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Axial free vibration analysis of a tapered nanorod using Adomian decomposition method

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Abstract. This study aimed to conduct an analysis of the axial free vibration of tapered nanorods based on nonlocal elasticity theory. The small-scale effect on the free axial vibration of a tapered nanorod was studied employing the Adomian decomposition method (ADM) and the finite difference method (FDM) as a checking tool where a contradiction existed between the results of this study and available results in one highly cited work in the literature, which was used for comparison purposes in this work. Different boundary conditions for the nanorod were considered: fixed-fixed nanorod, fixed-free nanorod, and fixed-linear spring nanorod. The governing equation of the problem is a variable coefficient differential equation for which analytical solutions are strictly limited. For this type of problem, analytical approximate methods are effective, and there are many studies available in the literature on the application of these methods to solve linear/nonlinear ordinary/partial differential equations. ADM is one of the methods and was successfully used in this study to analyze the free vibration of nanorods. The results obtained in this study have shown that the presented technique is so powerful and has potential for applications in nanomechanics based on nonlocal elasticity theory.
Keywords: Adomian decomposition method; axial vibration; free vibration; tapered nanorod

1. Introduction

Iijima (1991) explained the preparation of a new type of finite carbon structure, i.e., carbon nanotubes (CNTs), consisting of needle-like tubes that attracted worldwide attention to this new structural element, which became a popular research area in the last three decades and has been of great interest to many researchers because of their mechanical, electronic, electrochemical, and physical properties. Nanorods have been used for microelectron-mechanical (MEM) and nanoelectromechanical (NEM) devices. These industrial applications include the application of axial external forces that may lead to axial vibration. Hence, it is important to investigate the axial dynamic behavior of nanorods.

In the previous research, it was pointed out that the classical continuum mechanics, also called local theory, may give incorrect results due to the incapability of taking size-dependent effects into account. The theories considering that effect are molecular dynamic models and nonlocal continuum models. Nonlocal continuum theories were used in the modeling of carbon nanotubes and became prominent, such as the nonlocal elasticity theory, which was developed by Eringen (1972, 1976, 1983, 2002). In the theory, stress at a reference point is assumed to be a functional of the strain field at every point of the continuum. The nonlocal elasticity theory has been widely used by the researchers to investigate the properties of the micro/nano structures more precisely. For a decent introduction to nonlocal theory, sufficient references exist in literature related to application of the theory to nano-mechanics of structural elements (Wang and Liew 2007, Reddy 2007, Reddy and Pang 2008, Aydogdu 2009a, Thai 2012, Karličić *et al*. 2016, Chakraverty and Behera 2017).

However, the studies about the investigation of axial vibration of nanorods using any nonlocal continuum models are still limited. Aydogdu (2009b) developed a nonlocal elastic rod model and applied it to study the small-scale effect on axial vibration of nanorods. Narendar and Gopalakrishnan (2009) developed a nonlocal multiple Timoshenko beam model to analyze the wave propagation in multi-walled carbon nanotubes. Demir *et al*. (2010) studied free vibration analysis of carbon nanotubes based on Timoshenko beam theory using the discrete singular convolution method. Karaoglu and Aydogdu (2010) used a double elastic Euler beam model with and without non-local effects to analyze transverse forced vibration of double-walled carbon nanotubes by taking into account the non-coaxial displacements. Murmu and Adhikari (2010) presented an analytical method for investigation of nonlocal frequencies in longitudinal vibration of a double-nanorod system. Narendar and Gopalakrishnan (2011) studied the axial wave propagation of double nanorod systems. Murmu and Adhikari (2011) examined the nonlocal effects in the longitudinal dynamic properties of single-walled carbon nanotubes with attached buckyballs. Hsu *et al*. (2011) studied longitudinal vibration of a cracked nanobeam with

