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Dynamic characteristics of a discrete-time activator-inhibitor system

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Abstract. Enzymes are specialized proteins that act as biological catalysts, accelerating chemical reactions essential for the functioning of all living cells. They play a critical role in metabolism, and their activity is regulated by molecules known as activators, which enhance their function, and inhibitors, which reduce it. These regulatory mechanisms ensure that metabolic processes occur efficiently and at the right time. Systems biology applies mathematical modeling to study and simulate these complex biological networks, allowing researchers to better understand and predict the outcomes of various metabolic reactions, providing valuable insights into cellular behavior. So, this work investigates the dynamical properties of a discrete activator-inhibitor system. It proves the existence of an interior equilibrium solution and analyzes its local dynamics. The study explores possible bifurcations, showing that the system undergoes Neimark-Sacker and flip bifurcations. Chaos in the system is also examined. Finally, simulations are provided to validate the theoretical findings.

Keywords: bifurcations; chaos; discrete-time system; numerical simulation; simulation; stability

1. Introduction

Enzymes are biological molecules that stimulate all metabolisms in the living organisms. All metabolic reactions are balanced by enzyme inhibitors and activators. Velocity of enzymatic reaction is regulated by certain enzymes as well. Most of the enzymes are much more specific in nature and are different from other catalysts. Activity of enzymes can be exaggerated by the other molecules, i.e., activators and inhibitors. Inhibitors are the molecules that decrease enzyme activity while activators are molecules that enhance activity. Among activators ions, small organic molecules, lipids, proteins and peptides can be found. It is noted here that there are various enzymes that are directly activated by small inorganic molecules mostly by cations. There are many therapeutic drugs and poisons that act as enzyme inhibitors. Enzyme inhibitors are not only a functional tool for the study of enzymatic reaction but also used in drug designing. In few cases, activation of enzymes is the result of eradication of enzyme inhibitors. In overall this effect seems to be enzyme activation. Some cations together with heavy metal cations inhibit specific enzyme. An activity of enzyme decreases remarkably with optimal temperature and pH. There are numerous enzymes that are (permanently) denatured when exposed to extreme heat and lost their structure and catalytic behavior.

Enzymes are various chemical compounds (typically proteins) that are merged into a group because of their only property-they can put down enzyme efficiency. The repression of activity is due to the connecting of inhibitor to enzyme molecule that seize catalytic reaction. Since enzymes bring about maximum part of chemical reaction in living organism, the enzyme inhibitors play pivotal role in the evolution of different sciences and also in the related technologies.

Drug designing and drug discovery is also controlled by certain enzymes so there are several pharmacological medicines that are basically treated as enzyme inhibitors. The group of familiar pharmaceutical agents with name nonsteroidal anti-inflammatory drugs (NSAIDs) includes inhibitors of enzyme cyclooxygenase (COX) that catalyze a first step of synthesis of biologically active compounds prostaglandins that are used for the relief of pain, inflammation, fever, formation of blood clots and contraction of smooth muscles, etc. According to the mode of action enzyme inhibitors can be divided into two different groups (irreversible and reversible inhibitors). Reversible enzymes may further classified as mixed, competitive, non-competitive and uncompetitive inhibitors according to their kinetic demand. In recent years, many mathematicians have investigated the dynamical properties of continuous-time ctivator-inhibitor system. For instance, Gonpot *et al*. (2008) investigated the dynamical properties of Gierer-Meinhardt activator-inhibitor system. Prytula (2015) studied the dynamical properties in generalized activator-inhibitor systems with space fractional derivatives. Zhu (2018) formulated and analyzed the activator-inhibitor system for seashell pattern. Sun *et al*. (2021) studied turing-hopf bifurcation analysis of a diffusive activator-inhibitor system. Gu *et al*. (2020) investigated turning instability and hopf bifurcation for the depletion type Gierer-Meinhardt system theoretically. Wang *et al*. (2018) investigated the dynamical properties of an activator-inhibitor system with different sources. Yochelis *et al*. (2021) studied nonlinear initiation of side-branching by activator-inhibitor-substrate morphogenesis. On the other hand, in recent years, many scientists studied the dynamical behavior of different substances in several disciplines of applied science. For

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instance, Ahmed *et al*. (2020) studied the dynamic behavior of multi-phase crystalline porous shells. Ebrahimi *et al*. (2020) have investigated dynamic behavior of MEE curved nanobeams. Moreover, dynamic behavior of hygro-magneto-thermo-electrical nanobeam was studied by Ebrahimi *et al*. (2019). Benmansour *et al*. (2019) studied dynamic properties of isolated protein microtubules. Moreover, in recent years, many mathematicians have also investigated the dynamic characteristics for the discrete biological system. For example, Rana (2017) explored control and chaotic dynamics of discrete predator-prey system. Al-Basyouni and Khan (2020) explored the bifurcations and chaos in discrete predator-prey system. Further, bifurcation analysis for a discrete prey-predator model with square-root functional response and optimal harvesting is investigated by Chakraborty *et al*. (2020). Liu and Cai (2019) studied the dynamic characteristics of a discrete predator-prey system. Agiza *et al*. (2009) explored chaotic dynamics of a discrete model. Liu and Xiao (2007) investigated the complex dynamics of predator-prey system. Khan *et al*. (2016) explored the bifurcation analysis of a plant-herbivore system. Khan and Alayachi (2022) investigated the dynamics of a phytoplankton-zooplankton model incorporating a Holling type-II functional response and toxicity effects. They identified various bifurcations, including P-D and N-S bifurcations, leading to chaotic dynamics under certain parameter conditions. The results demonstrated that toxicity plays a critical role in the onset of complex behaviors such as chaos, significantly affecting the stability and long-term behavior of the system. For more results, we refer the reader to (Tunc and Tunc 2007, 2016a, b, Owolabi and Pindza 2022a, b, Tunc 2010, Owolabi *et al*. 2023, 2024, Owolabi and Jain 2023, Alqhtani *et al*. 2023). Motivation from aforementioned studies, our aim in this paper is to investigate dynamic characteristics of the activator-inhibitor system

$$\begin{aligned} x_{t+1} &= (1-h)x_t + h\rho + h\frac{x_t^2}{y_t}, \\ y_{t+1} &= (1-h\,\gamma)y_t + hx_t^2, \end{aligned} \tag{1}$$

which is discrete form, that is obtained by forward Euler formula, of the following continuous-time activator-inhibitor system (p. 305; Edelstein-Keshet 2005)

$$\frac{dx}{dt} = \rho + \frac{x^2}{y} - x, \frac{dy}{dt} = x^2 - \gamma y \tag{2}$$

where x and y represent the concentrations of the activator and inhibitor, respectively, ρ represents the strength of the self-activation of the activator, and γ measures the strength of the production of the activator and the degradation of the inhibitor. Our main contributions in this article include:

- Topological classifications around fixed point of the activator-inhibitor system Eq. (1).
- Comprehensive bifurcation analysis around fixed point by bifurcation theory.
- Investigation of chaos by feedback control method for the activator-inhibitor system Eq. (1).
- Validation of obtained results numerically.

The next section is about the study of local dynamical classifications around fixed point of the discrete activator-inhibitor system Eq. (1). The bifurcation analysis around equilibrium solution is given in Section 3. Section 4 is about the investigation of chaos by feedback control method for the activator-inhibitor system Eq. (1). Theoretical results are numerically verified in Section 5. Conclusion is given in Section 6.

2. Local dynamical classifications around equilibrium solution of the discrete activator-inhibitor system

In this section, local dynamical classifications around equilibrium solution of the system Eq. (1) are studied. For this, first we explored existence of equilibrium solution and variational matrix around equilibrium solution of the system Eq. (1). It is simple to check that $\forall\, \gamma, \rho, h > 0$ system Eq. (1) has an interior equilibrium solution $\mathcal{T} = E_{xy}^{+}\left(\rho + \gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$.

Remark 1. It is important here to mention that the interior equilibrium solution $\mathcal{T}$ of the discrete-time activator-inhibitor system Eq. (1) has notable connections with its applications. Specifically, this equilibrium reflects the balance between activation and inhibition dynamics in the system, where ρ and γ represent key parameters influencing the system's behavior. The equilibrium point $\mathcal{T}$ offers insight into stable states that the system may converge to under certain conditions. In applications, this can model scenarios such as biological or chemical interactions where activator-inhibitor dynamics are at play, providing an equilibrium configuration that captures the interaction between activating and inhibiting factors. The precise values of ρ and γ define the location of this equilibrium and determine how changes in these parameters affect the stability and nature of the system's response. Thus, this equilibrium point is crucial for understanding long-term system behavior in practical applications involving similar dynamics.

Now, the variation matrix $\Omega|_{E_{xy(x,y)}}$ around $E_{xy(x,y)}$ under the map

$$(f_1, f_2) \longmapsto (x_{t+1}, y_{t+1}) \tag{3}$$

where

$$f_1 = (1-h)x + h\rho + h\frac{x^2}{y}, f_2 = (1-h\,\gamma)y + hx^2 \tag{4}$$

is

$$\Omega|_{E_{xy(x,y)}} = \begin{pmatrix} 1-h+\frac{2h\,x}{y} & -\frac{hx^2}{y^2} \\ 2hx & 1-h\,\gamma \end{pmatrix} \tag{5}$$

Moreover, about interior equilibrium solution $\mathcal{T}$, Eq. (5) takes the form

$$\Omega|_{\mathcal{T}} = \begin{pmatrix} 1-h+\frac{2h\,\gamma}{\rho+\gamma} & -\frac{h\gamma^2}{(\rho+\gamma)^2} \\ 2h(\rho+\gamma) & 1-h\,\gamma \end{pmatrix} \tag{6}$$

From Eq. (6), characteristic equation of $\Omega|_{\mathcal{T}}$ around $\mathcal{T}$ is

$$\lambda^2 - p\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)\lambda + q\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right) = 0 \quad (7)$$

where

$$p = \frac{2\rho + 2\gamma - h\rho + h\gamma - \rho h\gamma - h\gamma^2}{\rho+\gamma},$$
$$q = \frac{\rho(1 - h - h\gamma + h^2\gamma) + \gamma(1 + h - h\gamma + h^2\gamma)}{\rho+\gamma}. \quad (8)$$

Finally, from Eq. (7), one gets:

$$\lambda_{1,2} = \frac{P \pm \sqrt{\Delta}}{2} \quad (9)$$

where

$$\Delta = \left(\frac{2\rho + 2\gamma - h\rho + h\gamma - \rho h\gamma - h\gamma^2}{\rho+\gamma}\right)^2 - 4\left(\frac{\rho(1 - h - h\gamma + h^2\gamma) + \gamma(1 + h - h\gamma + h^2\gamma)}{\rho+\gamma}\right). \quad (10)$$

Based on above computation local dynamical classifications around $\mathcal{T}$ of the system Eq. (1) are presented according to sign of Δ, i.e., $\Delta < 0$ and $\Delta \geq 0$, respectively.

