities are functionally graded as well. When an electric voltage is provided externally to NCP patches, carbon nanotubes (CNTs) are utilized to enhance their electro-mechanical performance. The kinematic relations are shown by applying von Karman's assumptions and the first-order shear deformation theory (FSDT). By utilizing Hamilton's principle and variational approach, the equations regulating motion are derived. The differential motion equations are solved using an analytical method based on Fourier series functions. After ensuring the accuracy of the results, the influence of various factors on the model's normalized frequencies is examined. These factors include the porosity index, patterns of pore distribution, patterns of CNT distribution, and other critical parameters. More efficient smart structures and devices might be in the works according to this study's results.
Keywords: carbon nanotubes-reinforced composites; electric field; porous materials; spherical shells; vibration analysis

1. Introduction

Nanoparticles have been widely studied as reinforcements for composite structures due to their unique properties that enable significant improvements in mechanical, thermal, and electrical performance (Arshid *et al*. 2025, Arshid *et al*. 2023a, b, c, Chen *et al*. 2023, Shen, 2012, Yao and Arshid 2024). Among these nanoparticles, carbon nanotubes (CNTs) have attracted considerable attention from researchers. Their exceptional characteristics have motivated their integration into composite materials, making them a subject of extensive investigation across various scientific and engineering disciplines. Carbon nanotubes are cylindrical structures composed of carbon atoms arranged in a hexagonal lattice. They are classified as single-walled carbon nanotubes (SWCNTs) or multi-walled carbon nanotubes (MWCNTs), depending on the number of concentric cylindrical layers (Arshid *et al*. 2024a, b, Givi *et al*. 2024, Rasooli Jazi *et al*. 2024, Soleimani-Javid *et al*. 2021). CNTs possess remarkable flexibility and resilience, enabling the absorption and distribution of mechanical stresses more effectively than many traditional materials. The nanoscale size of CNTs ensures that they can be uniformly dispersed within a polymer, ceramic, or metal matrix, creating composites with enhanced isotropic properties. Once dispersed effectively, CNTs provide a large interfacial area between the reinforcement and the matrix, which facilitates the efficient transfer of stress and enhances the overall mechanical properties of the composite (Amir *et al*. 2019, Arefi *et al*. 2018a, Arshid 2024, Hou *et al*. 2023, Kaveh *et al*. 2024, Melaibari *et al*. 2022, Song *et al*. 2016, Wang and Shen 2012, Zavari *et al*. 2024).

One of the primary advantages of CNT-reinforced composites lies in their application in fields where both high performance and reduced weight are crucial. For example, in the realm of sports equipment, CNT-based composites have revolutionized the design and functionality of various products (Arefi *et al*. 2019a, 2021, Bidgoli *et al*. 2022, Mohammad-Rezaei Bidgoli and Arefi 2019, 2023a). Tennis rackets, golf clubs, bicycles, and hockey sticks have been developed using CNT-enhanced composites, resulting in equipment that is not only lighter but also stronger and more durable. The improved stiffness and impact resistance provided by CNTs enssure better energy transfer during use, offering athletes an edge in performance. Moreover, the durability of CNT-reinforced composites allows for the development of sports equipment with longer lifespans, reducing the need for frequent replacements and contributing to sustainability efforts. In addition to mechanical benefits, the electrical conductivity of CNTs has been harnessed in the design of wearable sports gear that integrates sensors for performance monitoring, further broadening their application scope. The unique properties of CNTs have also been utilized to enhance the vibrational damping capabilities of sports equipment. By absorbing and dissipating energy, CNTs reduce vibrations transmitted to the user, enhancing comfort and reducing the risk of injury. This property is particularly valuable in sports like tennis or cycling, where repeated impacts and vibrations can lead to fatigue or strain. The versatility of CNT-reinforced composites extends beyond sports equipment, encompassing a wide range of industries, including aerospace, automotive, and biomedical sectors. Nonetheless, their use in sports represents a particularly

*Corresponding author, Ph.D.,
E-mail: Izw61503@163.com

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impactful application, as it combines advancements in material science with tangible benefits for end-users.

Sandwich structures incorporating nanocomposite layers as facesheets have garnered significant attention in materials science and engineering due to their unique combination of mechanical, thermal, and lightweight properties (Arshid *et al*. 2024c, Abaei *et al*. 2024, Frostig and Thomsen 2004, Kargar *et al*. 2021, Khoddami Maraghi and Arshid 2024, Li and Liu 2024, Shen *et al*. 2022, Soleimani-Javid *et al*. 2022). These structures, which typically consist of two outer facesheets bonded to a lightweight core, are widely recognized for their ability to provide high strength and stiffness while maintaining low overall weight. When nanocomposite materials are used as the facesheets, the performance of sandwich structures is significantly enhanced, motivating researchers to explore their potential across diverse scientific and technological fields. Nanocomposite layers, comprising a matrix material reinforced with nanoscale fillers such as CNTs, graphene, or other nano-particles, offer exceptional properties that make them particularly suitable for use in sandwich structures (Arefi *et al*. 2018b, Bidgoli and Arefi 2023, Khayat *et al*. 2021, Yang *et al*. 2021). By utilizing nanocomposite layers as facesheets, the mechanical strength, durability, and overall performance of the sandwich structure are substantially improved. The incorporation of nanoparticles imparts superior tensile strength, enhanced fracture toughness, and improved stiffness to the facesheets, enabling them to withstand higher loads and harsh operating conditions (Arefi *et al*. 2019b, Bidgoli and Arefi 2022, Mohammad-Rezaei Bidgoli and Arefi 2023b).

The advantages of nanocomposite layers in sandwich structures extend beyond mechanical properties. The inclusion of nanoparticles, such as CNTs or graphene, enhances thermal conductivity, enabling these structures to perform effectively in environments subjected to extreme temperatures (Ebrahimi-Mamaghani *et al*. 2023, 2024, Mamaghani *et al*. 2013). This thermal stability is particularly beneficial in applications where thermal cycling or exposure to high temperatures is prevalent. Furthermore, the electrical conductivity of certain nanocomposite materials introduces multifunctionality to the sandwich structures, allowing them to be used in applications requiring both structural integrity and electrical performance, such as electromagnetic shielding or energy storage systems (He *et al*. 2024, Long *et al*. 2024, Wang *et al*. 2024, Xie *et al*. 2025, Zhao *et al*. 2025). One of the unique properties of nanocomposite layers that has motivated their use in sandwich structures is their ability to provide superior damage resistance. Traditional materials used as facesheets, such as metals or fiber-reinforced polymers, often exhibit vulnerabilities to impact and fatigue. Nanocomposite layers, on the other hand, distribute stress more effectively due to the high aspect ratio and strength of the embedded nanoparticles, reducing the likelihood of crack initiation and propagation. This characteristic has made nanocomposite-enhanced sandwich structures a preferred choice in applications requiring high impact resistance and durability (Long *et al*. 2025).

The motivation to utilize nanocomposite layers in sandwich structures lies in the unparalleled combination of properties they offer. Through ongoing research and development, challenges such as nanoparticle dispersion and cost optimization are being addressed, paving the way for broader adoption of these advanced materials. By integrating the unique properties of nanocomposites into sandwich structures, scientists and engineers continue to push the boundaries of material performance, enabling innovations that benefit a wide array of fields (Deng *et al*. 2024, Ni *et al*. 2024, Xu *et al*. 2025, Yang *et al*. 2024, 2025). Numerous works have been provided in considering mechanical performances of engineering structures, such as beams, plates, and shell that can be seen in Refs. (Mousavi *et al*. 2021, Soleimani and Beni 2018).

Regarding the earlier statements, lightweight smart structures with wide-ranging applications in many fields of science and industry have received little attention in the literature, according to the authors findings. As a result of this finding, the authors set out to investigate how different kinds of materials and geometrical features affected the free vibration of spherical shells. We employed the first-order shear deformation theory (FSDT) to characterize the shell’s displacement components. By applying Hamilton’s principle, we were able to derive the motion equations and analyze how different parameters affected the shell’s frequencies. This led to the development of more practical and effective constructions. Our study can be seen as a benchmark for future research.

2. Mathematical formulations

2.1 Model’s description

The under-evaluation shell consists of three layers characterized by length a, width b, curvature radii R_{x_1} and R_{x_2} and an overall thickness of h. The middle layer consists of functionally graded porous materials (FGPMs), whereas the outer layers are composed of NCPMs that can be activated by an electric field. The thickness of each layer is represented as h_t, h_c, and h_b, respectively. The layers are presumed to be securely bonded, with no anticipated separation occurring between them.

2.2 Materials’ Relations

The core of the sandwich shell is composed of FGPMs. The constitutive law for the FGPMs core is expressed as follows (Alhaifi *et al*. 2021, Bekkaye *et al*. 2020):

$$\begin{Bmatrix} \sigma_{x_1x_1} \\ \sigma_{x_2x_2} \\ \tau_{x_2z} \\ \tau_{x_1z} \\ \tau_{x_1x_2} \end{Bmatrix}^c = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}^c \begin{Bmatrix} \varepsilon_{x_1x_1} \\ \varepsilon_{x_2x_2} \\ \gamma_{x_2z} \\ \gamma_{x_1z} \\ \gamma_{x_1x_2} \end{Bmatrix} \quad (1)$$

Stiffness matrix components, stress, and strain are all graphically represented in this equation. To be more precise, σ_{ij} stands for stress components and ε_{kl} for strain components. As described for the FGPM core in this specific scenario, Q_{ijkl} represents the stiffness matrix components (Amir *et al*. 2020, Luo *et al*. 2021):

$$\begin{gathered} Q_{11}^c = Q_{22}^c = E_c(z)/(1-\nu_c^2)\,, \\ Q_{12}^c = \nu_c E_c(z)/(1-\nu_c^2), \\ Q_{44}^c = Q_{55}^c = Q_{66}^c = G_c(z) \end{gathered} \tag{2}$$

In the following, we will talk about how the core's Poisson's ratio (ν_c) and Young's elasticity modulus ($E_c(z)$), which vary with thickness, are defined. The porosity of FGPMs is mostly determined by their e_0 value, which is the ratio of the bulk volume to the number of pores. The porosity value might be anywhere from 0 to 1, so keep that in mind. There are three possible configurations for the core's pores. There is no effect of thickness on the elasticity-density relationship in the first distribution (Distribution 1). Within this arrangement, we find (X. Shen *et al*. 2024):

$$\begin{Bmatrix} E_c \\ \rho_c \end{Bmatrix} = \begin{Bmatrix} E_1(1-e_0\gamma) \\ \rho_1\sqrt{1-e_m\gamma} \end{Bmatrix} \left(\frac{1}{e_0}\left[1 - \left(\frac{2}{\pi}\sqrt{1-e_0} - \frac{2}{\pi} + 1\right)^2\right]\right), \tag{3}$$

It is possible to determine the mass density coefficient (e_m) by utilizing the porosity coefficient. Here is the equation that represents this relationship (abaei *et al*. 2020):

$$e_m = 1 - \sqrt{1-e_0} \tag{4}$$

"Distribution 2" describes the second symmetrical pattern of pore implantation. The patterns of elasticity modulus, density, and porosity are described by the following equations (Kiran and Kattimani 2018):

$$\begin{Bmatrix} E_c(z) \\ \rho_c(z) \end{Bmatrix} = \begin{Bmatrix} E_1\left(1 - e_0\cos\left({}^{\pi z}/_{h_c}\right)\right) \\ \rho_1\left(1 - e_m\cos\left({}^{\pi z}/_{h_c}\right)\right) \end{Bmatrix}, \tag{5}$$

"Distribution 3" is the name given to the third, non-symmetrical pattern of pore implantation in this paper. The following functions determine the pattern's elasticity modulus and density as they change with core thickness (Ebrahimi *et al*. 2019):

$$\begin{Bmatrix} E_c(z) \\ \rho_c(z) \end{Bmatrix} = \begin{Bmatrix} E_1\left(1 - e_0\cos\left[\left({}^{\pi}/_{2h_c}\right)\left(z + {}^{h_c}/_2\right)\right]\right) \\ \rho_1\left(1 - e_m\cos\left[\left({}^{\pi}/_{2h_c}\right)\left(z + {}^{h_c}/_2\right)\right]\right) \end{Bmatrix}, \tag{6}$$

According to the constitutive legislation that follows, the NCPM facesheets behave as (Hamdia *et al*. 2018):

$$\begin{Bmatrix} \sigma_{x_1x_1} \\ \sigma_{x_2x_2} \\ \tau_{x_2z} \\ \tau_{x_1z} \\ \tau_{x_1x_2} \end{Bmatrix}^f = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}^f \begin{Bmatrix} \varepsilon_{x_1x_1} \\ \varepsilon_{x_2x_2} \\ \gamma_{x_2z} \\ \gamma_{x_1z} \\ \gamma_{x_1x_2} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^f \begin{Bmatrix} E_{x_1} \\ E_{x_2} \\ E_z \end{Bmatrix}, \tag{7}$$

Table 1 Values of CNTs' efficiency parameters based on different CNTs' volume fractions (Kaddari *et al*. 2020).