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different boundary conditions. Aydogdu (2012a) analyzed the axial vibration of single-walled carbon nanotubes embedded in an elastic medium. Aydogdu (2012b) presented an explicit equation for nonlocal coefficient using a general unimodal nonlocal rod in the solution of longitudinal wave propagation in nanorods. Akgoz and Civalek (2012) studied the longitudinal free vibration problem of a microscaled bar using the strain gradient elasticity theory. Şimşek (2012) considered free longitudinal vibration of axially functionally graded (AFG) tapered nanorods with variable cross-section based on the nonlocal elasticity theory. Huang (2012) presented a physically-based nonlocal model to investigate the influences of the nonlocal long-range interactions on the longitudinal vibration of nanorods. Danesh *et al*. (2012) used the differential quadrature method (DQM) to analyze small-scale effects in the axial vibration of nanorods. Lee and Lee (2012) performed modal analysis of single-walled carbon nanotubes and nanocones using a finite element method with ANSYS. Guo and Yang (2012) used the modified Wentzel-Brillouin-Kramers (WBK) method to obtain an asymptotic solution for the axial vibration of general nonuniform nanorods. Ciekot (2012) used the WKB method to solve the equation of motion for free axial vibration of the nanorod. Chang (2012, 2013) developed an elastic rod model to investigate the small-scale effect on longitudinal vibration of nonuniform and nonhomogeneous nanorods based on the nonlocal elasticity theory. Ciekot and Kukla (2013) solved the problem of the free longitudinal vibration of a double-nanorod system (DNRS) using Green's function method. Adhikari *et al*. (2013) investigated free and forced axial vibrations of damped nonlocal rods. Adhikari *et al*. (2014) presented a dynamic finite element method for a nonlocal rod embedded in an elastic medium and undergoing axial vibration. Aydogdu (2014) investigated longitudinal wave propagation in multiwalled carbon nanotubes. Aydogdu and Elishakoff (2014) studied the axial vibration of carbon nanotubes with an attached spring using nonlocal elasticity theory. Yayli (2014) investigated the free axial vibration response of carbon nanotubes (CNTs) with arbitrary boundary conditions using Fourier sine series together with Stokes' transformation. Li *et al*. (2015) used a hardening nonlocal approach to investigate the longitudinal dynamic behaviors of some common one-dimensional nanostructures. Aydogdu and Arda (2016) analyzed longitudinal forced vibration of nanorods. Li *et al*. (2016) developed a finite element method based on strain gradient theory to analyze the longitudinal vibration analysis of small-scaled rods. Yayli (2016) studied free vibration of single-walled carbon nanotubes with restrained boundary conditions. Heidari (2016) studied the dynamical properties of an axially vibrating uniform nanorod. Dinçkal (2016) presented a finite element model for vibration analysis of carbon nanotubes (CNTs) with both Euler-Bernoulli and Timoshenko beam theories. Li *et al*. (2017) analyzed longitudinal dynamic problems of nanorods or nanobars based on the nonlocal elasticity theory and Bishop's assumptions considering radial deformation and inertia. Yazdi *et al*. (2017) analyzed free longitudinal vibration of nanorods from a wave viewpoint. Xu *et al*. (2017) examined the size effects on the dynamic behaviors of rods within the framework of the nonlocal strain gradient elastic theory. Eren and Aydogdu (2018) studied the nonlinear free vibration of a nanorod subjected to a finite strain using the Galerkin method. Yayli (2018a) presented a hardening nonlocal approach for the free longitudinal vibration analysis of nanorods (carbon nanotubes) with arbitrary boundaries. Yayli (2018b) considered longitudinal vibration analysis of functionally graded restrained nanorods with two axial springs attached at both ends via non-local elasticity theory. Aydogdu *et al*. (2018) investigated the vibration of axially functionally graded nanorods based on strain gradient elasticity theory using the Ritz method. Nazemnezhad and Kamali (2018) considered nonlocal free longitudinal vibration of thick nanorods by focusing on the inertia of lateral motions and shear stiffness effects. Numanoglu *et al*. (2018) investigated longitudinal free vibration behaviors of one-dimensional nanostructures with various boundary conditions. Bao *et al*. (2019) presented a solution procedure for analyzing the free longitudinal vibration of nanorods in which the longitudinal displacement of the nanorod is sought as a linear combination of a Fourier series and auxiliary trigonometric functions. Hosseini *et al*. (2020) used a parametric analysis to analyze the axial vibration of a FG nanobeam in order to understand how length scale affects the dynamic behavior of the structure using nonlocal elasticity theory. The free and forced axial vibrations in zigzag single-walled carbon nanotubes (SWCNT) under two different linear and harmonic axial concentrated stresses were examined by Khosravi *et al*. (2020). In zigzag Single-Walled Carbon Nanotubes (SWCNT), Khosravi *et al*. (2020) examined both forced and free axial vibration under two different linear and harmonic axial concentrated forces. Using Eringen's nonlocal elasticity theory, Civalek *et al*. (2022) investigated the free torsional and axial vibrations of porous nanorods with torsional, axial elastic boundary conditions. Jin *et al*. (2023) examined the axial-free vibration of a rotating functionally graded (FG) piezoelectric nano-rod, which possesses continuously variable material properties along the thickness direction, to elucidate the dynamic behaviors and electromechanical coupling characteristics of rotating nonuniform nanocomponents in nanoelectron-mechanical systems. Uzun *et al*. (2023) utilized higher-order models to investigate size-dependent torsional and longitudinal free vibrations of constrained saturated porous nanorods through a higher-order strain gradient elasticity theory. Uzun *et al*. (2023) examined the free longitudinal vibrations of nanorods made of a functionally graded (FG) material with deformable borders using a hardening nonlocal elasticity framework. Uzun *et al*. (2024) examined the torsional dynamic response of functionally graded nanorods with torsional motion about the center of twist based on the theory of strain gradient nonlocal elasticity and Eringen's nonlocal elasticity theory. Civalek *et al*. (2024) formulated a comprehensive approach for analyzing the torsional vibrations of non-circular nanorods under diverse boundary conditions, employing second-order strain gradient theory.