Lemma 2.1. If $\Delta < 0$ then around $\mathcal{T}$ of the system Eq. (1), following dynamical classifications hold:

(i) $\mathcal{T}$ is stable focus if

$$0 < h < \frac{\rho + \rho\gamma - \gamma + \gamma^2}{\rho\gamma + \gamma^2} \quad (11)$$

with

$$\rho > \frac{\gamma - \gamma^2}{1+\gamma} \quad (12)$$

(ii) $\mathcal{T}$ is unstable focus if Eq. (12) holds and

$$h > \frac{\rho + \rho\gamma - \gamma + \gamma^2}{\rho\gamma + \gamma^2} \quad (13)$$

(iii) $\mathcal{T}$ is non-hyperbolic if

$$h = \frac{\rho + \rho\gamma - \gamma + \gamma^2}{\rho\gamma + \gamma^2} \quad (14)$$

Lemma 2.2. If $\Delta \geq 0$ then around $\mathcal{T}$ of the system Eq. (1), following dynamical classifications hold:

(i) $\mathcal{T}$ is stable node if

$$0 < h < \min\left\{\frac{\gamma^2+\rho\gamma+\rho-\gamma+\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2}, \frac{\gamma^2+\rho\gamma+\rho-\gamma-\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2}\right\}; \quad (15)$$

(ii) $\mathcal{T}$ is unstable node if

$$h > \max\left\{\frac{\gamma^2+\rho\gamma+\rho-\gamma+\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2}, \frac{\gamma^2+\rho\gamma+\rho-\gamma-\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2}\right\} \quad (16)$$

(ii) $\mathcal{T}$ is non-hyperbolic if

$$h = \frac{\gamma^2+\rho\gamma+\rho-\gamma+\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2} \quad (7)$$

or

$$h = \frac{\gamma^2+\rho\gamma+\rho-\gamma-\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}}{\rho\gamma+\gamma^2} \quad (18)$$

3. Bifurcation analysis

We will study bifurcation analysis about $\mathcal{T}$ of the system Eq. (1) in the present section by bifurcation theory (Guckenheimer and Holmes 1983, Kuznetsov 2004).

3.1 Hope bifurcation analysis around $\mathcal{T}$

If $\Delta < 0$ then in view of Eq. (14) and Eq. (9), the simple computation yields $\left|\lambda_{1,2}\right|_{(14)} = 1$. This implies that system Eq. (1) may undergoes hopf bifurcation if (γ, h, ρ) are locate in the set:

$$N|_{\mathcal{T}} = \left\{(\gamma, h, \rho), h = \frac{\rho + \rho\gamma - \gamma + \gamma^2}{\rho\gamma + \gamma^2}\right\} \quad (19)$$

But following theorem ensure the presence of hope bifurcation about $\mathcal{T}$ of the system Eq. (1).

Theorem 3.1. Around $\mathcal{T}$, system Eq. (1) undergoes the hopf bifurcation if $(\gamma, h, \rho) \in N|_{\mathcal{T}}$ by choosing h as a bifurcation parameter.

Proof. If h is a bifurcation parameter then in the neighborhood of h^* that is $h = h^* + \epsilon$, system Eq. (1) becomes

$$x_{t+1} = \left(1 - (h^* + \epsilon)\right)x_t + \rho(h^* + \epsilon) + (h^* + \epsilon)\frac{x_t^2}{y_t},$$
$$y_{t+1} = (1 - (h^* + \epsilon)\,\gamma)y_t + (h^* + \epsilon)x_t^2. \quad (20)$$

Further, for ϵ-dependence system Eq. (20), from Eq. (9), one gets:

$$\lambda_{1,2} = \frac{P(\epsilon) \pm \iota\sqrt{4q(\epsilon) - p^2(\epsilon)}}{2} \quad (21)$$

where

$$p(\epsilon) = \frac{2\rho + 2\gamma + (h^* + \epsilon)(\gamma - \rho - \rho\gamma - \gamma^2)}{\rho+\gamma},$$
$$q(\epsilon) = \frac{\rho + \gamma + (h^* + \epsilon)(-\rho - \rho\gamma + \gamma - \gamma^2) + (h^* + \epsilon)^2(\rho\gamma + \gamma^2)}{\rho+\gamma}. \quad (22)$$

From Eq. (21), the following computation shows that non-degenerate condition holds, i.e.,

$$\left.\frac{\mathrm{d}\left|\lambda_{1,2}\right|}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{\gamma^2 + \rho + \rho\gamma - \gamma}{2(\rho+\gamma)} \neq 0 \quad (23)$$

Further, for the existence of hopf bifurcation around $\mathcal{T} = E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the discrete activator-inhibitor system Eq. (1), it is also require that $\lambda^m_{1,2} \neq 1, m = 1,\cdots,4$ if $\epsilon = 0$ that is corresponds to $p(0) \neq -2,0,1,2$. But if (14) holds then from Eq. (22), one get $q(0) = 1$. Therefore, $p(0) \neq -2,2$, and so, for the existence of hopf bifurcation, it is only requires that $p(0) \neq 0,1$. For this, the straightforward computation yields

$$\rho \neq \frac{\left\{4\gamma^2 - 2\gamma^3 + 2\gamma \pm \sqrt{(4\gamma^2+2\gamma-2\gamma^3)^2 - 4(1+\gamma^2)(-4\gamma^3+\gamma^4+\gamma^2)}\right\}}{2(1+\gamma^2)}, \frac{\left\{2\gamma - 2\gamma^3 + 2\gamma^2 \pm \sqrt{(2\gamma-2\gamma^3+2\gamma^2)^2 - 4(1+\gamma^2+\gamma)(\gamma^4+\gamma^2-3\gamma^3)}\right\}}{2(\gamma^2+\gamma+1)} \quad (24)$$

Now, it is noted that Eq. (20) takes the form

$$\begin{aligned} u_{t+1} &= \big(1-(h^*+\epsilon)\big)(u_t+x^*) + \rho(h^*+\epsilon) + (h^*+\epsilon)\frac{(u_t+x^*)^2}{v_t+y^*} - x^*, \\ v_{t+1} &= (1-(h^*+\epsilon)\gamma)(v_t+y^*) + (h^*+\epsilon)\times (u_t+x^*)^2 - y^*, \end{aligned} \quad (25)$$

by

$$u_t = x_t - x^*, v_t = y_t - y^* \quad (26)$$

where $x^* = \rho+\gamma, y^* = \frac{(\rho+\gamma)^2}{\gamma}$. Hereafter, if $\epsilon = 0$ normal form is investigated. So, by Taylor series about $O = E_{00}(0,0)$, Eq. (25) takes the form

$$\begin{aligned} u_{t+1} &= \alpha_{11}u_t + \alpha_{12}v_t + \alpha_{13}u_t^2 + \alpha_{14}u_tv_t \\ &+\alpha_{15}v_t^2 + \alpha_{16}u_tv_t^2 + \alpha_{17}u_t^2v_t + \alpha_{18}v_t^3 \\ v_{t+1} &= \alpha_{21}u_t + \alpha_{22}v_t + \alpha_{23}u_t^2 \end{aligned} \quad (27)$$

with

$$\begin{aligned} &\alpha_{11} = 1-h+2h\frac{x^*}{y^*}, \quad \alpha_{12} = -h\frac{x^{*2}}{y^{*2}}, \\ &\alpha_{13} = \frac{h}{y^*}, \quad \alpha_{14} = -2h\frac{x^*}{y^{*2}}, \\ &\alpha_{15} = h\frac{x^{*2}}{y^{*3}}, \quad \alpha_{16} = 2h\frac{x^*}{y^{*3}}, \\ &\alpha_{17} = -\frac{h}{y^{*2}}, \quad \alpha_{18} = -h\frac{x^{*2}}{y^{*4}}, \\ &\alpha_{21} = 2hx^*, \quad \alpha_{22} = 1-h\gamma, \quad \alpha_{23} = h \end{aligned} \quad (28)$$

Now, in order to get linear part of Eq. (27) in to canonical form one use the following transformation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha_{12} & 0 \\ \eta-\alpha_{11} & -\zeta \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad (29)$$

with

$$\eta = \frac{2\rho+2\gamma-h\rho+h\gamma-\rho h\gamma-h\gamma^2}{2(\rho+\gamma)}, \quad (30)$$

$$\zeta = \frac{1}{2}\sqrt{4\left(\frac{\rho(1-h-h\gamma+h^2\gamma) + \gamma(1+h-h\gamma+h^2\gamma)}{\rho+\gamma}\right) - \left(\frac{2\rho+2\gamma-h\rho+h\gamma-\rho h\gamma-h\gamma^2}{\rho+\gamma}\right)^2}.$$