η_3	η_2	η_1	V_{CNT}^*
0.715	1.022	0.137	0.12
1.138	1.626	0.142	0.17
1.109	1.585	0.141	0.28

And the electric displacements are as (Razavi *et al*. 2017):

$$\begin{Bmatrix} D_{x_1} \\ D_{x_2} \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & 0 & 0 & 0 \end{bmatrix}^f \begin{Bmatrix} \varepsilon_{x_1x_1} \\ \varepsilon_{x_2x_2} \\ \gamma_{x_2z} \\ \gamma_{x_1z} \\ \gamma_{x_1x_2} \end{Bmatrix} + \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix}^f \begin{Bmatrix} E_{x_1} \\ E_{x_2} \\ E_z \end{Bmatrix}, \tag{8}$$

When the electric field components are denoted by E_k and the piezoelectric coefficients are represented by e_{kij}. Additionally, the electric displacement vector is denoted by D_i, and the dielectric permeability coefficients are denoted by κ_{ik}. Additional notation for the facesheets is the superscript *f*. These relations can be used to show the facesheets' stiffness components (Alhaifi *et al*. 2023, Arefi *et al*. 2022):

$$\begin{aligned} &Q_{11} = E_{11}/(1-\nu_{12}\nu_{21})\,, \\ &Q_{12} = Q_{21} = \nu_{21}E_{11}/(1-\nu_{12}\nu_{21})\,, \\ &Q_{22} = E_{22}/(1-\nu_{12}\nu_{21})\,, \\ &Q_{44} = G_{23}, \\ &Q_{55} = G_{13}, \\ &Q_{66} = G_{12}, \end{aligned} \tag{9}$$

In the given relationships, ν stands for Poisson's ratio, and E and G, respectively, represent Young's modulus and shear modulus. To forecast the same characteristics of nanocomposites, the extended rule of mixture (ERM) approach is frequently employed. It follows that (M. Babaei *et al*. 2022):

$$E_{11} = \eta_1 V_{CNT} E_{11}^{CNT} + V_m E_m, \tag{10}$$

$$^{\eta_2}/_{E_{22}} = {}^{V_{CNT}}\!/_{E_{22}^{CNT}} + {}^{V_m}/_{E_m}, \tag{11}$$

$$^{\eta_3}/_{G_{12}} = {}^{V_{CNT}}\!/_{G_{12}^{CNT}} + {}^{V_m}/_{G_m}, \tag{12}$$

The shear modulus, longitudinal and transverse Young's moduli, and E_{11}^{CNT}, E_{22}^{CNT}, G_{12}^{CNT}, E_m, and G_m, respectively, represent the CNTs and the matrix. The volume fraction of the matrix material is denoted by V_m, whereas the volume fraction of CNTs in the composite is represented by V_{CNT}. Ensuring the correctness of the ERM technique requires precise determination of the efficiency parameters η_1, η_2, and η_3 for the CNTs. This was accomplished by contrasting the elasticity moduli calculated by molecular dynamics simulations with those calculated by the ERM approach. Table 1 displays the outcomes.

Table 2 Material properties of single-walled CNTs at 300 Kelvin (Khoa *et al*. 2019)

Properties	Value
$\rho(kg/m^3)$	1400
$E_{11}(TPa)$	5.6466
$E_{22}(TPa)$	7.0800
$G_{12}(TPa)$	1.9445

Table 3 Material properties of PVDF matrix at 300 Kelvin

Properties	Value
$\rho(kg/m^3)$	1780
$E(GPa)$	2.2
ν	0.384
$e_{31}(C/m^2)$	-0.13
$e_{32}(C/m^2)$	-0.45
$e_{24}(C/m^2)$	-0.276
$e_{15}(C/m^2)$	-0.009
$\kappa_{11}=\kappa_{22}=\kappa_{33}(C/Vm)$	16×10^{-9}

On the other hand, a different equation can be used to get the skins' Poisson's ratio, which is not dependent on the CNT distribution pattern (Kiarasi *et al*. 2022):

$$\nu_{12}=V_{CNT}^{*}\nu_{12}^{CNT}+V_m\nu_m, \tag{13}$$

The formula V_{CNT}^{*}, which represents the volume fraction of CNTs, is defined as (Zhong *et al*. 2018):

$$V_{CNT}^{*}=\frac{w_{CNT}}{w_{CNT}+\left(\rho_{CNT}/\rho_m\right)-\left(\rho_{CNT}/\rho_m\right)w_{CNT}}, \tag{14}$$

In this case, w_{CNT} represents the weight percent of CNTs, and ρ_{CNT} and ρ_m denote the densities of CNTs and matrix, respectively. Table 2 lists the material parameters of single-walled carbon nanotubes, while Table 3 lists the same for the PVDF matrix.

This study explores various dispersion patterns of CNTs. Consequently, the following functions are established:

$$V_{CNT}^{i}=\begin{cases} V_{CNT}^{*} & \text{U}\\ \left[1-\frac{2}{h_i}\left(z\mp\frac{h_c+h_i}{2}\right)\right]V_{CNT}^{*} & \text{FG A}\\ \left[1+\frac{2}{h_i}\left(z\mp\frac{h_c+h_i}{2}\right)\right]V_{CNT}^{*} & \text{FG V}\\ 2\left[1-\frac{2}{h_i}\left(z\mp\frac{h_c+h_i}{2}\right)\right]V_{CNT}^{*} & \text{FG O}\\ \frac{4}{h_i}\left[\left|z\mp\frac{h_c+h_i}{2}\right|\right]V_{CNT}^{*} & \text{FG X}\end{cases}, \quad i=t,b \tag{15}$$

In order to guarantee the satisfaction of Maxwell's relations, a certain variation in the electric potential across the thickness direction of the face sheets is proposed (Karami *et al*. 2018, Zhang *et al*. 2025):

$$\bar{\Psi}(x_1,x_2,z,t)=\frac{2z_{b(or)t}V_0}{h_{b(or)t}} -\psi(x_1,x_2,t)\cos\left(\pi z_{b(or)t}/h_{b(or)t}\right) \tag{16}$$

By plugging the value of the applied voltage (V_0) into Eq. (16), we can get the electric field components. So (Huo *et al*. 2024):

$$\begin{aligned} E_{x_1}&=-\frac{\partial\bar{\Psi}}{\partial x_1}=\cos\left(\pi z_{b(or)t}/h_{b(or)t}\right)\frac{\partial\psi}{\partial x_1},\\ E_{x_2}&=-\frac{\partial\bar{\Psi}}{\partial x_2}=\cos\left(\pi z_{b(or)t}/h_{b(or)t}\right)\frac{\partial\psi}{\partial x_2},\\ E_z&=-\frac{\partial\bar{\Psi}}{\partial z}=-\frac{2V_0}{h_{b(or)t}}-\psi\frac{\pi}{h_{b(or)t}}\sin\left(\pi z_{b(or)t}/h_{b(or)t}\right)\end{aligned} \tag{17}$$

2.3 Kinematic relations

The following equation demonstrates how the FSDT can be used to express the displacements of any point in the shell when the shear deformation effect is taken into account (Liu *et al*. 2018):

$$\begin{aligned} \bar{u}(x_1,x_2,z,t)&=\left(1+\frac{z}{R_{x_1}}\right)u(x_1,x_2,t)+z\phi_{x_1}(x_1,x_2,t)\\ \bar{v}(x_1,x_2,z,t)&=\left(1+\frac{z}{R_{x_2}}\right)v(x_1,x_2,t)+z\phi_{x_2}(x_1,x_2,t)\\ \bar{w}(x_1,x_2,z,t)&=w(x_1,x_2,t)\end{aligned} \tag{18}$$

The structure's displacements along the x_1, x_2, and z axes are denoted by the symbols $\bar{u}$, $\bar{v}$, and $\bar{w}$, respectively. In addition, the rotations about the x_2 and x_1 axes are denoted by ϕ_{x_1} and ϕ_{x_2} accordingly. The strain components are determined using von Karman's assumptions:

$$\begin{aligned} \varepsilon_{x_1x_1}&=\frac{\partial u}{\partial x_1}+\frac{w}{R_{x_1}}+z\frac{\partial\phi_{x_1}}{\partial x_1}\\ \varepsilon_{x_2x_2}&=\frac{\partial v}{\partial x_2}+\frac{w}{R_{x_2}}+z\frac{\partial\phi_{x_2}}{\partial x_2}\\ \gamma_{x_1x_2}&=\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}+z\left(\frac{\partial\phi_{x_1}}{\partial x_2}+\frac{\partial\phi_{x_2}}{\partial x_1}-c_0\left(\frac{\partial v}{\partial x_1}-\frac{\partial u}{\partial x_2}\right)\right)\\ \gamma_{x_1z}&=\phi_{x_1}+\frac{\partial w}{\partial x_1}-\frac{u}{R_{x_1}}\\ \gamma_{x_2z}&=\phi_{x_2}+\frac{\partial w}{\partial x_2}-\frac{v}{R_{x_2}}\end{aligned} \tag{19}$$

In which:

$$c_0=\frac{1}{2}\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right) \tag{20}$$

The following is the formalized version of Hamilton's principle, which is used to derive the governing motion equations (Amirabadi *et al*. 2021):

$$\int_t\delta(U-T-W)\,dt=0 \tag{21}$$

U and T are the strain and kinetic energy, respectively, and W is the external work, as defined in Eq. (21). Thinking about the kinetic energy and how it changes is the initial step. To do this, we can calculate the fluctuations of kinetic energy using the following equation:

$$\delta T=\int_V\left(\rho\vec{V}.\delta\vec{V}\right)dV=\int_V \rho\left[\begin{matrix}\left(c_1\frac{\partial u}{\partial t}+z\frac{\partial\phi_{x_1}}{\partial t}\right)\left(c_1\frac{\partial\delta u}{\partial t}+z\frac{\partial\delta\phi_{x_1}}{\partial t}\right)\\+\left(c_2\frac{\partial v}{\partial t}+z\frac{\partial\phi_{x_2}}{\partial t}\right)\left(c_2\frac{\partial\delta v}{\partial t}+z\frac{\partial\delta\phi_{x_2}}{\partial t}\right)\end{matrix}+\frac{\partial w}{\partial t}\frac{\partial\delta w}{\partial t}\right]dV \quad (22)$$