In this study, the small-scale effect on the free axial vibration of a tapered nanorod is examined using ADM. For comparison purposes, Danesh *et al*. (2012) are chosen as a reference study in which a small-scale effect in a tapered nanorod undergoing axial vibration was investigated using the differential quadrature method. A significant difference was observed while comparing the results with the solutions obtained in this study via ADM. Hence, a finite difference solution was also carried out to evaluate the performance of ADM. The results showed that ADM results are more accurate when compared to the results computed by Danesh *et al*. (2012). Hence, the results of the current research will be useful for further research on the axial vibration of tapered nanorods.

2. Formulation of the problem

Consider a one-dimensional nonlocal tapered rod under free axial vibration without any external applied source shown in the following figure.

The equation of motion for the nanorod shown in Fig. 1 can be obtained using Newton's second law as follows (Karličić *et al*. 2016).

$$-P + (P + dP) = \rho A dx \frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $P(x,t)$ is the axial stress resultant and $u(x,t)$ is axial displacement. Eq. (1) leads to:

$$\frac{\partial P}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{2}$$

where stress resultant P is defined as $(x,t) = \int_A \sigma_{xx} dA$. The stress-strain relation for a nonlocal elastic body is given as (Karličić *et al*. 2016);

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \tag{3}$$

where $(e_0 a)^2$ denotes the nonlocal parameter, E is Young's modulus, and $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ is axial strain. Substituting the definition of stress resultant P into Eq. (3), the axial internal force for the non-local theory is obtained as

$$P - (e_0 a)^2 \frac{\partial^2 P}{\partial x^2} = EA \frac{\partial u}{\partial x} \tag{4}$$

Now, the governing equation of motion can be expressed in terms of the axial displacement u(x,t) for the nonlocal elastic constitutive relation. Introducing Eq. (2) into the (4) equation of motion can be obtained as follows:

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) = (e_0 a)^2 \frac{\partial^2}{\partial x^2}\left(\rho A \frac{\partial^2 u}{\partial t^2}\right) \tag{5}$$

Eq. (5) can be rearranged as

$$\frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) = \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right]\left(\rho A \frac{\partial^2 u}{\partial t^2}\right) \tag{6}$$

Boundary conditions (BCs) for the nanorod shown in Fig. 1 considered in the study are summarized in a table given below for different end conditions. Nonlocal axial force P at the end of a nanorod can be obtained from Eqs. (2) and (4) as follows:

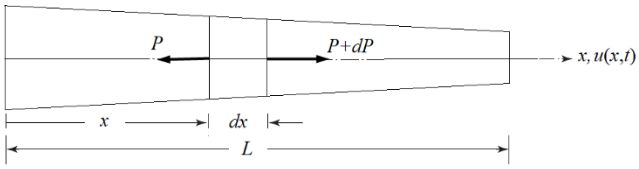
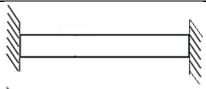
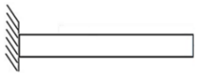
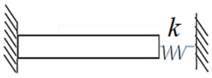