From Eq. (27) and Eq. (29), one gets

$$\begin{aligned} x_{t+1} &= \eta x_t - \zeta y_t + \bar{P}(x_t,y_t), \\ y_{t+1} &= \zeta x_t + \eta y_t + \bar{Q}(x_t,y_t), \end{aligned} \quad (31)$$

where

$$\begin{aligned} \bar{P} &= r_{11}x_t^3 + r_{12}x_t^2 + r_{13}x_t^2y_t + r_{14}x_ty_t + r_{15}x_ty_t^2 \\ &+r_{16}y_t^3 + r_{17}y_t^2, \\ \bar{Q} &= r_{21}x_t^3 + r_{22}x_t^2 + r_{23}x_t^2y_t + r_{24}x_ty_t + r_{25}x_ty_t^2 \\ &+r_{26}y_t^3 + r_{27}y_t^2, \end{aligned} \quad (32)$$

and

$$\begin{aligned} r_{11} &= (\eta-\alpha_{11})\left(\alpha_{17}\alpha_{12} + \alpha_{16}(\eta-\alpha_{11}) + \frac{\alpha_{18}}{\alpha_{12}}\right), \\ r_{12} &= \alpha_{13}\alpha_{12} + (\eta-\alpha_{11})\left(\alpha_{14} + \frac{\alpha_{15}(\eta-\alpha_{11})}{\alpha_{12}}\right), \\ r_{13} &= -\zeta(\eta-\alpha_{11})\left(2\alpha_{16} + \frac{3\alpha_{18}(\eta-\alpha_{11})}{\alpha_{12}}\right) - \zeta\alpha_{17}\alpha_{12}, \\ r_{14} &= -\alpha_{14}\zeta - \frac{2\alpha_{15}\zeta(\eta-\alpha_{11})}{\alpha_{12}}, \\ r_{15} &= \zeta^2\left(\alpha_{16} + \frac{3\alpha_{18}}{\alpha_{12}}\right), r_{16} = -\frac{\zeta^2\alpha_{18}}{\alpha_{12}}, r_{17} = \frac{\zeta^2\alpha_{15}}{\alpha_{12}}, \\ r_{21} &= \frac{(\eta-\alpha_{11})^2}{\zeta}\left(\alpha_{16}(\eta-\alpha_{11}) + \frac{\alpha_{18}}{\alpha_{12}} + \alpha_{17}\alpha_{12}\right), \\ r_{22} &= \frac{(\eta-\alpha_{11})}{\zeta}\left(\alpha_{13}\alpha_{12} + \alpha_{14}(\eta-\alpha_{11}) + \frac{\alpha_{15}(\eta-\alpha_{11})^2}{\alpha_{12}}\right) - \frac{\alpha_{23}{\alpha_{12}}^2}{\zeta}, \\ r_{23} &= -(\eta-\alpha_{11})\left(2\alpha_{16}(\eta-\alpha_{11}) + \alpha_{17}\alpha_{12} + \frac{3\alpha_{18}(\eta-\alpha_{11})^2}{\alpha_{12}}\right), \\ r_{24} &= -(\eta-\alpha_{11})\left(\alpha_{14} + \frac{2\,\alpha_{15}(\eta-\alpha_{11})}{\alpha_{12}}\right), \\ r_{25} &= \zeta(\eta-\alpha_{11})\left(\alpha_{16} + \frac{3\alpha_{18}}{\alpha_{12}}\right), \\ r_{26} &= \frac{-\alpha_{18}\zeta(\eta-\alpha_{11})}{\alpha_{12}}, r_{27} = \frac{\alpha_{15}\zeta(\eta-\alpha_{11})}{\alpha_{12}}. \end{aligned} \quad (33)$$

From Eq. (32), the computation yields

$$\begin{aligned} &\left.\frac{\partial^2\bar{P}}{\partial x_t^2}\right|_O = 2r_{12}, \left.\frac{\partial^2\bar{P}}{\partial x_t\partial y_t}\right|_O = r_{14}, \left.\frac{\partial^2\bar{P}}{\partial y_t^2}\right|_O = 2r_{17}, \\ &\left.\frac{\partial^3\bar{P}}{\partial x_t^3}\right|_O = 6r_{11}, \left.\frac{\partial^3\bar{P}}{\partial x_t^2\partial y_t}\right|_O = 2r_{13}, \left.\frac{\partial^3\bar{P}}{\partial x_t\partial y_t^2}\right|_O = 2r_{15}, \\ &\left.\frac{\partial^3\bar{P}}{\partial y_t^3}\right|_O = 6r_{16}, \left.\frac{\partial^2\bar{Q}}{\partial x_t^2}\right|_O = 2r_{22}, \left.\frac{\partial^2\bar{Q}}{\partial x_t\partial y_t}\right|_O = r_{24}, \\ &\left.\frac{\partial^2\bar{Q}}{\partial y_t^2}\right|_O = 2r_{27}, \left.\frac{\partial^3\bar{Q}}{\partial x_t^3}\right|_O = 6r_{21}, \left.\frac{\partial^3\bar{Q}}{\partial x_t^2\partial y_t}\right|_O = 2r_{23}, \\ &\left.\frac{\partial^3\bar{Q}}{\partial x_t\partial y_t^2}\right|_O = 2r_{25}, \left.\frac{\partial^3\bar{Q}}{\partial y_t^3}\right|_O = 6r_{26} \end{aligned} \quad (34)$$

Finally, to determine Eq. (31) undergo N-S bifurcation following quantity required to be non-zero:

$$\chi=-\Re\left(\frac{(1-2\bar{\lambda})\bar{\lambda}^{2}\rho_{11}\rho_{20}}{1-\lambda}\right)-\frac{\|\rho_{11}\|^{2}}{2}-\|\rho_{02}\|^{2}+\Re(\bar{\lambda}\rho_{21}) \tag{35}$$

where

$$\begin{aligned}\rho_{02}&=\frac{1}{8}\left(\frac{\partial^{2}\bar{P}}{\partial x_t^{2}}-\frac{\partial^{2}\bar{P}}{\partial y_t^{2}}+2\frac{\partial^{2}\bar{Q}}{\partial x_t\partial y_t}+\iota\left(\frac{\partial^{2}\bar{Q}}{\partial x_t^{2}}-\frac{\partial^{2}\bar{Q}}{\partial y_t^{2}}+2\frac{\partial^{2}\bar{P}}{\partial x_t\partial y_t}\right)\right)\Bigg|_{O}\\ \rho_{11}&=\frac{1}{4}\left(\frac{\partial^{2}\bar{P}}{\partial x_t^{2}}+\frac{\partial^{2}\bar{P}}{\partial y_t^{2}}+\iota\left(\frac{\partial^{2}\bar{Q}}{\partial x_t^{2}}+\frac{\partial^{2}\bar{Q}}{\partial y_t^{2}}\right)\right)\Bigg|_{O}\\ \rho_{20}&=\frac{1}{8}\left(\frac{\partial^{2}\bar{P}}{\partial x_t^{2}}-\frac{\partial^{2}\bar{P}}{\partial y_t^{2}}+2\frac{\partial^{2}\bar{Q}}{\partial x_t\partial y_t}+\iota\left(\frac{\partial^{2}\bar{Q}}{\partial x_t^{2}}-\frac{\partial^{2}\bar{Q}}{\partial y_t^{2}}-2\frac{\partial^{2}\bar{P}}{\partial x_t\partial y_t}\right)\right)\Bigg|_{O}\end{aligned} \tag{36}$$

and

$$\rho_{21}=\frac{1}{16}\left(\frac{\partial^{3}\bar{P}}{\partial x_t^{3}}+\frac{\partial^{3}\bar{P}}{\partial y_t^{3}}+\frac{\partial^{3}\bar{Q}}{\partial x_t^{2}\partial y_t}+\frac{\partial^{3}\bar{Q}}{\partial y_t^{3}}+\iota\left(\frac{\partial^{3}\bar{Q}}{\partial x_t^{3}}+\frac{\partial^{3}\bar{Q}}{\partial x_t\partial y_t^{2}}-\frac{\partial^{3}\bar{P}}{\partial x_t^{2}\partial y_t}-\frac{\partial^{3}\bar{P}}{\partial y_t^{3}}\right)\right)\Bigg|_{O} \tag{37}$$

In view of Eq. (34), from Eq. (36) and Eq. (37), one gets

$$\begin{aligned}\rho_{02}&=\frac{1}{4}\left(r_{12}-r_{17}+r_{24}+\iota(r_{22}-r_{27}+r_{14})\right),\\ \rho_{11}&=\frac{1}{2}\left(r_{12}+r_{17}+\iota(r_{22}+r_{27})\right),\\ \rho_{20}&=\frac{1}{4}\left(r_{12}-r_{17}+r_{24}+\iota(r_{22}-r_{27}-r_{14})\right),\\ \rho_{21}&=\frac{1}{8}\left(3r_{11}+3r_{16}+r_{23}+3r_{26}+\iota(3r_{21}+r_{25}-r_{15}-3r_{16})\right).\end{aligned} \tag{38}$$

Finally, utilizing Eq. (38) into Eq. (35) if one get $\chi\neq 0$ as $(\gamma,h,\rho)\in N|_{\mathcal{T}}$ then around $\mathcal{T}$, system Eq. (1) undergo N-S bifurcation. Further, supercritical (resp. subcritical) N-S bifurcation occurs if $\chi<0$ (resp. $\chi>0$).

3.2 Flip bifurcation analysis around $\mathcal{T}$

If $\Delta>0$ then from Eqs. (17) and (9) one gets $\lambda_1|_{(17)}=-1$ but $\lambda_2|_{(17)}=\frac{\left\{3\gamma(\rho+\gamma)^2-(\gamma^2+\rho\gamma+\rho-\gamma)^2-(\gamma^2+\rho\gamma+\rho-\gamma)\times\sqrt{(\gamma-r^2+\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}\right\}}{\gamma(\rho+\gamma)^2}\neq 1 \text{ or } -1$. This implies that system Eq. (1) may undergoes flip bifurcation if (γ,h,ρ) are locate in the set:

$$F|_{\mathcal{T}}=\left\{(\gamma,h,\rho),h=\frac{\left\{\gamma^2+\rho\gamma+\rho-\gamma+\sqrt{(\gamma-r^2-\rho\gamma-\rho)^2-4\gamma(\rho+\gamma)^2}\right\}}{\rho\gamma+\gamma^2}\right\} \tag{39}$$

But following theorem ensure the presence of flip bifurcation about $\mathcal{T}$ of the system Eq. (1).

Theorem 3.2. If $(\gamma,h,\rho)\in F|_{\mathcal{T}}$ then around $\mathcal{T}$, system Eq. (1) undergoes the flip bifurcation.