Which leads to:

$$\begin{aligned}\delta T=\int_A\Bigg[&\left(-I_1\frac{\partial^2u}{\partial t^2}-I_2\frac{\partial^2\phi_{x_1}}{\partial t^2}\right)\delta u\\&+\left(-I_4\frac{\partial^2v}{\partial t^2}-I_5\frac{\partial^2\phi_{x_2}}{\partial t^2}\right)\delta v\\&+\left(-I_2\frac{\partial^2u}{\partial t^2}-I_3\frac{\partial^2\phi_{x_1}}{\partial t^2}\right)\delta\phi_{x_1}\\&+\left(-I_5\frac{\partial^2v}{\partial t^2}-I_3\frac{\partial^2\phi_{x_2}}{\partial t^2}\right)\delta\phi_{x_2}+\left(-I_0\frac{\partial^2w}{\partial t^2}\right)\delta w\Bigg]dA\end{aligned} \quad (23)$$

That:

$$(I_0,I_1,I_2,I_3,I_4,I_5)=\int_{-h/2}^{h/2}\rho\left(1,(c_1)^2,zc_1,z^2,(c_2)^2,zc_2\right)dz \quad (24)$$

Note that:

$$c_1=1+\frac{z}{R_{x_1}},\qquad c_2=1+\frac{z}{R_{x_2}} \quad (25)$$

At this point, we should have the structure's strain energy. Both the core and the facesheets contribute to the total strain energy of a shell. The strain energy can be calculated using the following formula:

$$\begin{aligned}U=\frac{1}{2}\int_V\big[&\sigma_{x_1x_1}\varepsilon_{x_1x_1}+\sigma_{x_2x_2}\varepsilon_{x_2x_2}+k_s\tau_{x_2z}\gamma_{x_2z}\\&+k_s\tau_{x_1z}\gamma_{x_1z}+\tau_{x_1x_2}\gamma_{x_1x_2}-D_{x_1}E_{x_1}\\&-D_{x_2}E_{x_2}-D_zE_z\big]dAdz\end{aligned} \quad (26)$$

You can ignore the last three phrases since they only apply to the piezoelectric facesheets and not the FGPM core. This leads us to the following expression for the strain energy variations:

$$\begin{aligned}\delta U=\int_A\int_z\Bigg[&\sigma_{x_1x_1}\left(\frac{\partial\delta u}{\partial x_1}+\frac{\delta w}{R_{x_1}}+z\frac{\partial\delta\phi_{x_1}}{\partial x_1}\right)\\&+\sigma_{x_2x_2}\left(\frac{\partial\delta v}{\partial x_2}+\frac{\delta w}{R_{x_2}}+z\frac{\partial\delta\phi_{x_2}}{\partial x_2}\right)\\&+k_s\tau_{x_2z}\left(\delta\phi_{x_2}+\frac{\partial\delta w}{\partial x_2}-\frac{\delta v}{R_{x_2}}\right)\\&+k_s\tau_{x_1z}\left(\delta\phi_{x_1}+\frac{\partial\delta w}{\partial x_1}-\frac{\delta u}{R_{x_1}}\right)\\&+\tau_{x_1x_2}\left(\frac{\partial\delta u}{\partial x_2}+\frac{\partial\delta v}{\partial x_1}+z\left(\begin{matrix}\frac{\partial\delta\phi_{x_1}}{\partial x_2}+\frac{\partial\delta\phi_{x_2}}{\partial x_1}\\-c_0\left(\frac{\partial\delta v}{\partial x_1}-\frac{\partial\delta u}{\partial x_2}\right)\end{matrix}\right)\right)\\&-D_{x_1}\left(\cos\left(\frac{\pi z_{b(or)t}}{h_{b(or)t}}\right)\frac{\partial\delta\psi}{\partial x_1}\right)-D_{x_2}\left(\cos\left(\frac{\pi z_{b(or)t}}{h_{b(or)t}}\right)\frac{\partial\delta\psi}{\partial x_2}\right)\\&-D_z\left(-\delta\psi\frac{\pi}{h_{b(or)t}}\sin\left(\frac{\pi z_{b(or)t}}{h_{b(or)t}}\right)\right)\Bigg],\end{aligned} \quad (27)$$

The shear correction factor of FSDT, denoted as k_s, is 5/6 in this report. Here is another way to express the same equation:

$$\begin{aligned}\delta U=\int_A\Bigg[&N^1_{x_1x_1}\frac{\partial\delta u}{\partial x_1}+N^2_{x_1x_1}\delta w+M_{x_1x_1}\frac{\partial\delta\phi_{x_1}}{\partial x_1}\\&+N^1_{x_2x_2}\frac{\partial\delta v}{\partial x_2}+N^2_{x_2x_2}\delta w+M_{x_2x_2}\frac{\partial\delta\phi_{x_2}}{\partial x_2}\\&+N^1_{x_2z}\left(\delta\phi_{x_2}+\frac{\partial\delta w}{\partial x_2}\right)\\&-N^2_{x_2z}\delta v+N^1_{x_1z}\left(\delta\phi_{x_1}+\frac{\partial\delta w}{\partial x_1}\right)-N^2_{x_1z}\delta u\\&+N_{x_1x_2}\left(\frac{\partial\delta u}{\partial x_2}+\frac{\partial\delta v}{\partial x_1}\right)+M^1_{x_1x_2}\left(\frac{\partial\delta\phi_{x_1}}{\partial x_2}+\frac{\partial\delta\phi_{x_2}}{\partial x_1}\right)\\&-M^2_{x_1x_2}\left(\frac{\partial\delta v}{\partial x_1}-\frac{\partial\delta u}{\partial x_2}\right)-\bar{D}_{x_1}\frac{\partial\delta\psi}{\partial x_1}-\bar{D}_{x_2}\frac{\partial\delta\psi}{\partial x_2}\\&+\bar{D}_z\delta\psi\Bigg]dA\end{aligned} \quad (28)$$

In the previous equation, the hired stress is defined as:

$$\begin{aligned}&\left(N^1_{x_1x_1},N^2_{x_1x_1},M_{x_1x_1}\right)=\int_{-h/2}^{h/2}\sigma_{x_1x_1}\left(1,\frac{1}{R_{x_1}},z\right)dz\\&\left(N^1_{x_2x_2},N^2_{x_2x_2},M_{x_2x_2}\right)=\int_{-h/2}^{h/2}\sigma_{x_2x_2}\left(1,\frac{1}{R_{x_2}},z\right)dz,\\&\left(N^1_{x_2z},N^2_{x_2z}\right)=\int_{-h/2}^{h/2}k_s\tau_{x_2z}\left(1,\frac{1}{R_{x_2}}\right)dz,\\&\left(N^1_{x_1z},N^2_{x_1z}\right)=\int_{-h/2}^{h/2}k_s\tau_{x_1z}\left(1,\frac{1}{R_{x_1}}\right)dz,\\&\left(N_{x_1x_2},M^1_{x_1x_2},M^2_{x_1x_2}\right)=\int_{-h/2}^{h/2}\tau_{x_1x_2}(1,z,zc_0)\,dz,\\&\left(\bar{D}_{x_1},\bar{D}_{x_2}\right)=\int_{-h/2}^{h/2}\cos\left(\frac{\pi z_{b(or)t}}{h_{b(or)t}}\right)\left(D_{x_1},D_{x_2}\right)dz,\\&\bar{D}_z=\int_{-h/2}^{h/2}\frac{\pi}{h_{b(or)t}}\sin\left(\frac{\pi z_{b(or)t}}{h_{b(or)t}}\right)D_z\,dz\end{aligned} \quad (29)$$

Finally, it yields:

$$\delta U = \int_A \left[\begin{array}{l} \left(-\frac{\partial N^1_{x_1x_1}}{\partial x_1} - N^2_{x_1z} - \frac{\partial N_{x_1x_2}}{\partial x_2} - \frac{\partial M^2_{x_1x_2}}{\partial x_2}\right)\delta u \\ + \left(-\frac{\partial N^1_{x_2x_2}}{\partial x_2} - N^2_{x_2z} - \frac{\partial N_{x_1x_2}}{\partial x_1} + \frac{\partial M^2_{x_1x_2}}{\partial x_1}\right)\delta v \\ + \left(N^2_{x_1x_1} + N^2_{x_2x_2} - \frac{\partial N^1_{x_2z}}{\partial x_2} - \frac{\partial N^1_{x_1z}}{\partial x_1}\right)\delta w \\ + \left(-\frac{\partial M_{x_1x_1}}{\partial x_1} + N^1_{x_1z} - \frac{\partial M^1_{x_1x_2}}{\partial x_2}\right)\delta \phi_{x_1} \\ + \left(-\frac{\partial M_{x_2x_2}}{\partial x_2} + N^1_{x_2z} - \frac{\partial M^1_{x_1x_2}}{\partial x_1}\right)\delta \phi_{x_2} \\ + \left(\frac{\partial \overline{D}_{x_1}}{\partial x_1} + \frac{\partial \overline{D}_{x_2}}{\partial x_2} + \overline{D}_z\right)\delta\psi \end{array} \right] dA \tag{30}$$

The piezoelectric skins are, as said before, supplied with voltage. By following these steps, you can determine how much energy has changed due to the external load:

$$\delta W = \int_A \left[-N_{x_1}\frac{\partial^2 w}{\partial x_1^2} - N_{x_2}\frac{\partial^2 w}{\partial x_2^2}\right] \delta w \, \mathrm{d}A, \tag{31}$$

N_{x_1} and N_{x_2} represent the applied electric voltage in the *x* and *y* courses, respectively:

$$N_{x_1} = N_{x_2} = -2e_{31}V_0 \tag{32}$$

We can obtain the equations that regulate motion in terms of stress resultants by substituting the derived formulas for changes in strain energy, kinetic energy, and external work into Hamilton's principle:

$$\begin{array}{l} \delta u: -\frac{\partial N^1_{x_1x_1}}{\partial x_1} - N^2_{x_1z} - \frac{\partial N_{x_1x_2}}{\partial x_2} - \frac{\partial M^2_{x_1x_2}}{\partial x_2} \\ +I_1\frac{\partial^2 u}{\partial t^2} + I_2\frac{\partial^2 \phi_{x_1}}{\partial t^2} = 0, \end{array} \tag{33}$$

$$\begin{array}{l} \delta v: -\frac{\partial N^1_{x_2x_2}}{\partial x_2} - N^2_{x_2z} - \frac{\partial N_{x_1x_2}}{\partial x_1} + \frac{\partial M^2_{x_1x_2}}{\partial x_1} \\ +I_4\frac{\partial^2 v}{\partial t^2} + I_5\frac{\partial^2 \phi_{x_2}}{\partial t^2} = 0, \end{array} \tag{34}$$

$$\begin{array}{l} \delta w: N^2_{x_1x_1} + N^2_{x_2x_2} - \frac{\partial N^1_{x_2z}}{\partial x_2} - \frac{\partial N^1_{x_1z}}{\partial x_1} \\ +I_0\frac{\partial^2 w}{\partial t^2} - N_{x_1}\frac{\partial^2 w}{\partial x_1^2} - N_{x_2}\frac{\partial^2 w}{\partial x_2^2} = 0, \end{array} \tag{35}$$

$$\begin{array}{l} \delta \phi_{x_1}: -\frac{\partial M_{x_1x_1}}{\partial x_1} + N^1_{x_1z} - \frac{\partial M^1_{x_1x_2}}{\partial x_2} + I_2\frac{\partial^2 u}{\partial t^2} \\ +I_3\frac{\partial^2 \phi_{x_1}}{\partial t^2} = 0, \end{array} \tag{36}$$

$$\begin{array}{l} \delta \phi_{x_2}: -\frac{\partial M_{x_2x_2}}{\partial x_2} + N^1_{x_2z} - \frac{\partial M^1_{x_1x_2}}{\partial x_1} + I_5\frac{\partial^2 v}{\partial t^2} \\ +I_3\frac{\partial^2 \phi_{x_2}}{\partial t^2} = 0, \end{array} \tag{37}$$

$$\delta\psi: \frac{\partial \overline{D}_{x_1}}{\partial x_1} + \frac{\partial \overline{D}_{x_2}}{\partial x_2} + \overline{D}_z = 0, \tag{38}$$