Fig. 1 A tapered nonlocal rod

Table 1 Boundary conditions for the nanorod considered in the study

Rod	BC at Left End	BC at Right End
	$u(0,t) = 0$	$u(L,t) = 0$
	$u(0,t) = 0$	$P(L,t) = 0$
	$u(0,t) = 0$	$EA(L)\frac{\partial u}{\partial x}(L,t) + ku(L,t) = 0$

$$P(x,t) = (e_0 a)^2 \frac{\partial}{\partial x}\left(\rho A \frac{\partial^2 u}{\partial t^2}\right) + EA \frac{\partial u}{\partial x} \tag{7}$$

Since the rod is tapered, the cross-section is assumed to be variable such that $A = A_0 \psi(x)$. Assuming the displacement is defined based on the separation of variables technique as $u(x,t) = U(x)e^{i\omega t}$, then eq. (6) takes the following form.

$$\frac{\partial}{\partial x}\left(\psi \frac{\partial U}{\partial x}\right) + \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right]\left(\frac{\omega}{c}\right)^2 \psi U = 0 \tag{8}$$

where $c = \sqrt{E/\rho}$. Expanding the parentheses, Eq. (8) can be rearranged as follows:

$$\begin{aligned}&\left[1 - \left(\frac{\omega}{c}\right)^2 (e_0 a)^2\right]\psi \frac{d^2 U}{dx^2} + \left[1 - 2\left(\frac{\omega}{c}\right)^2 (e_0 a)^2\right]\frac{d\psi}{dx}\frac{dU}{dx} \\ &+ \left(\frac{\omega}{c}\right)^2 \left[\psi - (e_0 a)^2 \frac{d^2 \psi}{dx^2}\right] U = 0\end{aligned} \tag{9}$$

Defining nondimensional parameters $\bar{U} = \frac{U}{L}$, $\xi = \frac{x}{L}$, $\gamma = \frac{e_0 a}{L}$ and $\Omega = (\frac{\omega L}{c})^2$ as nondimensional axial displacement function, nondimensional coordinate, nondimensional nonlocal (scale) parameter, and nondimensional vibration frequency, respectively, Eq. (9) takes the nondimensional form given below.

$$\begin{aligned}&[1 - \Omega^2\gamma^2]\psi \frac{d^2 \bar{U}}{d\xi^2} + [1 - 2\Omega^2\gamma^2]\frac{d\psi}{d\xi}\frac{d\bar{U}}{d\xi} \\ &+ \Omega^2 \left[\psi - \gamma^2 \frac{d^2 \psi}{d\xi^2}\right]\bar{U} = 0\end{aligned} \tag{10}$$

Eq. (10) is the nondimensional governing equation for tapered nanorods undergoing axial vibration. ADM is employed to solve the equation in this study. Eq. (10) is also solved via the finite difference method to support the results produced by ADM.

3. ADM and application to the problem

ADM (Adomian 1994) is a powerful technique for the solution of linear/nonlinear ordinary/partial differential equations and has attracted great attention in applied sciences. The convergence of the method is discussed in (Abbaoui and Cherruault 1994a, b). Consider an equation of the following form.

$$Ly + Ny + Ry = f(x) \tag{11}$$

where L is the linear operator of maximum order, N is the nonlinear operator, and R is the operator for remaining terms. Assume that L is a second-order derivative, then the inverse operator for L becomes a double integration. If all the terms in the left-hand side of the equation in Eq.(11) are taken to the right-hand side except the term Ly, then by the application of the inverse operator to both sides of the equation, the following relation is obtained.

$$y(x) = y(0) + xy'(0) + g(x) - L^{-1}Ry - L^{-1}Ny \tag{12}$$

where $g(x)$ is obtained by the application of the inverse operator to the function $f(x)$. The solution in ADM is defined with the following definitions:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{13}$$

$$Ny(x) = \sum_{n=0}^{\infty} A_n(x) \tag{14}$$

In Eq. (14) $A_n(x)$ is the n^{th} Adomian polynomial defined as follows:

$$A_k = \frac{1}{k!}\frac{\partial}{\partial \lambda^k}\left[N\left(\sum_{n=0}^{\infty} y_n \lambda^n\right)\right]_{\lambda=0} \tag{15}$$