Proof. If h is a bifurcation parameter then in the neighborhood of h^* that is $h=h^*+\epsilon$, discrete activator-inhibitor system Eq. (1) takes the form Eq. (20). Further, system Eq. (1) takes the following form:

$$\begin{aligned}u_{t+1}&=\widehat{\alpha_{11}}u_t+\widehat{\alpha_{12}}v_t+\widehat{\alpha_{13}}u_t^2+\widehat{\alpha_{14}}u_tv_t+\widehat{\alpha_{15}}v_t^2\\&\quad+\gamma_{01}u_t\epsilon+\gamma_{02}v_t\epsilon+\gamma_{03}u_t^2\epsilon+\gamma_{04}u_tv_t\epsilon+\gamma_{05}v_t^2\epsilon,\\ v_{t+1}&=\widehat{\alpha_{21}}u_t+\widehat{\alpha_{22}}v_t+\widehat{\alpha_{23}}u_t^2\\&\quad+\gamma_{06}u_t\epsilon+\gamma_{07}v_t\epsilon+\gamma_{08}u_t^2\end{aligned} \tag{40}$$

where

$$\begin{aligned}&\widehat{\alpha_{11}}=1-h^*+2h^*\frac{x^*}{y^*},\widehat{\alpha_{12}}=-\frac{h^*x^{*2}}{y^{*2}},\widehat{\alpha_{13}}=\frac{h^*}{y^*},\\ &\widehat{\alpha_{14}}=-2h^*\frac{x^*}{y^{*2}},\widehat{\alpha_{15}}=h^*\frac{x^{*2}}{y^{*3}},\gamma_{01}=-1+2\frac{x^*}{y^*},\\ &\gamma_{02}=-\frac{x^{*2}}{y^{*2}},\gamma_{03}=\frac{1}{y^*},\gamma_{04}=-2\frac{x^*}{y^{*2}},\gamma_{05}=\frac{x^{*2}}{y^{*3}},\\ &\widehat{\alpha_{21}}=2h^*x^*,\widehat{\alpha_{22}}=1-h^*\gamma,\widehat{\alpha_{23}}=h^*,\gamma_{06}=2x^*,\\ &\gamma_{07}=-\gamma,\gamma_{08}=1,\end{aligned} \tag{41}$$

by Eq. (26). Again, Eq. (40) becomes

$$\begin{pmatrix}x_{t+1}\\ y_{t+1}\end{pmatrix}=\begin{pmatrix}-1&0\\0&\lambda_2\end{pmatrix}\begin{pmatrix}x_t\\ y_t\end{pmatrix}+\begin{pmatrix}\hat{X}\\ \hat{Y}\end{pmatrix} \tag{42}$$

For nontrivial solution,

$$\begin{aligned}\hat{X}&=\frac{\widehat{\alpha_{13}}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\widehat{\alpha_{23}}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t^2\\&+\frac{\widehat{\alpha_{14}}(\lambda_2-\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}u_tv_t+\frac{\widehat{\alpha_{15}}(\lambda_2-\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t^2\\&+\frac{\gamma_{01}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{06}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t\epsilon\\&+\frac{\gamma_{02}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{07}}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t\epsilon\\&+\frac{\gamma_{03}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{08}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t^2\epsilon\\&+\frac{\gamma_{04}(\lambda_2-\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}u_tv_t\epsilon+\frac{\gamma_{05}(\lambda_2-\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t^2\epsilon\\ \hat{Y}&=\frac{\widehat{\alpha_{13}}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{12}}\widehat{\alpha_{23}}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t^2\\&+\frac{\widehat{\alpha_{14}}(1+\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}u_tv_t+\frac{\widehat{\alpha_{15}}(1+\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t^2\\&+\frac{\gamma_{01}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{12}}\gamma_{06}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t\epsilon\\&+\frac{\gamma_{02}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{12}}\gamma_{07}}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t\epsilon+\frac{\gamma_{04}(1+\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}u_tv_t\epsilon\\&+\frac{\gamma_{03}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{12}}\gamma_{08}}{\widehat{\alpha_{12}}(1+\lambda_2)}u_t^2\epsilon+\frac{\gamma_{05}(1+\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}(1+\lambda_2)}v_t^2\epsilon\\ u_t&=\widehat{\alpha_{12}}(x_t+y_t),\\ v_t&=-(1+\widehat{\alpha_{11}})x_t+(\lambda_2-\widehat{\alpha_{11}})y_t\\ {u_t}^2&=\widehat{\alpha_{12}}^{\,2}({x_t}^2+2x_ty_t+{y_t}^2),\end{aligned} \tag{43}$$

$$v_t{}^2 = (1+\widehat{\alpha_{11}})^2 x_t{}^2 + (\lambda_2 - \widehat{\alpha_{11}})^2 y_t{}^2 - 2(1+\widehat{\alpha_{11}})(\lambda_2 - \widehat{\alpha_{11}}) x_t y_t,$$
$$u_t v_t = -\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}}) x_t{}^2 + \begin{pmatrix} \widehat{\alpha_{12}}(\lambda_2 - \widehat{\alpha_{11}}) - \\ \widehat{\alpha_{12}}(1+\widehat{\alpha_{11}}) \end{pmatrix} x_t y_t + \widehat{\alpha_{12}}(\lambda_2 - \widehat{\alpha_{11}}) y_t{}^2,$$
$$v_t{}^2 \epsilon = (1+\widehat{\alpha_{11}})^2 x_t{}^2 \epsilon + (\lambda_2 - \widehat{\alpha_{11}})^2 y_t{}^2 \epsilon - 2(1+\widehat{\alpha_{11}})(\lambda_2 - \widehat{\alpha_{11}}) x_t y_t \epsilon$$

by

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \widehat{\alpha_{12}} & \widehat{\alpha_{12}} \\ -1-\widehat{\alpha_{11}} & \lambda_2 - \widehat{\alpha_{11}} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad (44)$$

Hereafter, for Eq. (42), center manifold $M^C E_{00}(0,0)$ around $E_{00}(0,0)$ is explored in the neighborhood of ϵ, and hence, one can write $M^C E_{00}(0,0)$ as a following mathematical expression:

$$M^C E_{00}(0,0) = \begin{Bmatrix} (x_t, y_t): y_t = C_0 \epsilon + C_1 x_t{}^2 + C_2 x_t \epsilon \\ + C_3 \epsilon^3 + O(|x_t| + |\epsilon|)^3 \end{Bmatrix} \quad (45)$$

The computation yields

$$C_0 = 0 = C_3,$$
$$C_1 = \frac{1}{1-\lambda_2{}^2} \begin{pmatrix} \widehat{\alpha_{12}}\widehat{\alpha_{13}}(1+\widehat{\alpha_{11}}) + \widehat{\alpha_{23}}\widehat{\alpha_{12}}^2 \\ -\widehat{\alpha_{14}}(1+\widehat{\alpha_{11}})^2 \end{pmatrix} + \frac{\widehat{\alpha_{15}}(1+\widehat{\alpha_{11}})^3}{\widehat{\alpha_{12}}(1-\lambda_2{}^2)},$$
$$C_2 = \frac{1}{1-\lambda_2{}^2} \begin{pmatrix} \gamma_{01}(1+\widehat{\alpha_{11}}) + \widehat{\alpha_{12}}\gamma_{06} - \\ \dfrac{\begin{Bmatrix} \gamma_{02}(1+\widehat{\alpha_{11}}) + \\ \gamma_{07}\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}}) \end{Bmatrix}}{\widehat{\alpha_{12}}} \\ + \widehat{\alpha_{14}}(\lambda_2 - \widehat{\alpha_{11}})(1+\widehat{\alpha_{11}}) + \\ 2\widehat{\alpha_{12}} C_0 \begin{pmatrix} \widehat{\alpha_{13}}(1+\widehat{\alpha_{11}}) + \widehat{\alpha_{12}}\widehat{\alpha_{23}} \\ -\widehat{\alpha_{14}}(1+\widehat{\alpha_{11}})^2 \end{pmatrix} \end{pmatrix} \quad (46)$$

Finally, we will express Eq. (42) restrict to $M^C E_{00}(0,0)$ as follows:

$$f(x_t) = -x_t + h_1 x_t{}^2 + h_2 x_t \epsilon + h_3 x_t{}^2 \epsilon + h_4 x_t \epsilon^2 + h_5 x_t{}^3 + O(|x_t| + |\epsilon|)^4, \quad (47)$$