Eqs. (33)-(38) can be used to derive the displacement-based governing equations for motion once the stress resultants have been defined:

$$\begin{array}{l} \delta u: -\left(\begin{array}{c} A_1\frac{\partial^2 u}{\partial x_1^2} + A_2\frac{\partial w}{\partial x_1} + A_3\frac{\partial^2 \phi_{x_1}}{\partial x_1^2} + A_4\frac{\partial^2 v}{\partial x_1 \partial x_2} \\ +A_5\frac{\partial w}{\partial x_1} + A_6\frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} + A_7\frac{\partial \psi}{\partial x_1} \end{array} \right) \\ -\left(A_8\phi_{x_1} + A_8\frac{\partial w}{\partial x_1} - A_9 u - A_{10}\frac{\partial \psi}{\partial x_1}\right) \\ -\left(\begin{array}{c} A_{11}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{11}\frac{\partial^2 u}{\partial x_2^2} + A_{12}\frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} \\ +A_{12}\frac{\partial^2 \phi_{x_1}}{\partial x_2^2} - A_{13}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{13}\frac{\partial^2 u}{\partial x_2^2} \end{array} \right) \\ -\left(\begin{array}{c} A_{14}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{14}\frac{\partial^2 u}{\partial x_2^2} + A_{15}\frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} \\ +A_{15}\frac{\partial^2 \phi_{x_1}}{\partial x_2^2} - A_{16}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{16}\frac{\partial^2 u}{\partial x_2^2} \end{array} \right) \\ +I_1\frac{\partial^2 u}{\partial t^2} + I_2\frac{\partial^2 \phi_{x_1}}{\partial t^2} = 0 \end{array} \tag{39}$$

$$\begin{array}{l} \delta v: -\left(\begin{array}{c} A_{17}\frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{18}\frac{\partial w}{\partial x_2} + A_{19}\frac{\partial^2 \phi_{x_1}}{\partial x_1 \partial x_2} \\ +A_{20}\frac{\partial^2 v}{\partial x_2^2} + A_{21}\frac{\partial w}{\partial x_2} + A_{22}\frac{\partial^2 \phi_{x_2}}{\partial x_2^2} + A_{23}\frac{\partial \psi}{\partial x_2} \end{array} \right) \\ -\left(A_{24}\phi_{x_2} + A_{24}\frac{\partial w}{\partial x_2} - A_{25} v - A_{26}\frac{\partial \psi}{\partial x_2}\right) \\ -\left(\begin{array}{c} A_{11}\frac{\partial^2 v}{\partial x_1^2} + A_{11}\frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{12}\frac{\partial^2 \phi_{x_2}}{\partial x_1^2} \\ +A_{12}\frac{\partial^2 \phi_{x_1}}{\partial x_1 \partial x_2} - A_{13}\frac{\partial^2 v}{\partial x_1^2} + A_{13}\frac{\partial^2 u}{\partial x_1 \partial x_2} \end{array} \right) \\ +\left(\begin{array}{c} A_{14}\frac{\partial^2 v}{\partial x_1^2} + A_{14}\frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{15}\frac{\partial^2 \phi_{x_2}}{\partial x_1^2} \\ +A_{15}\frac{\partial^2 \phi_{x_1}}{\partial x_1 \partial x_2} - A_{16}\frac{\partial^2 v}{\partial x_1^2} + A_{16}\frac{\partial^2 u}{\partial x_1 \partial x_2} \end{array} \right) \\ +I_4\frac{\partial^2 v}{\partial t^2} + I_5\frac{\partial^2 \phi_{x_2}}{\partial t^2} = 0, \end{array} \tag{40}$$

$$\begin{array}{l} \delta \phi_{x_1}: -\left(\begin{array}{c} A_{27}\frac{\partial^2 u}{\partial x_1^2} + A_{28}\frac{\partial w}{\partial x_1} + A_{29}\frac{\partial^2 \phi_{x_1}}{\partial x_1^2} + A_{30}\frac{\partial^2 v}{\partial x_1 \partial x_2} \\ +A_{31}\frac{\partial w}{\partial x_1} + A_{32}\frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} + A_{33}\frac{\partial \psi}{\partial x_1} \end{array} \right) \\ +\left(A_{34}\phi_{x_1} + A_{34}\frac{\partial w}{\partial x_1} - A_{35} u - A_{36}\frac{\partial \psi}{\partial x_1}\right) \\ -\left(A_{37}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{37}\frac{\partial^2 u}{\partial x_2^2} + A_{38}\frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} + A_{38}\frac{\partial^2 \phi_{x_1}}{\partial x_2^2} \right. \\ \left. - A_{39}\frac{\partial^2 v}{\partial x_1 \partial x_2} + A_{39}\frac{\partial^2 u}{\partial x_2^2}\right) + I_2\frac{\partial^2 u}{\partial t^2} + I_3\frac{\partial^2 \phi_{x_1}}{\partial t^2} = 0, \end{array} \tag{41}$$

$$\delta \phi_{x_2}: -\left(\begin{array}{c} A_{40}\frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{41}\frac{\partial w}{\partial x_2} + A_{42}\frac{\partial^2 \phi_{x_1}}{\partial x_1 \partial x_2} \\ +A_{43}\frac{\partial^2 v}{\partial x_2^2} + A_{44}\frac{\partial w}{\partial x_2} + A_{45}\frac{\partial^2 \phi_{x_2}}{\partial x_2^2} + A_{46}\frac{\partial \psi}{\partial x_2} \end{array} \right) \tag{42}$$

$$
+\left(A_{47}\phi_{x_2}+A_{47}\frac{\partial w}{\partial x_2}-A_{48}v-A_{49}\frac{\partial \psi}{\partial x_2}\right)
-\left(A_{37}\frac{\partial^2 v}{\partial x_1^2}+A_{37}\frac{\partial^2 u}{\partial x_1\partial x_2}+A_{38}\frac{\partial^2\phi_{x_2}}{\partial x_1^2}
+A_{38}\frac{\partial^2\phi_{x_1}}{\partial x_1\partial x_2}-A_{39}\frac{\partial^2 v}{\partial x_1^2}+A_{39}\frac{\partial^2 u}{\partial x_1\partial x_2}\right)
+I_5\frac{\partial^2 v}{\partial t^2}+I_3\frac{\partial^2\phi_{x_2}}{\partial t^2}=0,
$$

$$
\begin{aligned}
\delta w:&\begin{pmatrix} A_{50}\frac{\partial u}{\partial x_1}+A_{51}w+A_{52}\frac{\partial\phi_{x_1}}{\partial x_1}+A_{53}\frac{\partial v}{\partial x_2}\\ +A_{54}w+A_{55}\frac{\partial\phi_{x_2}}{\partial x_2}+A_{56}\psi\end{pmatrix}\\
&+\begin{pmatrix} A_{57}\frac{\partial u}{\partial x_1}+A_{58}w+A_{59}\frac{\partial\phi_{x_1}}{\partial x_1}\\ +A_{60}\frac{\partial v}{\partial x_2}+A_{61}w+A_{62}\frac{\partial\phi_{x_2}}{\partial x_2}+A_{63}\psi\end{pmatrix}\\
&-\left(A_{47}\frac{\partial\phi_{x_2}}{\partial x_2}+A_{47}\frac{\partial^2 w}{\partial {x_2}^2}-A_{48}\frac{\partial v}{\partial x_2}-A_{49}\frac{\partial^2\psi}{\partial {x_2}^2}\right)\\
&-\left(A_{34}\frac{\partial\phi_{x_1}}{\partial x_1}+A_{34}\frac{\partial^2 w}{\partial {x_1}^2}-A_{35}\frac{\partial u}{\partial x_1}-A_{36}\frac{\partial^2\psi}{\partial {x_1}^2}\right)\\
&+N_E\left(\frac{\partial^2 w}{\partial {x_1}^2}+\frac{\partial^2 w}{\partial {x_2}^2}\right)+I_0\frac{\partial^2 w}{\partial t^2}=0
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\delta\psi:&\,A_{64}\frac{\partial\phi_{x_1}}{\partial x_1}+A_{64}\frac{\partial^2 w}{\partial {x_1}^2}-A_{65}\frac{\partial u}{\partial x_1}+A_{66}\frac{\partial^2\psi}{\partial {x_1}^2}\\
&+A_{67}\frac{\partial\phi_{x_2}}{\partial x_2}+A_{67}\frac{\partial^2 w}{\partial {x_2}^2}-A_{68}\frac{\partial v}{\partial x_2}+A_{69}\frac{\partial^2\psi}{\partial {x_2}^2}\\
&+A_{70}\frac{\partial u}{\partial x_1}+A_{71}w+A_{72}\frac{\partial\phi_{x_1}}{\partial x_1}+A_{73}\frac{\partial v}{\partial x_2}\\
&+A_{74}w+A_{75}\frac{\partial\phi_{x_2}}{\partial x_2}-A_{76}\psi=0
\end{aligned}
\tag{44}
$$

The Appendix defines the variables that are utilized in these equations.

3. Solution technique

The resultant differential equations are solved using Navier's method. This Fourier series-based closed-form approach meets supported boundary criteria. To be more precise, the following operations are expected to (Zenkour and Hafed 2020):

$$u(x_1,x_2,t)=U\cos\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{45}$$

$$v(x_1,x_2,t)=V\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{46}$$

$$w(x_1,x_2,t)=W\sin\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{47}$$

$$\phi_{x_1}(x_1,x_2,t)=\Phi_{x_1}\cos\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{48}$$

$$\phi_{x_2}(x_1,x_2,t)=\Phi_{x_2}\sin\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{49}$$

$$\psi(x_1,x_2,t)=\Psi\sin\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi x_2}{b}\right)e^{i\omega t}, \tag{50}$$

We have m for the x_1 axial wavenumber and n for the x_2 axial wavenumber.

The following equations are produced by replacing the assumed functions with the corresponding components of displacement in the governing motion equations:

$$([K]-\omega^2[M])\{X\}=\{0\}, \tag{51}$$

The matrixes [K] and [M] indicate stiffness and mass, respectively, and the natural frequency is denoted by ω in the resultant equations (Eq. (51)). Finding the structure's natural frequencies is as simple as solving the eigenvalue issue in Eq. (51).

4. Results and discussion

4.1 Validation study

The study checked the reliability of the results by comparing them to the previous results to make sure they were consistent. A spherical composite shell reinforced with one layer of FG-CNT was the object of the study. The normalized frequencies ($\hat{\omega}=\omega a^2/h\sqrt{\rho_m/E_m}$) were calculated and compared with those of Kiani (2017) based on various V_{CNT}^* and CNTs' distribution patterns, and the results were listed in Table 4 for the first two modes. Also, in Table 4, $a/b=1$, $a/R=0.5$ and $a/h=20$. It is clear that there was a high level of concordance between this study and the one conducted by Kiani (Kiani, 2017). The natural frequencies remained relatively close despite minor variations in the displacement field selection and solution approach. All things considered, the formulations and solution method were proven to be reliable, and the outcomes of the current investigation will be presented in the sub-section that follow.

4.2 Case study

Here we shall now give the study's results about the frequency domain. Keep in mind that the core material possesses specific characteristics, including a density (ρ_1) of 7850 kg/m^3, a Young's modulus (E_1) of 200 GPa, and a Poisson's ratio (ν) of 0.33. Additionally, it is worth mentioning that all the reported results are normalized with the aid of below relation:

$$Dimensionless\ Frequency=\omega h\sqrt{\rho_m/E_m}$$

where ω is the dimensional natural frequency. Moreover, for a spherical shell, $R_{x_1}=R_{x_2}=R$.