Inserting Eqs. (13-14) into Eq.(12) following successive relations is obtained.

$$y_0(x) = y(0) + xy'(0) + g(x) \tag{16}$$

$$y_k(x) = -L^{-1}Ry_{k-1}(x) - L^{-1}A_{k-1}(x) \qquad k \geq 1 \tag{17}$$

Finally, the solution is defined in terms of an infinite series, as given below.

$$y(x) = \lim_{n\to\infty} \sum_{k=0}^{n} y_k(x) \tag{18}$$

After calculation of N terms, the solution by Eq. (16) with these terms is called the N^{th}-order solution.

The formulation given above is now applied to Eq. (10) by rearranging the equation in the following form.

$$\frac{d^2\bar{U}}{d\xi^2} + \frac{[1-2\Omega^2\gamma^2]}{[1-\Omega^2\gamma^2]}\frac{\frac{d\psi}{d\xi}}{\psi}\frac{d\bar{U}}{d\xi} + \frac{\Omega^2}{[1-\Omega^2\gamma^2]}\left[\psi - \gamma^2\frac{d^2\psi}{d\xi^2}\right]\bar{U} = 0 \tag{19}$$

In Eq. (19) linear operator L is a second-order derivative. Hence, an algorithm based on Eqs. (16-17) may be constructed as follows:

$$v_0(x) = \bar{U}(0) + \xi\bar{U}'(0) \tag{20}$$

$$v_k(\xi) = -L^{-1}\left\{\begin{array}{l}[1-\lambda^2\gamma^2]\frac{\frac{d\psi}{d\xi}}{\psi}\frac{dv_{k-1}}{d\xi} \\ +\lambda^2\left[\psi - \gamma^2\frac{d^2\psi}{d\xi^2}\right]v_{k-1}\end{array}\right\} \qquad k \geq 1 \tag{21}$$

where $\lambda^2 = \frac{\Omega^2}{(1-\Omega^2\gamma^2)}$ and the inverse operator L^{-1} is double integration. After constructing N terms from Eq. (21), the N^{th}-order solution is obtained from the following relation.

$$\bar{U}^N(\xi) = \sum_{i=0}^{N} v_i(\xi) \tag{22}$$

Once l is determined from the solution given in Eq. (22) together with the boundary conditions given in Table 1, free vibration frequencies can be computed from the following.

$$\Omega = \sqrt{\frac{\lambda^2}{1+\lambda^2\gamma^2}} \tag{23}$$

4. Case study

The study by Danesh *et al*. (2012) is used as the case study for the formulation given above. In the study, the differential quadrature method (DQM) was used to analyze small-scale effects in the axial vibration of tapered nanorods. Danesh *et al*. (2012) assumed a variation function for the area as

$$\psi(x) = 1 - \frac{3}{4}\frac{x}{L} \tag{24}$$

where its non-dimensional form is $\psi(\xi) = 1 - 0.75\xi$. Non-dimensional boundary conditions for the problem are given in the following table.

In Table 2, the non-dimensional spring constant is defined as $\bar{k} = \frac{kL}{EA(1)}$. Danesh *et al*. (2012) defined a parameter called the frequency ratio to illustrate the small-scale effect on the vibration response, which is defined as:

$$\text{Frequency Ratio (FR)} = \frac{\text{Natural Frequency Using Nonlocal Theory}}{\text{Natural Frequency Using Local Theory}} \tag{25}$$

First, natural frequency using local theory is required. These values can be determined by taking $\gamma = 0$ in Eq. (21). Using the ADM algorithm given in Eqs. (20-21) natural frequencies using local theory for fixed-fixed and fixed-free rods are 3.0733 and 2.0009 respectively. FR values for fixed-fixed and fixed-free nanorods by taking the length of nanorod as *5 nm* are compared in the table below. ADM solutions are of *15^th^* order.

From Table 3, it can be observed that there are significant differences between the results of Danesh *et al*. (2012) and this study (ADM), especially for fixed-free rod. In fixed-free rod, the results of Danesh *et al*. (2012) decrease with increasing scale parameter e_0a. However, for 2.0 nm FR value increase, which is an unexpected behavior. ADM results seem consistent, but a third method should be used for checking if ADM results are accurate or erroneous. To this aim, a second-order central difference method (CDM) is employed. Since it is a fundamental numerical technique, no detail of the method will be explained in this article. Any numerical analysis textbook may be visited for theoretical details. Applying second-order CDM to Eq. (19), the following relation is obtained.