where

$$h_1 = \frac{1}{1+\lambda_2} \begin{pmatrix} (\lambda_2 - \widehat{\alpha_{11}})\widehat{\alpha_{13}}\widehat{\alpha_{12}} - \widehat{\alpha_{23}}\widehat{\alpha_{12}}^2 - \\ \widehat{\alpha_{14}}(\lambda_2 - \widehat{\alpha_{11}})(1+\widehat{\alpha_{11}}) + \\ \dfrac{\widehat{\alpha_{15}}(\lambda_2 - \widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})^2}{\widehat{\alpha_{12}}} \end{pmatrix},$$
$$h_2 = \frac{1}{1+\lambda_2} \begin{pmatrix} \gamma_{01}(\lambda_2 - \widehat{\alpha_{11}}) - \gamma_{06}\widehat{\alpha_{12}} - \\ (1+\widehat{\alpha_{11}}) \\ \left(\dfrac{\gamma_{02}(\lambda_2 - \widehat{\alpha_{11}}) - \gamma_{07}\widehat{\alpha_{12}}}{\widehat{\alpha_{12}}} \right) \end{pmatrix},$$
$$h_3 = \frac{1}{\widehat{\alpha_{12}}(1+\lambda_2)} \begin{pmatrix} 2C_2\widehat{\alpha_{12}}^2 \begin{pmatrix} \widehat{\alpha_{13}}(\lambda_2 - \widehat{\alpha_{11}}) \\ -\widehat{\alpha_{12}}\widehat{\alpha_{23}} \end{pmatrix} \\ + \; C_2\widehat{\alpha_{12}}\widehat{\alpha_{14}} \begin{pmatrix} \lambda_2 - \widehat{\alpha_{11}} - \\ (1+\widehat{\alpha_{11}}) \end{pmatrix} \\ (\lambda_2 - \widehat{\alpha_{11}}) \\ -2C_2\widehat{\alpha_{15}}(\lambda_2 - \widehat{\alpha_{11}})^2 \\ (1+\widehat{\alpha_{11}}) + \end{pmatrix}$$
$$\frac{1}{\widehat{\alpha_{12}}(1+\lambda_2)} \times \begin{pmatrix} + C_1(\lambda_2 - \widehat{\alpha_{11}}) \\ \begin{pmatrix} \gamma_{02}(\lambda_2 - \widehat{\alpha_{11}}) \\ -\widehat{\alpha_{12}}\gamma_{07} \end{pmatrix} + \\ \begin{pmatrix} \gamma_{01}(\lambda_2 - \widehat{\alpha_{11}}) - \\ \widehat{\alpha_{12}}\gamma_{06} \end{pmatrix} C_1\widehat{\alpha_{12}} \\ \begin{pmatrix} \gamma_{03}(\lambda_2 - \widehat{\alpha_{11}}) - \\ \widehat{\alpha_{12}}\gamma_{08} \end{pmatrix} \widehat{\alpha_{12}}^2 + \\ \gamma_{05}(\lambda_2 - \widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})^2 \\ -\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}}) \\ \gamma_{04}(\lambda_2 - \widehat{\alpha_{11}}) \end{pmatrix},$$
$$h_4 = \frac{C_2}{1+\lambda_2} \begin{pmatrix} \gamma_{01}(\lambda_2 - \widehat{\alpha_{11}}) \\ -\widehat{\alpha_{12}}\gamma_{06} + \\ \dfrac{\left\{ (\lambda_2 - \widehat{\alpha_{11}}) \begin{pmatrix} \gamma_{02}(\lambda_2 - \widehat{\alpha_{11}}) \\ -\widehat{\alpha_{12}}\gamma_{07} \end{pmatrix} \right\}}{\widehat{\alpha_{12}}} \end{pmatrix}, \quad (48), (49)$$
$$h_5 = \frac{C_1}{1+\lambda_2} \begin{pmatrix} 2\widehat{\alpha_{12}} \begin{pmatrix} \widehat{\alpha_{13}}(\lambda_2 - \widehat{\alpha_{11}}) \\ -\widehat{\alpha_{12}}\widehat{\alpha_{23}} \end{pmatrix} \\ + \widehat{\alpha_{14}}(\lambda_2 - \widehat{\alpha_{11}})(\lambda_2 - 2\widehat{\alpha_{11}} - 1) \\ - \dfrac{2\widehat{\alpha_{15}}(1+\widehat{\alpha_{11}})(\lambda_2 - \widehat{\alpha_{11}})^2}{\widehat{\alpha_{12}}} \end{pmatrix}$$

Now, in order for map Eq, (47) undergoes flip bifurcation around $\mathcal{T} = E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the discrete activator-inhibitor system Eq. (1) it should be required that following discriminatory quantities Γ_1 and Γ_2 are required to be non-zero:

$$\Gamma_1 = \left.\left(\frac{\partial^2 f}{\partial x_t \partial \epsilon} + \frac{1}{2}\frac{\partial f}{\partial \epsilon}\frac{\partial^2 f}{\partial x_t{}^2} \right)\right|_{E_{00}(0,0)},$$
$$\Gamma_2 = \left.\left(\frac{1}{6}\frac{\partial^3 f}{\partial x_t{}^3} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_t{}^2} \right)^2 \right)\right|_{E_{00}(0,0)}. \quad (50)$$

Therefore, the calculation yields

$$\Gamma_1 = \frac{\gamma(\rho+\gamma)^2}{\begin{Bmatrix} 4\gamma(\rho+\gamma)^2 - (\gamma^2+\rho\gamma+\rho-\gamma)^2 \\ -B(\gamma^2+\rho\gamma-\gamma) \end{Bmatrix}} \frac{1}{\gamma(\rho+\gamma)^3} \times \begin{pmatrix} 2(\rho+\gamma)^2(\gamma^2-\rho\gamma) - \\ B \begin{pmatrix} (\rho-\gamma)(\gamma-\rho^2-\gamma\rho-\rho) + \\ (\gamma-\rho)(\gamma^2+\rho\gamma+\rho-\gamma) \times \\ (\rho^2-\rho\gamma-\rho+\gamma) \end{pmatrix} + \\ \begin{pmatrix} 2\gamma^2(\gamma^2+\rho\gamma+\rho-\gamma) \\ +2\gamma^2 B \end{pmatrix} (\rho+\gamma) \\ - \dfrac{\begin{Bmatrix} (2\gamma(\rho+\gamma)^2+\gamma-\rho) \\ (B+(\gamma^2+\rho\gamma+\gamma-\rho)) \end{Bmatrix}}{\begin{Bmatrix} \gamma^3(\rho+\gamma)^3 \begin{pmatrix} \gamma^2+\rho\gamma+ \\ \rho-\gamma+B \end{pmatrix} \end{Bmatrix}} \times \\ \begin{pmatrix} -2\gamma^3(\rho+\gamma)^2 + \\ B \begin{pmatrix} -\gamma^3+\rho^2\gamma^2 \\ +\rho^2\gamma-\gamma^4 \end{pmatrix} \\ -(\gamma^2+\rho\gamma+\rho-\gamma) \\ \begin{pmatrix} \rho\gamma^4+\gamma^4-\rho^2\gamma^2+ \\ \rho\gamma^3+\rho\gamma^2+\gamma^3 \end{pmatrix} \end{pmatrix} \end{pmatrix}, \quad (51)$$

and,

$$\Gamma_2 = \frac{C_1}{\left\{\begin{pmatrix} 4\gamma(\rho+\gamma)^2 - (\gamma^2+\rho\gamma+\rho-\gamma)^2 \\ -B(\gamma^2+\rho\gamma+\rho-\gamma) \end{pmatrix}\gamma(\rho+\gamma)^6\right\}}\Phi \quad (52)$$

with

$$\Phi = \left(\begin{array}{c}
(-2\gamma^2(\gamma^2+\rho\gamma+\rho-\gamma+B)^2) \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + \\ (\gamma-\rho^2-\gamma\rho-\rho)B + \\ (\gamma^2+\rho\gamma+\rho-\gamma)\times \\ (\rho^2+\rho\gamma-\rho+\gamma) \end{pmatrix} \\
-2\gamma^3(\gamma^2+\rho\gamma+\rho-\gamma+B) \\
\begin{pmatrix} (\rho+\gamma)(\gamma^2+\rho\gamma+\rho-\gamma)^2 + \\ 2B(\rho+\gamma)(\gamma^2+\rho\gamma+\rho-\gamma) \\ +(\rho+\gamma)B \end{pmatrix} \\
-\begin{pmatrix} 2\gamma(\rho+\gamma)^2(\rho^2-\rho\gamma) \\ (\gamma^2+\rho\gamma+\rho-\gamma) - \\ \rho B(\rho+\gamma) \\ (\gamma^2+\rho\gamma+\rho-\gamma+B) \end{pmatrix} \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + \\ (\gamma-\rho^2-\rho\gamma-\rho)B + \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ (\rho^2+\rho\gamma+\rho-\gamma) \end{pmatrix} + \\
\begin{pmatrix} 4\gamma(\rho+\gamma)^2 + 2(\gamma-\rho) \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ +2(\gamma-\rho)B \end{pmatrix} \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + \\ (\gamma-\rho^2-\rho\gamma-\rho)B + \\ (\gamma^2+\rho\gamma+\rho-\gamma)\times \\ (\rho^2+\rho\gamma+\rho-\gamma) \end{pmatrix}
\end{array}\right)^2 \\
\left(\begin{array}{c}
\left(\dfrac{(\gamma^2(\rho+\gamma)^6)^{-1}}{\begin{array}{c}4\gamma(\rho+\gamma)^2-(\gamma^2+\rho\gamma+\rho-\gamma)^2 \\ -(\gamma^2+\rho\gamma+\rho-\gamma)B\end{array}}\right)^2 \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + \\ (\gamma-\rho^2-\rho\gamma-\rho)B \\ +(\gamma^2+\rho\gamma+\rho-\gamma) \\ (\rho^2+\rho\gamma-\rho+\gamma) \end{pmatrix} \\
\begin{pmatrix} -2\gamma^2(\rho+\gamma)^2(\gamma^2+\rho\gamma+\rho-\gamma)^2 \\ +2B(\gamma^2+\rho\gamma+\rho-\gamma) - B \end{pmatrix} - \gamma^4 \\
+(\rho+\gamma)\begin{pmatrix} (\gamma^2+\rho\gamma+\rho-\gamma)^3 + 3B\times \\ (\gamma^2+\rho\gamma+\rho-\gamma)^2 + 3B^2\times \\ (\gamma^2+\rho\gamma+\rho-\gamma) + B^3 \end{pmatrix} \\
(\rho+\gamma)^2\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + \\ B(\gamma-\rho^2-\rho\gamma-\rho) + \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ (\rho^2+\rho\gamma-\rho+\gamma) \end{pmatrix} \\
\begin{pmatrix} -4\gamma(\rho+\gamma)^2(\gamma^2+\rho\gamma+\rho-\gamma) \\ (1+4B) - 2\gamma\times \\ (\gamma-\rho)(\gamma^2+\rho\gamma+\rho-\gamma)^2 + B^2 \end{pmatrix} \\
\begin{pmatrix} -2\gamma^2(\rho+\gamma)^2 - \\ \gamma B(\gamma^2-\rho^2-\gamma\rho-\rho) - \\ (\gamma^3+\rho\gamma^2+\gamma\rho-\gamma^2) \\ (\rho^2-\gamma\rho-\rho+\gamma) \end{pmatrix} 2\gamma(\rho+\gamma)^2 \\
+(\gamma-\rho)(\gamma^2+\rho\gamma+\rho-\gamma) + B(\gamma-\rho)^2
\end{array}\right), \quad (53)$$

where

$$B = \sqrt{(\gamma-\gamma^2-\rho\gamma-\rho)^2 - 4\gamma(\rho+\gamma)} \quad (54)$$