Table 5 illustrates the impact of the porosity coefficient and varying porosity distribution patterns on the normalized natural frequencies of the spherical shell. The data indicate that an increase in porosity correlates with a general decrease in frequencies, attributable to alterations in the shell's stiffness, which is inversely related to the stiffness-to-density ratio. Furthermore, the table highlights the effect of different pore placement patterns within the FGP core on the shell's frequencies, revealing that "Distribution 2" yields

Table 4 First two natural frequency parameters for spherical panels with SSSS boundary conditions

V^*_{CNT}	Distribution	$\widehat{\omega}_1$		$\widehat{\omega}_2$	
		Present	Kiani (2017)	Present	Kiani (2017)
0.12	U	17.9058	17.9031	23.5324	23.5431
	FG-X	19.8065	19.7988	25.1829	25.1890
	FG-O	15.2784	15.2777	21.5136	21.5338
	FG-V	16.4823	16.4618	22.9657	22.9602
	FG-A	16.2341	16.2611	22.0231	22.0731
0.17	U	22.3069	22.3035	29.8122	29.7966
	FG-X	24.7699	24.7601	32.1985	32.1017
	FG-O	19.0987	19.0980	27.2473	27.2808
	FG-V	20.5726	20.5414	29.2196	29.1698
	FG-A	20.2774	20.3214	28.0455	28.1247
0.28	U	25.5837	25.5868	32.8458	32.8576
	FG-X	28.7191	28.7121	36.4841	36.4174
	FG-O	21.8216	21.8389	29.6173	29.6657
	FG-V	23.5838	23.5241	32.0952	32.0265
	FG-A	23.4972	23.5648	31.3721	31.4888

Table 5 Effect of porosity distribution patterns and porosity index on the results

e_0	Porosity distribution		
	Distribution 1	Distribution 2	Distribution 3
0	0.2297	0.2297	0.2297
0.1	0.2267	0.2293	0.2271
0.2	0.2236	0.2291	0.2244
0.3	0.2203	0.2291	0.2215
0.4	0.2167	0.2295	0.2182
0.5	0.2129	0.2303	0.2145
0.6	0.2086	0.2317	0.2102
0.7	0.2039	0.2342	0.2052
0.8	0.1984	0.2386	0.1992
0.9	0.1917	0.2471	0.1923

Table 6 Effect of CNTs' distribution patterns and porosity index on the results

e_0	Pattern of CNTs				
	FG-UU	FG-OO	FG-VA	FG-AV	FG-XO
0	0.22966	0.22960	0.23141	0.22792	0.22967
0.1	0.22715	0.22708	0.22897	0.22533	0.22716
0.2	0.22443	0.22436	0.22633	0.22253	0.22444
0.3	0.22147	0.22140	0.22346	0.21948	0.22148
0.4	0.21819	0.21811	0.22029	0.21609	0.21820
0.5	0.21449	0.21441	0.21672	0.21226	0.21450
0.6	0.21024	0.21015	0.21263	0.20784	0.21025
0.7	0.20524	0.20515	0.20785	0.20262	0.20526
0.8	0.19925	0.19914	0.20215	0.19631	0.19926
0.9	0.19225	0.19213	0.19564	0.18882	0.19227

Table 7 Effect of CNTs' distribution patterns and R/h ratio on the results

R/h	Pattern of CNTs				
	FG-UU	FG-OO	FG-VA	FG-AV	FG-XO
10	0.48203	0.48201	0.48275	0.48133	0.48204
20	0.31071	0.31066	0.31205	0.30938	0.31072
30	0.26318	0.26312	0.26482	0.26155	0.26319
40	0.24407	0.24400	0.24585	0.24228	0.24408
50	0.23465	0.23458	0.23651	0.23278	0.23466
60	0.22937	0.22930	0.23128	0.22745	0.22938
70	0.22612	0.22605	0.22806	0.22418	0.22613
80	0.22399	0.22392	0.22596	0.22203	0.22400
90	0.22253	0.22245	0.22450	0.22054	0.22253
100	0.22147	0.22140	0.22346	0.21948	0.22148

the highest frequencies, whereas "Distribution 1" results in the lowest frequencies. The numerical values used to generate this table are as follows: $m = n = 1$, $R = 100h$, $V^*_{CNT} = 0.12$, $V_0 = 0$, $FG - UU$.

Table 6 indicates that the FG-VA pattern typically yields greater frequencies than the distribution patterns of other CNTs, while the FG-AV pattern exhibits the lowest values. The previous findings about the impact of the porosity index are corroborated. The numerical values used to generate this table are as follows: $m = n = 1$, $R = 100h$, $V^*_{CNT} = 0.12$, $V_0 = 0$, Porosity distribution 3.

Both the impact of the CNTs' distribution patterns and the radius-to-thickness ratio on the fundamental frequencies are taken into account in Table 7. The frequencies drop dramatically as the aforementioned ratio is increased. It happens because the shell's rigidity decreases. The numerical values used to generate this table are as follows: $m = n = 1$, $V^*_{CNT} = 0.12$, $V_0 = 0$, $e_0 = 0.3$, Porosity distribution 3.

Table 8 shows the same effect for distinct types of porosity distributions, the results show that "Distribution 2" produces the highest frequencies and "Distribution 1" produces the lowest. The numerical values used to generate this table are as follows: $m = n = 1$, $V^*_{CNT} = 0.12$, $V_0 = 0$, $e_0 = 0.3$, $\mathrm{FG - UU}$.

In Table 9, the impact of the thickness-to-width ratio on the outcomes for different pore distribution patterns is examined. A rise in the h_c/h_b ratio is accompanied with an improvement in the frequencies. The increase in the shell's rigidity is the cause of this. Reason being, the ratio of stiffness to mass, when squared, gives rise to the natural frequency. The distribution of all CNT kinds shown in the table follows this pattern, which is important to remember. It appears that the mechanical properties of the structure are more affected by changes in the shell's geometry than by the exact arrangement of CNTs within it. Nevertheless, the distribution type of CNTs utilized can affect the amount of the frequency augmentation. The numerical values used to generate this table are as follows: $m = n = 1$, $h = Constant$, $R = 100h$, $V^*_{CNT} = 0.12$, $V_0 = 0$, $e_0 = 0.3$, $\mathrm{FG - UU}$.

Table 8 Effect of porosity distribution patterns and R/h ratio on the results

R/h	Porosity distribution		
	Distribution 1	Distribution 2	Distribution 3
10	0.4814	0.4861	0.4820
20	0.3100	0.3166	0.3107
30	0.2623	0.2699	0.2632
40	0.2431	0.2512	0.2441
50	0.2336	0.2420	0.2346
60	0.2283	0.2369	0.2294
70	0.2250	0.2337	0.2261
80	0.2228	0.2316	0.2240
90	0.2214	0.2302	0.2225
100	0.2203	0.2291	0.2215

Table 9 Effect of porosity distribution patterns and h_c/h_b ratio on the results

h_c/h_b ($h_b = h_t$)	Porosity distribution		
	Distribution 1	Distribution 2	Distribution 3
2	0.1841	0.1888	0.1846
4	0.2110	0.2189	0.2120
6	0.2276	0.2371	0.2289
8	0.2383	0.2488	0.2397
10	0.2456	0.2567	0.2472
12	0.2510	0.2625	0.2526
14	0.2550	0.2669	0.2567
16	0.2582	0.2703	0.2599
18	0.2608	0.2730	0.2625
20	0.2629	0.2753	0.2646
22	0.2646	0.2772	0.2664
24	0.2661	0.2788	0.2679

Table 10 Effect of porosity distribution patterns and externally applied electric voltage on the results

$V_0(volt)$	Porosity distribution		
	Distribution 1	Distribution 2	Distribution 3
-500	0.1705637	0.1725112	0.1707299
-400	0.1705630	0.1725104	0.1707292
-300	0.1705622	0.1725097	0.1707284
-200	0.1705615	0.1725090	0.1707277
-100	0.1705608	0.1725083	0.1707270
0	0.1705601	0.1725076	0.1707263
100	0.1705594	0.1725069	0.1707256
200	0.1705586	0.1725062	0.1707249
300	0.1705579	0.1725055	0.1707241
400	0.1705572	0.1725048	0.1707234
500	0.1705565	0.1725040	0.1707227
-500	0.1705637	0.1725112	0.1707299

Table 10 shows the impact on the shell's natural frequencies of a change from a negative to a positive externally supplied electric voltage. The natural frequencies

Table 11 Effect of porosity distribution patterns, CNTs' distribution patterns, CNTs' volume fractions, and mode numbers on the results

Pattern of CNTs	V_{CNT}^*	m, n	Porosity distribution		
			Distribution 1	Distribution 2	Distribution 3
FG-UU	0.12	1,1	0.2203	0.2291	0.2215
		1,2	0.4862	0.5089	0.4894
		2,2	0.8101	0.8408	0.8147
	0.17	1,1	0.2294	0.2379	0.2306
		1,2	0.4926	0.5150	0.4957
		2,2	0.8408	0.8699	0.8452
	0.28	1,1	0.2452	0.2531	0.2462
		1,2	0.5007	0.5227	0.5037
		2,2	0.8915	0.9182	0.8956
FG-VA	0.12	1,1	0.2223	0.2311	0.2235
		1,2	0.4871	0.5097	0.4903
		2,2	0.8168	0.8472	0.8214
	0.17	1,1	0.2323	0.2406	0.2334
		1,2	0.4940	0.5163	0.4971
		2,2	0.8500	0.8787	0.8544
	0.28	1,1	0.2495	0.2573	0.2505
		1,2	0.5033	0.5251	0.5062
		2,2	0.9048	0.9309	0.9089

are lowered as the applied voltage is increased because the shell rigidity is reduced. The table clearly demonstrates that the natural frequencies tend to decrease as the applied voltage rises. There may be ramifications for the shell's behavior and performance due to the decrease in natural frequencies, even though the changes are minor. The numerical values used to generate this table are as follows: $m = n = 1\,, R = 100h\ , V_{CNT}^* = 0.12\ , e_0 = 0.3\,, \mathrm{FG-UU}$.

Table 11 provides a thorough analysis of how different parameters impact the natural frequencies of the under-evaluation shell. The most important thing to take away from this table is that the frequencies go up dramatically when the volume % of CNTs goes up. This is because CNTs are very stiff. The numerical values used to generate this table are as follows: $R = 100h\,, e_0 = 0.3$.

5. Conclusions

An FGP spherical shell with two NCPMs facesheets is tested for vibrational performance under an externally applied electric voltage in this study. The FSDT is used to characterize the displacements, and the layers' attributes are functionally graded. The natural frequencies of the shell with simply supported edges are extracted using Navier's solution method after the governing motion equations are determined by applying Hamilton's principle. Among the three patterns for porosity distribution that were evaluated, symmetric patterns had the highest frequency values, and the study examined and discussed the impacts of different parameters on the natural frequencies. It was shown that increasing porosity generally reduces the outcomes.

lowering the frequency is achieved by raising the external voltage applied to the NCP facesheets. Along the thickness path of the faces, the FG-VA pattern has the highest frequencies and the FG-AV pattern the lowest frequencies of the patterns of CNTs dispersion that were considered.

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References

Alhaifi, K., Arshid, E. and Khorshidvand, A.R. (2021), “Large deflection analysis of functionally graded saturated porous rectangular plates on nonlinear elastic foundation via GDQM”, *Steel Compos. Struct.*, **39**(6), 795-809. https://doi.org/10.12989/scs.2021.39.6.795.

Alhaifi, K., Khorshidvand, A.R., Al-Masoudy, M.M., Arshid, E. and Madani, S.H. (2023), “A shooting method for buckling and post-buckling analyses of FGSP circular plates considering various patterns of Pores’ placement”, *Struct. Eng. Mech.*, **85**(3), 419-432. https://doi.org/10.12989/sem.2023.85.3.419.