Table 3 Some FR values for fixed-fixed and fixed-free nanorods

e_0a (nm)	fixed-fixed rod		fixed-free rod	
	Danesh *et al*. (2012)	ADM	Danesh *et al*. (2012)	ADM
0.5	0.9561	0.95605	0.9936	0.97232
1.0	0.8553	0.85394	0.9770	0.90172
1.5	0.7486	0.74197	0.9592	0.81282
2.0	0.6741	0.64604	0.9684	0.72444

Table 4 FR values including CDM results

e_0a (nm)	fixed-fixed rod			fixed-free rod		
	Danesh *et al*. (2012)	ADM	CDM	Danesh *et al*. (2012)	ADM	CDM
0.5	0.9561	0.95605	0.95603	0.9936	0.97232	0.97229
1.0	0.8553	0.85394	0.85393	0.9770	0.90172	0.90169
1.5	0.7486	0.74197	0.74197	0.9592	0.81282	0.81280
2.0	0.6741	0.64604	0.64603	0.9684	0.72444	0.72443

Table 5 FR values for fixed-fixed nanorod

L	e_0a = 0.5 nm	e_0a = 1.0 nm	e_0a = 1.5 nm	e_0a = 2.0 nm
10 nm	0.98841	0.95605	0.90891	0.85394
15 nm	0.99480	0.97969	0.95605	0.92583
20 nm	0.99706	0.98841	0.97451	0.95605
25 nm	0.99812	0.99253	0.98344	0.97115
30 nm	0.99869	0.99480	0.98841	0.97969

Table 6 FR values for fixed-free nanorod

L	e_0a = 0.5 nm	e_0a = 1.0 nm	e_0a = 1.5 nm	e_0a = 2.0 nm
10 nm	0.99285	0.97232	0.94087	0.90172
15 nm	0.99680	0.98740	0.97232	0.95238
20 nm	0.99820	0.99285	0.98414	0.97232
25 nm	0.99885	0.99541	0.98976	0.97902
30 nm	0.99920	0.99680	0.99285	0.98740

$$\frac{\bar{U}_{i+1}-2\bar{U}_i+\bar{U}_{i-1}}{(\Delta\xi)^2}+[1-\lambda^2\gamma^2]\frac{\left(\frac{d\psi}{d\xi}\right)_i}{\psi_i}\frac{\bar{U}_{i+1}-\bar{U}_{i-1}}{2\Delta\xi}+\lambda^2\left[\psi_i-\gamma^2\left(\frac{d^2\psi}{d\xi^2}\right)_i\right]\bar{U}_i=0 \tag{26}$$

From Table 4, it can be easily seen that ADM and CDM values agree very well. Hence, the results by Danesh *et al*. (2012) are poor and should be improved. This consequence is expected, which was also the case for the solution of some nonlinear differential equations (Firoozjae and Yazdani 2015). Additional FR values computed using ADM with *15th* order solution for different nanorod lengths are given in the following tables.

FR values for fixed-fixed and fixed-free nanorods are also compared graphically in the following figure. In Fig. 2, solid curves are used for fixed-fixed nanorods and dashed curves are used for fixed-free nanorods.

As the last case, a linear spring at the right hand is considered. The effect of the spring constant on frequency ration is investigated. First, we require natural frequencies by local theory. Hence, boundary conditions given in Table 3 are employed together with $e_0a=0$. Natural frequencies by local theory for different non-dimensional spring constants using *15th* order ADM solution are summarized below.

From Table 7, the natural frequency for $\bar{k}=10\,000$ is very close to 3.0733 that is for the nanorod with the right end fixed. Hence, $\bar{k}\geq 10\,000$ successfully simulates rigid end. By using the natural frequency values given in Table 7, FR values calculated according to Eq. (25) are compared in the table below. As it was in Table 4, there are still differences between the results of this study and Danesh *et al*. (2012). Hence, CDM is again employed, and it can be observed that CDM results support the solution obtained using ADM. 15th order solution is used in ADM while 100 subdivisions are used in CDM.