$$C_1 = \frac{-1}{A}\left\{\begin{array}{c}
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho)\times \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ +B(\gamma-\rho) \end{pmatrix} \\
(-\gamma^2(\gamma^2+\rho\gamma+\rho-\gamma) + B) - \\
4\gamma^2(\rho+\gamma)^4(\gamma^2+\rho\gamma+\rho-\gamma) - \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho)\times \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ +B(\gamma-\rho) \end{pmatrix} \\
\begin{pmatrix} 4\gamma B(\rho+\gamma)(\gamma^2-\rho^2) \\ (\gamma^2+\rho\gamma+\rho-\gamma) \end{pmatrix} \\
-(2\rho\gamma+2\gamma^2)(\gamma^2-\rho^2) \\
(\gamma^2+\rho\gamma+\rho-\gamma)^2 \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho)\times \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ +B(\gamma-\rho) \end{pmatrix} - \\
\begin{pmatrix} 4\gamma^2(\rho+\gamma)^4 - 2\gamma B^2(\rho+\gamma) \\ (\gamma^2-\rho^2) \end{pmatrix} \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho)\times \\ (\gamma^2+\rho\gamma+\rho-\gamma) \\ +B(\gamma-\rho) \end{pmatrix} - \\
\begin{pmatrix} 4\gamma^2(\rho+\gamma)^4 - (\gamma-\rho)^2 \\ (\gamma^2+\rho\gamma+\rho-\gamma) - B^2(\gamma-\rho) \end{pmatrix} - \\
\begin{pmatrix} (4\gamma^2-4\rho\gamma)(\rho+\gamma)^2(\gamma^2+\rho\gamma+\rho-\gamma) \\ -2\gamma B(\gamma-\rho)^2(\gamma^2+\rho\gamma+\rho-\gamma) \end{pmatrix} \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho) \\ (\gamma^2+\rho\gamma+\rho-\gamma) + B(\gamma-\rho) \end{pmatrix} - \\
(4\gamma B(\gamma-\rho)(\rho+\gamma)^2) \\
\begin{pmatrix} 2\gamma(\rho+\gamma)^2 + (\gamma-\rho) \\ (\gamma^2+\rho\gamma+\rho-\gamma) + B(\gamma-\rho) \end{pmatrix} + \\
\gamma^3(\rho+\gamma)\begin{pmatrix} (\gamma^2+\rho\gamma+\rho-\gamma)^3 + \\ B(\gamma^2+\rho\gamma+\rho-\gamma)^2\gamma^3 \\ (\rho+\gamma) \end{pmatrix} \\
\begin{pmatrix} B^2(\gamma^2+\rho\gamma+\rho-\gamma) + B^3 + \\ +2B(\gamma^2+\rho\gamma+\rho-\gamma)^2 + 2B^2 \end{pmatrix}
\end{array}\right\} \quad (55)$$

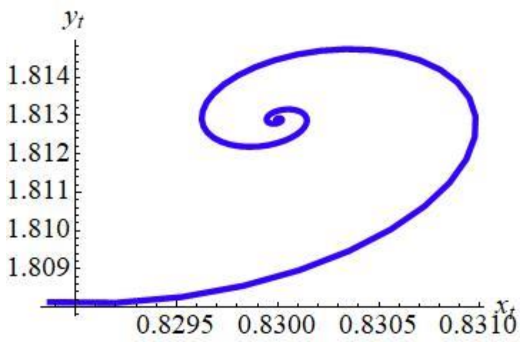
$$\begin{aligned} A = {} & 8\gamma^2(\rho+\gamma)^4 + (\gamma^2+\rho\gamma+\rho-\gamma)^4 - \\ & (\gamma^2+\rho\gamma+\rho-\gamma)^2(B^2 - 6\gamma(\rho+\gamma)^2) - \\ & B\begin{pmatrix} 2(\gamma^2+\rho\gamma+\rho-\gamma)^3 - 6\gamma(\rho+\gamma)^2 \\ (\gamma^2+\rho\gamma+\rho-\gamma) \end{pmatrix}(\rho+\gamma)^4. \end{aligned} \quad (56)$$

Finally, from Eq. (52) along with Eq. (53), Eq. (54), Eq. (55) and Eq. (56) if $\Gamma_2 \neq 0$ as $(\gamma, h, \rho) \in F|_{\mathcal{T}}$ then around $\mathcal{T}$, system Eq. (1) undergoes flip bifurcation. Further, if $\mathcal{T} > 0$ (respectively, $\mathcal{T} < 0$) then period-2points bifurcate from $\mathcal{T}$ are stable (respectively unstable).

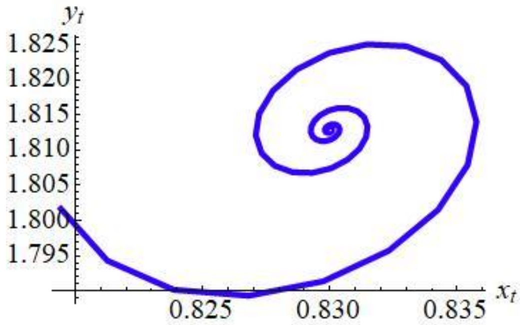
4. Chaos control

In this section, feedback control method is utilized to stabilize chaos at the state of unstable trajectories by adding control forces u_t to system (Elaydi 1996, Lynch 2007):

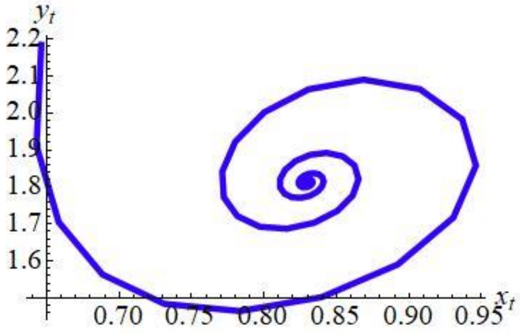
$$\begin{aligned} x_{t+1} &= (1-h)x_t + h\rho + h\frac{x_t^2}{y_t} + u_t, \\ y_{t+1} &= (1-h\,\gamma)y_t + hx_t^2, \end{aligned} \quad (57)$$



(a) $h = 0.3$ with $(0.74, 0.95)$



(b) $h = 0.56$ with $(0.74, 0.95)$



(c) $h = 0.66$ with $(0.74, 0.95)$

Fig.1 Stable focus of the system Eq. (1)

where $u_t = -k_1(x_t - x) - k_2(y_t - y)$, k_1, k_2 denotes feedback gains and $x = \rho + \gamma$, $y = \frac{(\rho+\gamma)^2}{\gamma}$. Now, for control system Eq. (57), the variational matrix $\Omega^C|_{E_{xy(x,y)}}$ takes the following form:

$$\Omega^C|_{E_{xy(x,y)}} = \begin{pmatrix} l_{11} - k_1 & l_{12} - k_2 \\ l_{21} & l_{22} \end{pmatrix} \tag{58}$$

where

$$l_{11} = 1 - h + \frac{2h\,\gamma}{\rho + \gamma}, l_{12} = -\frac{h\gamma^2}{(\rho+\gamma)^2}, \quad l_{21} = 2h(\rho + \gamma), l_{22} = 1 - h\,\gamma. \tag{59}$$

Now, if roots of the characteristic equation of $\Omega^C|_{E_{xy(x,y)}}$ are $\lambda_{1,2}$ then

$$\lambda_1 + \lambda_2 = l_{11} + l_{22} - k_1, \tag{60}$$

and

$$\lambda_1\lambda_2 = l_{22}(l_{11} - k_1) - l_{21}(l_{12} - k_2) \tag{61}$$

Now, it is noted here that marginal stability determine from the restrictions $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$, which give the fact that $|\lambda_{1,2}| < 1$. If $\lambda_1\lambda_2 = 1$ then from Eq. (61), one gets:

$$L_1: (1 - h\gamma)\big(\rho + \gamma - \rho h + h\gamma - k_1(\rho + \gamma)\big) + 2h^2\gamma^2 + 2h(\rho+\gamma)^2 k_2 - (\rho + \gamma) = 0. \tag{62}$$

If $\lambda_1 = 1$ then from Eq. (60) and Eq. (61), one gets:

$$L_2: h\gamma k_1 + (2h\rho + 2h\gamma)k_2 + h^2\gamma = 0 \tag{63}$$

Finally, if $\lambda_1 = -1$ then from Eq. (60) and Eq. (61), one gets:

$$L_3: (2 - h\gamma)k_1 - (2h\rho + 2h\gamma)k_2 - 4 + 2h\gamma + 2h - h^2\gamma - \frac{4h\gamma}{\rho + \gamma} = 0. \tag{64}$$

Therefore, from Eq. (62), Eq. (63) and Eq. (64) lines L_1, L_2 and L_3 in $(k_1, k_2)-$ plane gives the triangular region, which further give the fact that $|\lambda_{1,2}| < 1$.

5. Numerical simulations

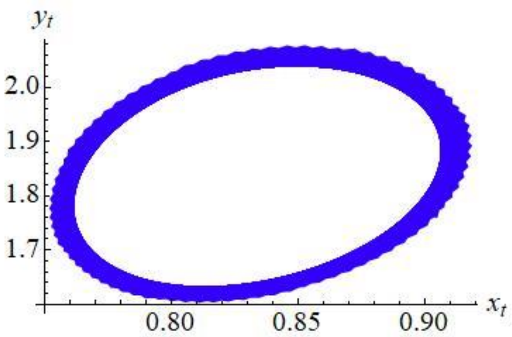
Case 1: Let $\gamma = 0.38$ and $\rho = 0.45 > \frac{\gamma - \gamma^2}{1+\gamma} = 0.1707246.3768115945$ then from Eq. (14), one gets $h = 1.22194039.31515533$. So, Lemma 2.1 implies that $\mathcal{T}$ of the system Eq. (1) is a stable focus if $h < 1.2219403931515533$, exchange stability if $h = 1.2219403931515533$ and meanwhile, it is an unstable focus if $h > 1.2219403931515533$. In order to show this fact deeply, if $h = 0.3 < 1.2219403931515533$ then Fig. 1a shows that $\mathcal{T} = (0.83, 1.8128947368421056)$ is a stable focus. Moreover, Fig. 1b-1c also shows that $\mathcal{T} = (0.83, 1.8128947368421056)$ is also stable focus if $h = 0.56, 0.66 < 1.221940393151553$. On the other hand, if $h = 1.23 > 1.2219403931515533$ then Fig. 2a shows that $\mathcal{T} = (0.83, 1.8128947368421056)$ of Eq. (1) changes behavior, that is, an unstable focus and as a consequence stable curves appear. Now, numerically we have to show that if $h = 1.23 > 1.2219403931515533$ then system Eq. (1) undergoes supercritical N-S bifurcation, that is, from Eq. (35) the discriminator quantity $\chi < 0$. So, if $h = 1.23$ then from Eq. (23), one gets $\left.\frac{d|\lambda_{1,2}|}{d\epsilon}\right|_{\epsilon=0} = 0.2321686746987952 > 0$. Moreover, from Eq. (21) and Eq. (38), one gets:

$$\lambda_{1,2} = 0.7144325301204819 \pm 0.7023910735100571\iota, \tag{65}$$

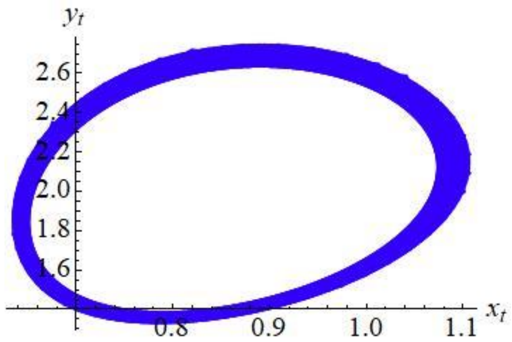
and

$$\begin{aligned} \rho_{02} &= 0.028862409717360205 + 0.032343341323668014\iota \\ \rho_{11} &= -0.1761665741440789 - 0.012595339001123307\iota \\ \rho_{20} &= 0.028862409717360205 - 0.11538819040106152\iota \\ \rho_{21} &= -0.06475013690933859 - 0.03520232314018237\iota. \end{aligned} \tag{66}$$

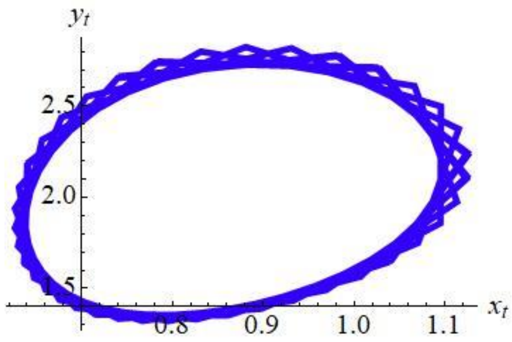
Using Eq. (65) and Eq. (66) in Eq. (35), one gets $\chi = -0.048819945164282855 < 0$, which confirm that our obtained results are mathematically correct and hence, system



(a) h = 1.23 with (0.74,0.95)



(b) h = 1.29 with (0.74,0.95)



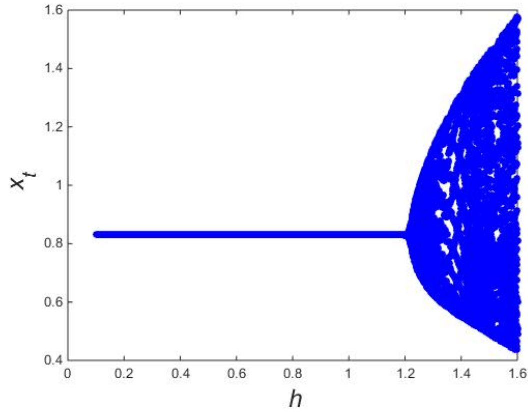
(c) h = 1.3 with (0.74,0.95)

Fig.2 Stable closed curve of the system Eq. (1)

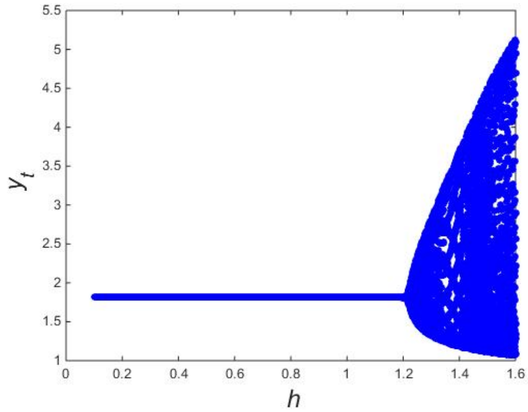
Eq. (1) undergoes a supercritical N-S bifurcation. Similarly, for others value $h = 1.29, 1.3 > 1.2219403931515533$ it is also from Fig. 2b-2c that stable curves appears and therefore, system Eq. (1) undergoes a supercritical N-S bifurcation, since $\chi = -0.050234587439895166, -0.05042921333662094 < 0$ respectively. Finally, bifurcation diagrams along with corresponding maximum Lyapunov exponents are drawn in Fig. 3.

Case 2: If $\gamma = 1.6$ and $\rho = 50.1$ then from Eq. (15), one gets $0 < h < min\{1.7143628069310204, 1.458267753997413\}$. From theoretical discussion, $\mathcal{T} = (51.7, 1670.55625)$ of the system Eq. (1) is a stable node if $0 < h < min\{1.7143628069310204, 1.458267753997413\}$, and by Eq. (16) it is an unstable node if $h > max\{1.7143628069310204, 1.458267753997413\}$

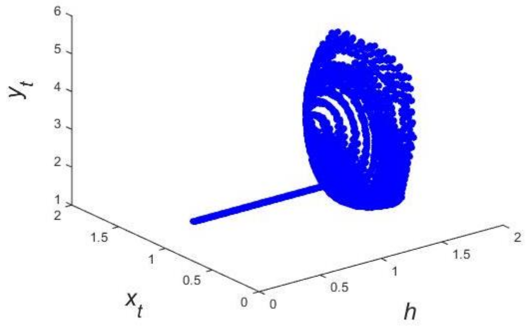
Moreover, from Eq. (17) if $h = 1.7143628069310204$ then system Eq. (1) changes stability and infact model undergoes flip bifurcation. So, in this case, flip bifurcation



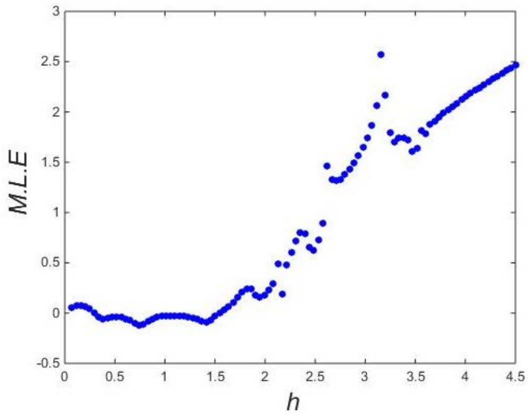
(a)



(b)



(c)



(d)

Fig. 3 N-S bifurcation diagrams along with corresponding maximum Lyapunov exponents of the system Eq. (1) with $h \in [0.1, 1.6]$ and initial value (0.74,0.95)

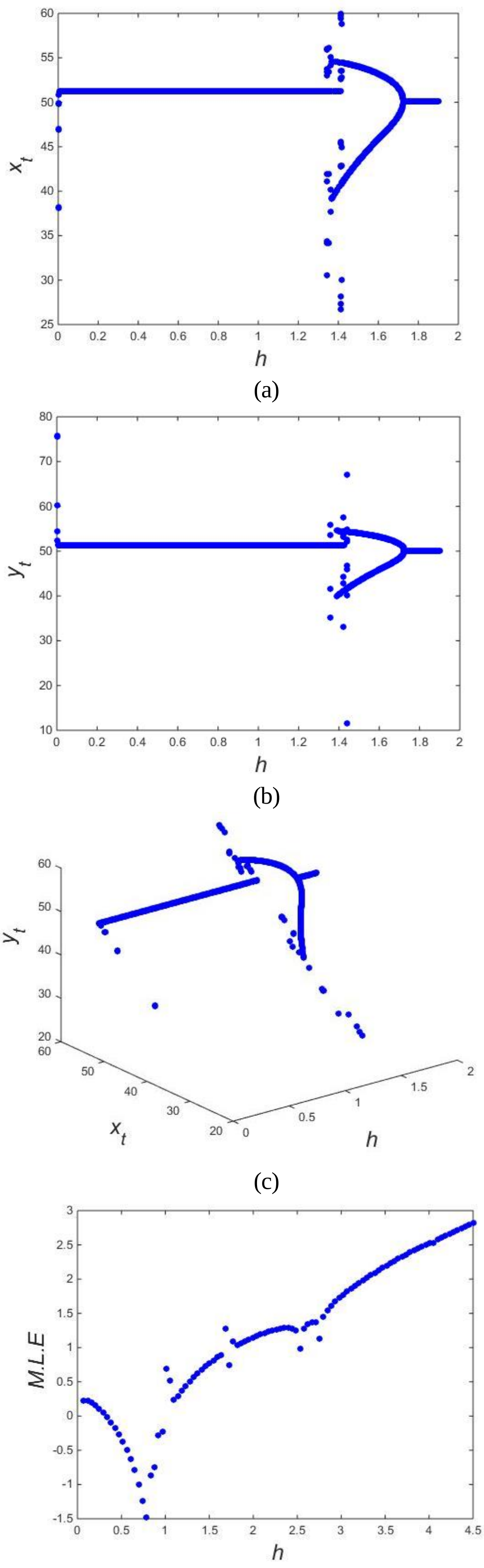


Fig. 4 Flip bifurcation diagrams along with corresponding maximum Lyapunov exponents of the system Eq. (1) with $h \in [0.001, 1.9]$ and initial value $(10.9, 20.9)$

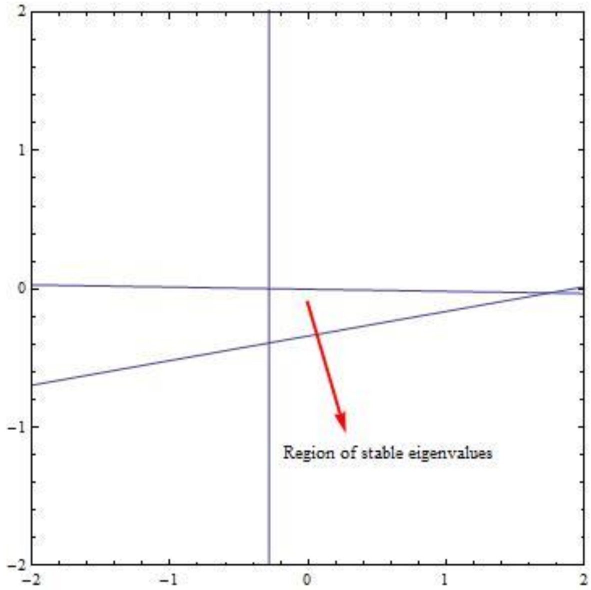


Fig. 5 Region of stability where $|\lambda_{1,2}| < 1$

diagram along with maximum Lyapunov exponents are plotted and presented in Fig. 4.