Amir, S., Arshid, E. and Maraghi, Z.K. (2020), “Free vibration analysis of magneto-rheological smart annular three-layered plates subjected to magnetic field in viscoelastic medium”, *Smart Struct. Syst.*, **25**(5), 581-592. https://doi.org/10.12989/sss.2020.25.5.581.

Amir, S., Soleimani-Javid, Z. and Arshid, E. (2019), “Size-dependent free vibration of sandwich micro beam with porous core subjected to thermal load based on SSDBT”, *ZAMM Zeitschrift Fur Angewandte Mathematik Und Mechanik*, **99**(9), 1-21. https://doi.org/10.1002/zamm.201800334.

Amirabadi, H., Farhatnia, F., Eftekhari, S.A. and Hosseini-Ara, R. (2021), “Wave propagation in rotating functionally graded GPL-reinforced cylindrical shells based on the third-order shear deformation theory”, *Waves Random Complex Med.*, **33**(2), 345-371. https://doi.org/10.1080/17455030.2021.1880031.

Arefi, M., Bidgoli, E.M.R. and Zenkour, A.M. (2018a), “Size-dependent free vibration and dynamic analyses of a sandwich microbeam based on higher-order sinusoidal shear deformation theory and strain gradient theory”, *Smart Struct. Syst.*, **22**(1), 27-40. https://doi.org/10.12989/sss.2018.22.1.027.

Arefi, M., Mohammad-Rezaei Bidgoli, E. and Zenkour, A.M. (2018b), “Free vibration analysis of a sandwich nano-plate including FG core and piezoelectric face-sheets by considering neutral surface”, *Mech. Adv. Mater. Struct.*, 1-12. https://doi.org/10.1080/15376494.2018.1455939.

Arefi, M., Mohammad-Rezaei Bidgoli, E. and Rabczuk, T. (2019a), “Effect of various characteristics of graphene nanoplatelets on thermal buckling behavior of FGRC micro plate based on MCST”, *Eur. J. Mech. A Solids*, **77**, 103802. https://doi.org/10.1016/j.euromechsol.2019.103802.

Arefi, M., Mohammad-Rezaei Bidgoli, E. and Rabczuk, T. (2019b), “Thermo-mechanical buckling behavior of FG GNP reinforced micro plate based on MSGT”, *Thin-Walled Struct.*, **142**, 444-459. https://doi.org/10.1016/j.tws.2019.04.054.

Arshid, E., Amir, S. and Loghman, A. (2023a), “On the vibrations of FG GNPs-RPN annular plates with piezoelectric/metallic coatings on Kerr elastic substrate considering size dependency and surface stress effects”, *Acta Mechanica*, 1-42. https://doi.org/10.1007/s00707-023-03593-4.

Arefi, M., Moghaddam, S.K., Bidgoli, E.M.R., Kiani, M. and Civalek, O. (2021), “Analysis of graphene nanoplatelet reinforced cylindrical shell subjected to thermo-mechanical loads”, *Compos. Struct.*, **255**, 112924. https://doi.org/10.1016/J.COMPSTRUCT.2020.112924.

Arefi, M., Mohammad-Rezaei Bidgoli, E. and Civalek, O. (2022), “Bending response of FG composite doubly curved nanoshells with thickness stretching via higher-order sinusoidal shear theory”, *Mech. Based Des. Struct.*, **50**(7), 2350-2378. https://doi.org/10.1080/15397734.2020.1777157.

Arshid, E., Ghorbani, M.A., Momeni Nia, M.J., Civalek, Ö. and Kumar, A. (2023b), “Thermo-elastic buckling behaviors of advanced fluid-infiltrated porous shells integrated with GPLs-reinforced nanocomposite patches”, *Mech. Adv. Mater. Struct.*, 1-17. https://doi.org/10.1080/15376494.2023.2251015.

Arshid, E., Momeni Nia, M.J., Ghorbani, M.A., Civalek, Ö. and Kumar, A. (2023c), “On the poroelastic vibrations of lightweight FGSP doubly-curved shells integrated with GNPs-reinforced composite coatings in thermal atmospheres”, *Appl. Math. Modell.*, **124**, 122-141. https://doi.org/10.1016/j.apm.2023.07.036.

Arshid, E., Amir, S. and Loghman, A. (2024a), “Aero-Hygro-Thermoelastic size-dependent analysis of NCMF-reinforced GNPs sector microplates located between piezoelectric patches in supersonic flow considering surface stress effects”, *Mech. Based Des. Struct.*, 1-62. https://doi.org/10.1080/15397734.2023.2295532.

Arshid, E., Amir, S., Loghman, A. and Civalek, Ö. (2024b), “Aerodynamic stability and free vibration of fgp-reinforced nano-fillers annular sector microplates exposed to supersonic flow”, *Thin Wall. Struct.*, 111610. https://doi.org/10.1016/j.tws.2024.111610.

Arshid, E., Azimi, M., Moradi, M., El Ouni, M.H. and Alashker, Y. (2024c), “Mathematical solution for vibrational response of shear and normal deformable advanced metal foam CMBs treated with nanocomposite actuators”, *Int. J. Struct. Stabil. Dyn.*, 2550118. https://doi.org/10.1142/S0219455425501184.

Arshid, E., Maraghi, Z.K. and Civalek, Ö. (2025), “Variable-thickness higher-order sandwich beams with FG cellular core and CNT-RC patches: Vibrational analysis in thermal environment”, *Arch. Appl. Mech.*, **95**(1), 2. https://doi.org/10.1007/s00419-024-02716-0.

Babaei, H., Zavari, S., Kaveh, A., Arshid, E. and Civalek, O. (2024), “Dynamic response of advanced lightweight porous plates integrated with nanocomposite face sheets resting on elastic substrate”, *Int. J. Struct. Stabil. Dyn.*, https://doi.org/10.1142/S0219455425501329.

Babaei, M., Hajmohammad, M. H. and Asemi, K. (2020), “Natural frequency and dynamic analyses of functionally graded saturated porous annular sector plate and cylindrical panel based on 3D elasticity”, *Aerosp. Sci. Technol.*, **96**, 105524. https://doi.org/10.1016/j.ast.2019.105524.

Babaei, M., Kiarasi, F., Asemi, K., Dimitri, R. and Tornabene, F. (2022), “Transient thermal stresses in FG porous rotating truncated cones reinforced by graphene platelets”, *Appl. Sci.*, **12**(8), 3932. https://doi.org/10.3390/app12083932.

Bekkaye, T.H.L., Fahsi, B., Bousahla, A.A., Bourada, F., Tounsi, A., Benrahou, K.H., Tounsi, A. and Al-Zahrani, M.M. (2020), “Porosity-dependent mechanical behaviors of FG plate using refined trigonometric shear deformation theory”, *Comput. Concr.*, **26**(5), 450. https://doi.org/10.12989/CAC.2020.26.5.439.

Bidgoli, E.M.R. and Arefi, M. (2022), *Free Vibration Analysis of Microplates Reinforced with Functionally Graded Graphene Nanoplatelets*, In *Innovations in Graphene-Based Polymer Composites*, 521-557, Elsevier. https://doi.org/10.1016/B978-0-12-823789-2.00003-0.

Bidgoli, E.M.R. and Arefi, M. (2023), “Effect of porosity and characteristics of carbon nanotube on the nonlinear characteristics of a simply-supported sandwich plate", *Arch. Civil Mech. Eng.*, **23**(3), 214. https://doi.org/10.1007/s43452-023-00752-1.

Bidgoli, E.M.R., Arefi, M. and Mohammadimehr, M. (2022), "Free vibration analysis of honeycomb doubly curved shell integrated with CNT-reinforced piezoelectric layers", *Mech. Based Des. Struct.*, **50**(12), 4409-4440. https://doi.org/10.1080/15397734.2020.1836969.

Chen, J., Yang, Y., Yu, S., Zhang, Y., Hou, J., Yu, N. and Fang, B. (2023), "MOF-derived nitrogen-doped porous carbon polyhedrons/carbon nanotubes nanocomposite for high-performance lithium-sulfur batteries", *Nanomaterials*, **13**(17), 2416. https://doi.org/10.3390/nano13172416.

Deng, K., Yang, L., Lu, Y. and Ma, S. (2024), "Multitype chatter detection via multichannelinternal and external signals in robotic milling", *Measurement*, **229**, 114417. https://doi.org/10.1016/j.measurement.2024.114417.

Ebrahimi-Mamaghani, A., Koochakianfard, O., Mostoufi, N. and Khodaparast, H.H. (2023), "Dynamics of spinning pipes conveying flow with internal elliptical cross-section surrounded by an external annular fluid by considering rotary inertia effects", *Appl. Math. Modell.*, **120**, 330-354. https://doi.org/10.1016/j.apm.2023.03.043.

Ebrahimi-Mamaghani, A., Koochakianfard, O., Rafiei, M., Alibeigloo, A., Dizaji, A.S. and Borjalilou, V. (2024), "Machine learning, analytical, and numerical techniques for vibration analysis of submerged porous functional gradient piezoelectric microbeams with movable supports", *Int. J. Struct. Stabil. Dyn.*, 2650054. https://doi.org/10.1142/S0219455426500549.

Ebrahimi, F., Jafari, A. and Mahesh, V. (2019), "Assessment of porosity influence on dynamic characteristics of smart heterogeneous magneto-electro-elastic plates", *Struct. Eng. Mech.*, **72**(1), 875-891. https://doi.org/10.12989/sem.2019.72.1.113.

Frostig, Y. and Thomsen, O.T. (2004), "High-order free vibration of sandwich panels with a flexible core", *Int. J. Solids Struct.*, **41**(5-6), 1697-1724. https://doi.org/10.1016/j.ijsolstr.2003.09.051.

Givi, S., Ghorbanpour Arani, A., Khoddami Maraghi, Z. and Arshid, E. (2024), "Free vibration and supersonic flutter analyses of a sandwich cylindrical shell with CNT-reinforced honeycomb core integrated with piezoelectric layers", *Mech. Based Des. Struct.*, 1-29. https://doi.org/10.1080/15397734.2024.2422413.

Hamdia, K.M., Ghasemi, H., Zhuang, X., Alajlan, N. and Rabczuk, T. (2018), "Sensitivity and uncertainty analysis for flexoelectric nanostructures", *Comput. Meth. Appl. Mech. Eng.*, **337**, 95-109. https://doi.org/10.1016/j.cma.2018.03.016.

He, L., Gao, Y., Liu, D., Hu, Y., Shi, J., Zhang, J., Li, X., Jin, B., Zhang, B., Wang, Z.L. and Wang, J. (2024), "Dynamic interfacial electrostatic energy harvesting via a single wire", *Sci. Adv.*, **10**(24). https://doi.org/10.1126/sciadv.ado5362.

Hou, Y., Choi, K.R., Ghazouani, N., Kaveh, A., Babaei, Z. and Kumar, A. (2023), "Static package design thorough thickness stretching and thermal environment effects for porous micro-scaled plates with piezoelectric nanocomposite patches", *Acta Mechanica*, **235**(2), 1235-1254. https://doi.org/10.1007/S00707-023-03794-X/METRICS.

Huo, S., Qu, H., Meng, F., Zhang, Z., Yang, Z., Zhang, S., Hu, X. and Wu, E. (2024), "Negative differential resistance with ultralow peak-to-valley voltage difference in Td-WTe 2 /2H-MoS 2 heterostructure", *Nano Lett.*, **24**(38), 11937-11943. https://doi.org/10.1021/acs.nanolett.4c03263.