From Table 8, it can be easily seen that CDM and ADM results agree very well and support each other. Danesh *et al*.

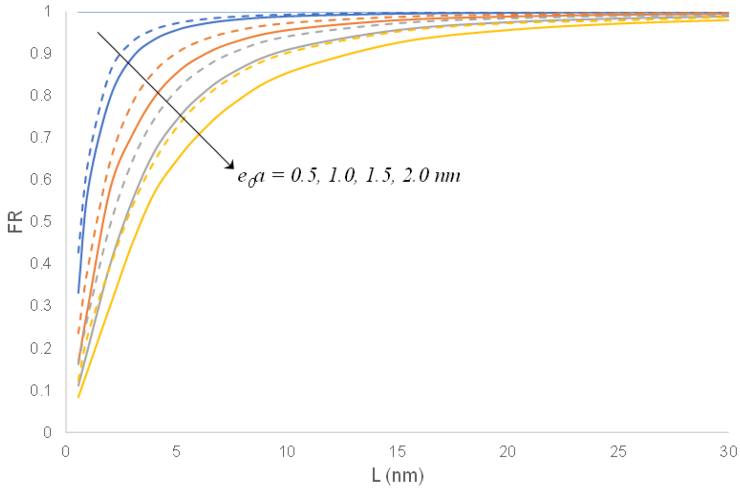


Fig. 2 FR values (fixed-fixed: solid lines, fixed-free: dashed lines)

Table 7 Natural frequencies for nanorod with spring at right end by local theory

$\bar{k} = 5$	$\bar{k} = 10$	$\bar{k} = 100$	$\bar{k} = 500$	$\bar{k} = 1000$	$\bar{k} = 10\ 000$
2.6409	2.8081	3.0407	3.0666	3.0700	3.0730

Table 8 FR values for nanorod with spring at right end

$\bar{k}$		e_0a = 0.5 nm	e_0a = 1.0 nm	e_0a = 1.5 nm	e_0a = 2.0 nm
	Danesh *et al.* (2012)	0.9604	0.8677	0.7678	0.6991
5	ADM	0.9651	0.8802	0.7818	0.6911
	CDM	0.9651	0.8802	0.7812	0.6911
	Danesh *et al.* (2012)	0.9583	0.8616	0.7584	0.6867
10	ADM	0.9622	0.8716	0.7683	0.6768
	CDM	0.9621	0.8716	0.7683	0.6768
	Danesh *et al.* (2012)	0.9564	0.8559	0.7496	0.6753
100	ADM	0.9569	0.8564	0.7456	0.6503
	CDM	0.9569	0.8564	0.7456	0.6503
	Danesh *et al.* (2012)	0.9562	0.8554	0.7489	0.6743
500	ADM	0.9562	0.8544	0.7427	0.6469
	CDM	0.9562	0.8544	0.7427	0.6469
	Danesh *et al.* (2012)	0.9562	0.8554	0.7488	0.6742
1000	ADM	0.9561	0.8542	0.7424	0.6465
	CDM	0.9561	0.8542	0.7423	0.6465
	Danesh *et al.* (2012)	0.9561	0.8553	0.7487	0.6741
10 000	ADM	0.9561	0.8539	0.7420	0.6461
	CDM	0.9560	0.8540	0.7420	0.6461

(2012) determined the results a bit differently as it was for fixed-fixed and fixed-free nanorods. Hence, ADM solutions can be defined as more accurate.

5. Conclusions

This study investigates the free axial vibration of tapered nanorods using ADM with a 15th-order solution. We chose highly cited research as a case study and observed a significant difference between the ADM results and the previous results of the selected reference. DQM numerical behavior may explain the discrepancy between the two studies. Another potential cause could be an overlooked small element in the DQM analysis code or the numerical values used as input. Therefore, we employed a second-order central difference method to verify the study's results and found a strong agreement between the ADM and CDM results. Therefore, we concluded that the results calculated using ADM were more accurate. This work enhances the accuracy of the free vibration analysis of tapered nanorods when compared to the DQM solution previously available in the literature. Consequently, it demonstrates the potency of ADM in the free vibration analysis of tapered nanorods, highlighting the potential of analytical approximation techniques in solving nanomechanics-related problems in science and engineering.

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