Case 3: Now, we will prove the validity of obtained results in Section 4 For instance, if $h = 1.9$, $\gamma = 1.6$ and $\rho = 50.1$ then from Eqs. (62)-(64), one gets:

$$L_1: -43.428k_1 + 534.5780000000001k_2 - 12.294800000000002 = 0, \tag{67}$$

$$L_2: 0.16k_1 + 10.340000000000002k_2 + 0.016000000000000004 = 0, \tag{68}$$

and

$$L_3: 1.8399999999999999k_1 - 10.340000000000002k_2 - 3.5083791102514503 = 0. \tag{69}$$

Hence, lines that are presented in Eq. (67), Eq. (68) and Eq. (69) determine triangular region that gives $|\lambda_{1,2}| < 1$ (see Fig. 5).

6. Conclusions

The work is about the topological classifications around interior equilibrium solution, bifurcations and chaos control in the discrete activator-inhibitor system Eq. (1). We studied the local stability around $\mathcal{T}$ of the system Eq. (1) by linear stability theory. We have also studied existence of possible bifurcations around $\mathcal{T}$, and proved that around $\mathcal{T}$ system Eq. (1) undergo both Neimark-Sacker and flip bifurcations. Further, state feedback control method is utilized in order to stabilize chaos exists in the system Eq. (1). Finally, obtained results verified numerically.

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References

Agiza, H.N., Elabbasy, E.M., El-Metwally, H. and Elsadany, A.A. (2009), "Chaotic dynamics of a discrete prey–predator model with Holling type II", *Nonlinear Anal. Real World Appl.*, **10**, 116-129. https://doi.org/10.1016/j.nonrwa.2007.08.029

Ahmed, R.A., Al-Maliki, A.F. and Faleh, N.M. (2020), "Dynamic characteristics of multi-phase crystalline porous shells with using strain gradient elasticity", *Adv. Nano Res.*, **8**(2), 157-167. https://doi.org/10.12989/anr.2020.8.2.157

Al-Basyouni, K.S. and Khan, A.Q. (2020), "Discrete-time predator-prey model with bifurcations and chaos'", *Math. Probl. Eng.*, **2020**, 1-14. https://doi.org/10.1155/2020/8845926

Alqhtani, M., Owolabi, K.M., Saad, K.M. and Pindza, E. (2023), "Spatiotemporal chaos in spatially extended fractional dynamical systems", *Commun. Nonlinear Sci. Numer. Simul.*, **119**, 107118. https://doi.org/10.1016/j.cnsns.2023.107118

Benmansour, D.L., Kaci, A., Bousahla, A.A., Heireche, H., Tounsi, A., Alwabli, A.S., Al-ghmady, K. and Mahmoud, S.R. (2019), "The nano scale bending and dynamic properties of isolated protein microtubules based on modified strain gradient theory", *Adv. Nano Res.*, **7**(6), 443-457. https://doi.org/10.12989/anr.2019.7.6.443

Chakraborty, P., Ghosh, U. and Sarkar, S. (2020), "Stability and bifurcation analysis of a discrete prey-predator model with square-root functional response and optimal harvesting", *J. Biol. Syst.*, **28**, 91-110. https://doi.org/10.1142/S0218339020500047

Ebrahimi, F., Barati, M.R. and Mahesh, V. (2019), "Dynamic modeling of smart magneto-electro-elastic curved nanobeams", *Adv. Nano Res.*, **7**(3), 145-155. http://doi.org/10.12989/anr.2019.7.3.145

Ebrahimi, F., Kokaba, M., Shaghaghi, G. and Selvamani, R. (2020), "Dynamic characteristics of hygro-magneto-thermo-electrical nanobeam with non-ideal boundary conditions", *Adv. Nano Res.*, **8**(2), 169-182. https://doi.org/10.12989/anr.2020.8.2.169

Edelstein-Keshet, L. (2005), *Mathematical Models in Biology. Society for Industrial and Applied Mathematics*, SIAM Publications, Pennsylvania, U.S.A.

Elaydi, S.N. (1996), *An Introduction to Difference Equations*, Springer-Verlag, New York, U.S.A.

Gonpot, P., Collet, J.S.A.J. and Sookia, N.U. (2008), "Gierer-Meinhardt model: bifurcation analysis and pattern formation", *Trends Appl. Sci. Res.*, **3**, 115-128. https://doi.org/10.3923/tasr.2008.115.128

Gu, L., Gong, P. and Wang, H. (2020), "Hopf bifurcation and turing instability analysis for the Gierer-Meinhardt model of the depletion type", *Discr. Dyn. Nat. Soc.*, **2020**, 5293748. https://doi.org/10.1155/2020/5293748

Guckenheimer, J. and Holmes, P. (1983), *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, Springer-Verlag, New York, U.S.A.

Khan, A.Q. and Alayachi, H.S. (2022), "Bifurcation and chaos in a phytoplankton-zooplankton model with holling type-II response and toxicity", *Int. J. Bifurcat. Chaos*, **32**, 2250176. https://doi.org/10.1142/S0218127422501760

Khan, A., Ma, J. and Xiao, D. (2016), "Bifurcations of a two-dimensional discrete time plant-herbivore system", *Commun. Nonlinear Sci. Numer. Simul.*, **39**, 185-198. https://doi.org/10.1016/j.cnsns.2016.02.037

Kuznetsov, Y.A. (2004), *Elements of Applied Bifurcation Theory*, Springer-Verlag New York, U.S.A.

Liu, X. and Xiao, D. (2007), "Complex dynamic behaviors of a discrete-time predator-prey system", *Chaos Solit. Fract.*, **32**, 80-94. https://doi.org/10.1016/j.chaos.2005.10.081

Liu, W. and Cai, D. (2019), "Bifurcation, chaos analysis and control in a discrete-time predator–prey system", *Adv. Differ. Eq.*, **2019**, 11. https://doi.org/10.1186/s13662-019-1950-6

Lynch, S. (2007), *Dynamical Systems with Applications Using Mathematica*, Mass, Boston, U.S.A. https://doi.org/10.1007/978-0-8176-4586-1

Owolabi, K.M. and Jain, S. (2023), "Spatial patterns through diffusion-driven instability in modified predator–prey models with chaotic behaviors", *Chaos Solit. Fract.*, **174**, 113839. https://doi.org/10.1016/j.chaos.2023.113839

Owolabi, K.M. and Pindza, E. (2022a), "Mathematical analysis and numerical simulation of time-fractional host-parasitoid system with Caputo operator", *J. Appl. Comput. Mech.*, **21**(1), 79-90. https://doi.org/10.17512/jamcm.2022.1.07

Owolabi, K.M. and Pindza, E. (2022b), "Numerical simulation of chaotic maps with the new generalized Caputo-type fractional-order operator", *Results Phys.*, **38**, 105563. https://doi.org/10.1016/j.rinp.2022.105563

Owolabi, K.M., Agarwal, R.P., Pindza, E., Bernstein, S. and Osman, M.S. (2023), "Complex Turing patterns in chaotic dynamics of autocatalytic reactions with the Caputo fractional derivative", *Neural Comput. Appl.*, **35**(15), 11309-11335. https://doi.org/10.1007/s00521-023-08298-2

Owolabi, K.M., Pindza, E., Karaagac, B. and Oguz, G. (2024), "Laplace transform-homotopy perturbation method for fractional time diffusive predator–prey models in ecology", *Part. Differ. Eq. Appl. Math.*, **9**, 100607. https://doi.org/10.1016/j.padiff.2023.100607

Prytula, Z. (2015), "Mathematical modelling of nonlinear dynamics in activator-inhibitor systems with superdiffusion", Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the National Academy of Sciences of Ukraine, **826**, 230-237.

Rana, S.M.S. (2017), "Chaotic dynamics and control of discrete ratio-dependent predator-prey system", *Discr. Dyn. Nat. Soc.*, **2017**, 1-13. https://doi.org/10.1155/2017/4537450

Sun, A., Wu, R. and Chen, M. (2021), "Turing-Hopf bifurcation analysis in a diffusive Gierer-Meinhardt model", *AIMS Math.*, **6**, 1920-1942. https://doi.org/10.3934/math.2021117

Tunc, C. (2010), "A note on boundedness of solutions to a class of non-autonomous differential equations of second order", *Appl. Anal. Discrete Math.*, **4**, 361-372. https://doi.org/10.2298/AADM100601026T

Tunc, C. and Tunc, E. (2007), "On the asymptotic behavior of solutions of certain second-order differential equations", *J. Frankl. Inst.*, **344**, 391-398. https://doi.org/10.1016/j.jfranklin.2006.02.011

Tunc, C. and Tunc, O. (2016a), "A note on the stability and boundedness of solutions to non-linear differential systems of second order", *J. Assoc. Arab Univ. Basic Appl. Sci.*, **24**, 169-175. https://doi.org/10.1016/j.jaubas.2016.12.004

Tunc, C. and Tunc, O. (2016b), "On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order", *J. Adv. Res.*, **7**, 165-168. https://doi.org/10.1016/j.jare.2015.04.005

Wang, J., Li, Y. and Hou, X. (2018), "Supercritical Hopf bifurcation and Turing patterns for an activator and inhibitor model with different sources", *Adv. Differ. Eq.*, **2018**, 1-23. https://doi.org/10.1186/s13662-018-1697-5

Yochelis, A. (2021), "The nonlinear initiation of side-branching by activator-inhibitor-substrate (Turing) morphogenesis", *Chaos*, **31**, 051102. https://doi.org/10.1063/5.0050630

Zhu, M. (2018), "Activator-Inhibitor model for seashell pattern formation", *Semantic Scholar*, Corpus ID 231685255. https://api.semanticscholar.org/CorpusID:231685255

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