Kaddari, M., Kaci, A., Bousahla, A.A., Tounsi, A., Bourada, F., Tounsi, A., Bedia, E.A.A. and Al-Osta, M.A. (2020), "A study on the structural behaviour of functionally graded porous plates on elastic foundation using a new quasi-3D model: Bending and free vibration analysis", *Comput. Concr.*, **25**(1), 57. https://doi.org/10.12989/CAC.2020.25.1.037.

Karami, B., Shahsavari, D. and Janghorban, M. (2018), "Wave propagation analysis in functionally graded (FG) nanoplates under in-plane magnetic field based on nonlocal strain gradient theory and four variable refined plate theory", *Mech. Adv. Mater. Struct.*, **25**(12), 1047-1057. https://doi.org/10.1080/15376494.2017.1323143.

Kargar, J., Ghorbanpour Arani, A., Arshid, E. and Irani Rahaghi, M. (2021), "Vibration analysis of spherical sandwich panels with MR fluids core and magneto-electro-elastic face sheets resting on orthotropic viscoelastic foundation", *Struct. Eng. Mech.*, **78**(5), 572. https://doi.org/10.12989/SEM.2021.78.5.557.

Kaveh, A., Babaei, H., Zavari, S., Arshid, E. and Civalek, Ö. (2024), "Vibrational response of a sandwich microplate considering the impact of flexoelectricity and based on a novel porous-FGM formulation", *Mech. Based Des. Struct.*, 1-22. https://doi.org/10.1080/15397734.2024.2337913.

Khayat, M., Baghlani, A. and Najafgholipour, M.A. (2021), "The propagation of uncertainty in the geometrically nonlinear responses of smart sandwich porous cylindrical shells reinforced with graphene platelets", *Compos. Struct.*, **258**, 113209. https://doi.org/10.1016/j.compstruct.2020.113209.

Khoa, N.D., Anh, V.M. and Duc, N.D. (2019), "Nonlinear dynamic response and vibration of functionally graded nano-composite cylindrical panel reinforced by carbon nanotubes in thermal environment", *J. Sandw. Struct. Mater.*, 1-32. https://doi.org/10.1177/1099636219847191.

Khoddami Maraghi, Z. and Arshid, E. (2024), "On the vibrational behavior of variable thickness FG porous beams with graphene-reinforced nanocomposite facesheets", *Acta Mechanica*, **235**(8), 5161-5185. https://doi.org/10.1007/s00707-024-03987-y.

Kiani, Y. (2017), "Free vibration of FG-CNT reinforced composite spherical shell panels using Gram-Schmidt shape functions", *Compos. Struct.*, **159**, 368-381. https://doi.org/10.1016/j.compstruct.2016.09.079.

Kiarasi, F., Asadi, A., Babaei, M., Asemi, K. and Hosseini, M. (2022), "Dynamic analysis of functionally graded carbon nanotube (FGCNT) reinforced composite beam resting on viscoelastic foundation subjected to impulsive loading", *J. Comput. Appl. Mech.*, **53**(1), 1-23. https://doi.org/10.22059/JCAMECH.2022.339008.693.

Kiran, M.C. and Kattimani, S.C. (2018), "Assessment of porosity influence on vibration and static behaviour of functionally graded magneto-electro-elastic plate: A finite element study", *Eur. J. Mech. A Solids*, **71**, 258-277. https://doi.org/10.1016/j.euromechsol.2018.04.006.

Li, J. and Liu, J. (2024), "Nonlinear optimized PID vibration control of thermal-dependent FG composite porous plates reinforced by agglomerated CNTs", *Int. J. Struct. Stabil. Dyn.*, **25**(4), 2550034. https://doi.org/10.1142/S0219455425500348.

Liu, S., Yu, T., Van Lich, L., Yin, S. and Bui, T.Q. (2018), "Size effect on cracked functional composite micro-plates by an XIGA-based effective approach", *Meccanica*, **53**(10), 2637-2658. https://doi.org/10.1007/s11012-018-0848-9.

Long, X., Dong, R., Li, J., Su, Y., Chang, C., Jia, F. and Wan, X. (2025), "Reverse analysis of film/substrate cohesion by indentation: A mesoscopic perspective", *Int. J. Solids Struct.*, **312**, 113285. https://doi.org/10.1016/j.ijsolstr.2025.113285.

Long, X., Li, Y., Shen, Z., Su, Y., Gu, T. and Siow, K.S. (2024), "Review of uniqueness challenge in inverse analysis of nanoindentation", *J. Manuf. Proc.*, **131**, 1897-1916. https://doi.org/10.1016/j.jmapro.2024.10.005.

Luo, Y., Xie, Y., Jiang, H., Chen, Y., Zhang, L., Sheng, X., Xie, D., Wu, H. and Mei, Y. (2021), "Flame-retardant and form-stable phase change composites based on MXene with high thermostability and thermal conductivity for thermal energy storage", *Chem. Eng. J.*, **420**, 130466. https://doi.org/10.1016/J.CEJ.2021.130466.

Mamaghani, A.E., Zohoor, H., Firoozbakhsh, K. and Hosseini, R.

(2013), “Dynamics of a running below-knee prosthesis compared to those of a normal subject”, *J. Solid Mech*., **5**(2), 152-160.

Melaibari, A., Daikh, A.A., Basha, M., Abdalla, A.W., Othman, R., Almitani, K.H., Hamed, M.A., Abdelrahman, A. and Eltaher, M.A. (2022), “Free vibration of FG-CNTRCs nano-plates/shells with temperature-dependent properties”, *Mathematics*, **10**(4), 583. https://doi.org/10.3390/math10040583.

Mohammad-Rezaei Bidgoli, E. and Arefi, M. (2019), “Free vibration analysis of micro plate reinforced with functionally graded graphene nanoplatelets based on modified strain-gradient formulation”, *J. Sand. Struct. Mater*., 109963621983930. https://doi.org/10.1177/1099636219839302.

Mohammad-Rezaei Bidgoli, E. and Arefi, M. (2023a), “Nonlinear vibration analysis of sandwich plates with honeycomb core and graphene nanoplatelet-reinforced face-sheets”, *Arch. Civil Mech. Eng*., **23**(1), 56. https://doi.org/10.1007/s43452-022-00589-0.

Mohammad-Rezaei Bidgoli, E. and Arefi, M. (2023b), “Size-dependent thermomechanical critical loads of GPL-reinforced nanobeams”, *Waves Random Complex Med*., 1-21. https://doi.org/10.1080/17455030.2023.2169385.

Mousavi, S.B., Amir, S., Jafari, A. and Arshid, E. (2021), “Analytical solution for analyzing initial curvature effect on vibrational behavior of PM beams integrated with FGP layers based on trigonometric theories”, *Adv. Nano Res*., **10**(3), 235-251. https://doi.org/10.12989/anr.2021.10.3.235.

Ni, L., Chen, J., Chen, G., Zhao, D., Wang, G. and Aphale, S.S. (2024), “An explainable neural network integrating Jiles-Atherton and nonlinear auto-regressive exogenous models for modeling universal hysteresis”, *Eng. Appl. Artif. Intell*., **136**, 108904. https://doi.org/10.1016/j.engappai.2024.108904.

Rasooli Jazi, F., Amir, S. and Arshid, E. (2024), “Vibration analysis of asymmetric sandwich rotating FG porous discs coated with agglomerated nanocomposite facesheets”, *Arch. Civil Mech. Eng*., **24**(4), 201. https://doi.org/10.1007/s43452-024-01009-1.

Razavi, H., Babadi, A.F. and Tadi Beni, Y. (2017), “Free vibration analysis of functionally graded piezoelectric cylindrical nanoshell based on consistent couple stress theory”, *Compos. Struct*., **160**, 1299-1309. https://doi.org/10.1016/j.compstruct.2016.10.056.

Shen, H.S. (2012), “Thermal buckling and postbuckling behavior of functionally graded carbon nanotube-reinforced composite cylindrical shells”, *Compos. Part B Eng*., **43**(3), 1030-1038. https://doi.org/10.1016/j.compositesb.2011.10.004.

Shen, X., Li, T., Xu, L., Kiarasi, F., Babaei, M. and Asemi, K. (2024), “Free vibration analysis of FG porous spherical cap reinforced by graphene platelet resting on Winkler foundation “, *Adv. Nano Res*., **16**(1), 26. https://doi.org/10.12989/ANR.2024.16.1.011.

Shen, Y., Yin, F. and Sun, P. (2022), “Vibration analysis and energy capability of sandwich axisymmetric curved panel rested on the novel viscoelastic substrate”, *Waves Random Complex Med*., **35**(1), 1833-1860. https://doi.org/10.1080/17455030.2022.2038809.

Soleimani-Javid, Z., Arshid, E., Amir, S. and Bodaghi, M. (2022), “On the higher-order thermal vibrations of FG saturated porous cylindrical micro-shells integrated with nanocomposite skins in viscoelastic medium”, *Defence Technol*., **18**(8), 1416-1434. https://doi.org/10.1016/j.dt.2021.07.007.

Soleimani-Javid, Z., Arshid, E., Khorasani, M., Amir, S. and Tounsi, A. (2021), “Size-dependent flexoelectricity-based vibration characteristics of honeycomb sandwich plates with various boundary conditions”, *Adv. Nano Res*., **10**(5), 449-460. https://doi.org/10.12989/anr.2021.10.5.449.

Soleimani, I. and Beni, Y.T. (2018), “Vibration analysis of nanotubes based on two-node size dependent axisymmetric shell element”, *Arch. Civil Mech. Eng*., **18**(4), 1345-1358. https://doi.org/10.1016/j.acme.2018.04.009.

Song, Z.G., Zhang, L.W. and Liew, K.M. (2016), “Active vibration control of CNT-reinforced composite cylindrical shells via piezoelectric patches”, *Compos. Struct*., **158**, 92-100. https://doi.org/10.1016/j.compstruct.2016.09.031.

Wang, J., Zhang, B., Zhao, Z., Gao, Y., Liu, D., Liu, X., Yang, P., Guo, Z., Wang, Z. L. and Wang, J. (2024), “Boosting the charge density of triboelectric nanogenerator by suppressing air breakdown and dielectric charge leakage”, *Adv. Energy Mater*., **14**(8). https://doi.org/10.1002/aenm.202303874.

Wang, Z.X. and Shen, H.S. (2012), “Nonlinear vibration and bending of sandwich plates with nanotube-reinforced composite face sheets”, *Compos. Part B Eng*., **43**(2), 411-421. https://doi.org/10.1016/j.compositesb.2011.04.040.

Xie, B., Li, Q., Zhang, Y. and Yuan, C. (2025), “High-frequency electromagnetic behavior, impedance modeling, and enhancement of low-permeability powder cores”, *IEEE T. Power Electr*., **40**(2), 3336-3356. https://doi.org/10.1109/TPEL.2024.3488117.

Xu, G., Guo, T., Li, A., Zhou, T. and Shuang, C. (2025), “Seismic performance of steel frame structures with novel self-centering beams: Shaking-table tests and numerical analysis”, *J. Struct. Eng*., **151**(3). https://doi.org/10.1061/JSENDH.STENG-13516.

Yang, M., Cai, C., Wang, D., Wu, Q., Liu, Z. and Wang, Y. (2024), “Symmetric differential demodulation-based heterodyne laser interferometry used for wide frequency-band vibration calibration”, *IEEE T Ind. Electr*., **71**(7), 8129-8137. https://doi.org/10.1109/TIE.2023.3299015.

Yang, Q., Zeng, X., Guo, K., Cao, S., Wei, K., Shan, W. and Tamura, Y. (2025), “Analysis of vortex-induced vibration in flexible risers using a physically-meaningful wake-oscillator model”, *Eng. Struct*., **325**, 119415. https://doi.org/10.1016/j.engstruct.2024.119415.

Yang, S.W., Hao, Y.X., Zhang, W., Yang, L. and Liu, L.T. (2021), “Free vibration and buckling of eccentric rotating FG-GPLRC cylindrical shell using first-order shear deformation theory”, *Compos. Struct*., **263**, 113728. https://doi.org/10.1016/j.compstruct.2021.113728.

Yao, Y. and Arshid, E. (2024), “Effects of folding degree and mass fraction on the static and natural frequency characteristics of functionally graded graphene origami-enabled auxetic metamaterials annular plates”, *Int. J. Struct. Stabil. Dyn*., Online Ready. https://doi.org/10.1142/S0219455426501282.

Zavari, S., Kaveh, A., Babaei, H., Arshid, E., Dimitri, R. and Tornabene, F. (2024), “A quasi-3D hyperbolic formulation for the buckling study of metal foam microplates layered with graphene nanoplatelets-embedded nanocomposite patches with temperature fluctuations”, *Compos. Struct*., 117876. https://doi.org/10.1016/j.compstruct.2024.117876.

Zenkour, A.M. and Hafed, Z.S. (2020), “Bending response of functionally graded piezoelectric plates using a two variable shear deformation theory”, *Adv. Aircr. Spacecr. Sci*., **7**(2), 115-134. https://doi.org/10.12989/aas.2020.7.2.115.

Zhang, X., Shen, S., Luo, Y., Elmaimouni, L. and Yu, J. (2025), “Dispersion and attenuation characteristics of Lamb waves in multilayered piezoelectric semiconductor plates with imperfect interfaces”, *Mech. Adv. Mater. Struct*., 1-14. https://doi.org/10.1080/15376494.2025.2456681.

Zhao, Y., Xing, S., Li, C., Yang, N., He, Y., Duan, K. and Zhang, J. (2025), “Frequency insensitive electromagnetic absorption core-shell sandwich structure with excellent electromagnetic damage tolerance”, *Compos. Part B Eng*., **289**, 111946. https://doi.org/10.1016/j.compositesb.2024.111946.

Zhong, R., Wang, Q., Tang, J., Shuai, C. and Qin, B. (2018), “Vibration analysis of functionally graded carbon nanotube reinforced composites (FG-CNTRC) circular, annular and sector plates”, *Compos. Struct*., **194**, 49-67. https://doi.org/10.1016/J.COMPSTRUCT.2018.03.104.

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Appendix

The used coefficients in Eqs. (39)-(44) are defined as:

$$(A_1,A_2,A_3)=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{11b}\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{11c}\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{11t}\left(1,\frac{1}{R_{x_1}},z\right)dz$$

$$(A_4,A_5,A_6)=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{12b}\left(1,\frac{1}{R_{x_2}},z\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{12c}\left(1,\frac{1}{R_{x_2}},z\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{12t}\left(1,\frac{1}{R_{x_2}},z\right)dz$$

$$A_7=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} e_{31}\frac{\pi}{h_b}\sin\left(\frac{\pi z_b}{h_b}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} e_{31}\frac{\pi}{h_t}\sin\left(\frac{\pi z_t}{h_t}\right)dz$$

$$(A_8,A_9)=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{k_s}{R_{x_1}}Q_{55b}\left(1,\frac{1}{R_{x_1}}\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{k_s}{R_{x_1}}Q_{55c}\left(1,\frac{1}{R_{x_1}}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{k_s}{R_{x_1}}Q_{55t}\left(1,\frac{1}{R_{x_1}}\right)dz$$

$$A_{10}=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{k_s}{R_{x_1}}e_{15}\cos\left(\frac{\pi z_b}{h_b}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{k_s}{R_{x_1}}e_{15}\cos\left(\frac{\pi z_t}{h_t}\right)dz$$

$$(A_{11},A_{12},A_{13})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{66b}\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{66c}\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{66t}\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz$$

$$(A_{14},A_{15},A_{16})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{1}{2}Q_{66b}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{1}{2}Q_{66c}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{1}{2}Q_{66t}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\left(1,z,\frac{1}{2}z\left(\frac{1}{R_{x_1}}-\frac{1}{R_{x_2}}\right)\right)dz$$

$$(A_{17},A_{18},A_{19})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{21b}\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{21c}\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{21t}\left(1,\frac{1}{R_{x_1}},z\right)dz$$

$$(A_{20},A_{21},A_{22})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{22b}\left(1,\frac{1}{R_{x_2}},z\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{22c}\left(1,\frac{1}{R_{x_2}},z\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{22t}\left(1,\frac{1}{R_{x_2}},z\right)dz$$

$$A_{23}=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} e_{32}\frac{\pi}{h_b}\sin\left(\frac{\pi z_b}{h_b}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} e_{32}\frac{\pi}{h_t}\sin\left(\frac{\pi z_t}{h_t}\right)dz$$

$$(A_{24},A_{25})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{k_s}{R_{x_2}}Q_{44b}\left(1,\frac{1}{R_{x_2}}\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{k_s}{R_{x_2}}Q_{44c}\left(1,\frac{1}{R_{x_2}}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{k_s}{R_{x_2}}Q_{44t}\left(1,\frac{1}{R_{x_2}}\right)dz$$

$$A_{26}=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{k_s}{R_{x_2}}e_{24}\cos\left(\frac{\pi z_b}{h_b}\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{k_s}{R_{x_2}}e_{24}\cos\left(\frac{\pi z_t}{h_t}\right)dz$$

$$(A_{27},A_{28},A_{29})=\int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{11b}z\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{11c}\left(1,\frac{1}{R_{x_1}},z\right)dz+\int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{11t}z\left(1,\frac{1}{R_{x_1}},z\right)dz$$

$$(A_{30}, A_{31}, A_{32}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{12b} z\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{12c} z\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{12t} z\left(1, \frac{1}{R_{x_2}}, z\right) dz$$

$$A_{33} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} e_{31} \frac{\pi}{h_b} z \sin\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} e_{31} \frac{\pi}{h_t} z \sin\left(\frac{\pi z_t}{h_t}\right) dz$$

$$(A_{34}, A_{35}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} k_s Q_{55b}\left(1, \frac{1}{R_{x_1}}\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} k_s Q_{55c}\left(1, \frac{1}{R_{x_1}}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} k_s Q_{55t}\left(1, \frac{1}{R_{x_1}}\right) dz$$

$$A_{36} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} k_s e_{15} \cos\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} k_s e_{15} \cos\left(\frac{\pi z_t}{h_t}\right) dz$$

$$(A_{37}, A_{38}, A_{39}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{66b} z\left(1, z, \frac{1}{2} z\left(\frac{1}{R_{x_1}} - \frac{1}{R_{x_2}}\right)\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{66c} z\left(1, z, \frac{1}{2} z\left(\frac{1}{R_{x_1}} - \frac{1}{R_{x_2}}\right)\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{66t} z\left(1, z, \frac{1}{2} z\left(\frac{1}{R_{x_1}} - \frac{1}{R_{x_2}}\right)\right) dz$$

$$(A_{40}, A_{41}, A_{42}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{21b} z\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{21c} z\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{21t} z\left(1, \frac{1}{R_{x_1}}, z\right) dz$$

$$(A_{43}, A_{44}, A_{45}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} Q_{22b} z\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} Q_{22c} z\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} Q_{22t} z\left(1, \frac{1}{R_{x_2}}, z\right) dz$$

$$A_{46} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} e_{32} \frac{\pi}{h_b} z \sin\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} e_{32} \frac{\pi}{h_t} z \sin\left(\frac{\pi z_t}{h_t}\right) dz$$

$$(A_{47}, A_{48}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} k_s Q_{44b}\left(1, \frac{1}{R_{x_2}}\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} k_s Q_{44c}\left(1, \frac{1}{R_{x_2}}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} k_s Q_{44t}\left(1, \frac{1}{R_{x_2}}\right) dz$$

$$A_{49} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} k_s e_{24} \cos\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} k_s e_{24} \cos\left(\frac{\pi z_t}{h_t}\right) dz$$

$$(A_{50}, A_{51}, A_{52}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{Q_{11b}}{R_{x_1}}\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{Q_{11c}}{R_{x_1}}\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{Q_{11t}}{R_{x_1}}\left(1, \frac{1}{R_{x_1}}, z\right) dz$$

$$(A_{53}, A_{54}, A_{55}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{Q_{12b}}{R_{x_1}}\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{Q_{12c}}{R_{x_1}}\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{Q_{12t}}{R_{x_1}}\left(1, \frac{1}{R_{x_2}}, z\right) dz$$

$$A_{56} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{e_{31}}{R_{x_1}} \frac{\pi}{h_b} \sin\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{e_{31}}{R_{x_1}} \frac{\pi}{h_t} \sin\left(\frac{\pi z_t}{h_t}\right) dz$$

$$(A_{57}, A_{58}, A_{59}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{Q_{21b}}{R_{x_2}}\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{Q_{21c}}{R_{x_2}}\left(1, \frac{1}{R_{x_1}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{Q_{21t}}{R_{x_2}}\left(1, \frac{1}{R_{x_1}}, z\right) dz$$

$$(A_{60}, A_{61}, A_{62}) = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{Q_{22b}}{R_{x_2}}\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \frac{Q_{22c}}{R_{x_2}}\left(1, \frac{1}{R_{x_2}}, z\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{Q_{22t}}{R_{x_2}}\left(1, \frac{1}{R_{x_2}}, z\right) dz$$

$$A_{63} = \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{e_{32}}{R_{x_2}} \frac{\pi}{h_b} \sin\left(\frac{\pi z_b}{h_b}\right) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{e_{32}}{R_{x_2}} \frac{\pi}{h_t} \sin\left(\frac{\pi z_t}{h_t}\right) dz$$

$$\begin{aligned}(A_{64}, A_{65}, A_{66}, A_{67}, A_{68}, A_{69}) &= \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \cos\left(\frac{\pi z_b}{h_b}\right) \begin{pmatrix} e_{15}, \frac{e_{15}}{R_{x_1}}, \kappa_{11} \cos\left(\frac{\pi z_b}{h_b}\right), \\ e_{24}, \frac{e_{24}}{R_{x_2}}, \kappa_{22} \cos\left(\frac{\pi z_b}{h_b}\right) \end{pmatrix} dz \\ &+ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \cos\left(\frac{\pi z_t}{h_t}\right) \begin{pmatrix} e_{15}, \frac{e_{15}}{R_{x_1}}, \kappa_{11} \cos\left(\frac{\pi z_t}{h_t}\right), \\ e_{24}, \frac{e_{24}}{R_{x_2}}, \kappa_{22} \cos\left(\frac{\pi z_t}{h_t}\right) \end{pmatrix} dz\end{aligned}$$

$$\begin{aligned}(A_{70}, A_{71}, A_{72}, A_{73}, A_{74}, A_{75}, A_{76}) &= \int_{-\frac{h_c}{2}-h_b}^{-\frac{h_c}{2}} \frac{\pi}{h_b} \sin\left(\frac{\pi z_b}{h_b}\right) \begin{pmatrix} e_{31}, \frac{e_{31}}{R_{x_1}}, e_{31} z, e_{32}, \frac{e_{32}}{R_{x_2}}, \\ e_{32} z, \kappa_{33} \frac{\pi}{h_b} \sin\left(\frac{\pi z_b}{h_b}\right) \end{pmatrix} dz \\ &+ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_t} \frac{\pi}{h_t} \sin\left(\frac{\pi z_t}{h_t}\right) \begin{pmatrix} e_{31}, \frac{e_{31}}{R_{x_1}}, e_{31} z, e_{32}, \frac{e_{32}}{R_{x_2}}, \\ e_{32} z, \kappa_{33} \frac{\pi}{h_t} \sin\left(\frac{\pi z_t}{h_t}\right) \end{pmatrix} dz\end{aligned}$$

Where:

$$z_b = z + \frac{h_c}{2} + \frac{h_b}{2},$$
$$z_t = z - \frac{h_c}{2} - \frac{h_t}{2}$$