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Static bending study of AFG nanobeam using local stress-and strain-driven nonlocal integral models

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Abstract. In this paper, the problem of static bending of axially functionally graded (AFG) nanobeam is formulated with the local stress(Lσ)- and strain-driven(εD) two-phase local/nonlocal integral models (TPNIMs). The novelty of the present study aims to compare the size-effects of nonlocal integral models on bending deflections of AFG Euler–Bernoulli nano-beams. The integral relation between strain and nonlocal stress components based on two types nonlocal integral models is transformed unitedly and equivalently into differential form with constitutive boundary conditions. Purely LσD- and εD-NIMs would lead to ill-posed mathematical formulation, and Purely εD- and LσD-nonlocal differential models (NDM) may result in inconsistent size-dependent bending responses. The general differential quadrature method is applied to obtain the numerical results for bending deflection and moment of AFG nanobeam subjected to different boundary and loading conditions. The influence of AFG index, nonlocal models, and nonlocal parameters on the bending deflections of AFG Euler–Bernoulli nanobeams is investigated numerically. A consistent softening effects can be obtained for both LσD- and εD-TPNIMs. The results from current work may provide useful guidelines for designing and optimizing AFG Euler-Bernoulli beam based nano instruments.

Keywords: size-effect; axially functionally graded nanobeam; nonlocal integral model; general differential quadrature method

1. Introduction

Functionally graded materials (FGMs) have attracted to lots of attention of scientific and engineering communities due to the superior mechanical and thermomechanical properties since it was proposed by Japanese material scientists in 1980s (Koizumi 1997). Meanwhile, with the development of science and technology, nanoscale FGMs are widely applied in micro- and nano-electro-mechanical-systems, such as electrically actuated devices (Zhang and Fu 2012), atomic force microscope cantilevers (Rahaeifard *et al.* 2009), capacitive sensors (Shabana *et al.* 2021), actuators (Komijani *et al.* 2014), switches (Gorgani *et al.* 2019) and so on. A distinct feature of nanoscale structures is size-dependent behaviors compared with macroscale structures. The understanding of size-dependent behavior of nanostructures is extreme important in practical applications. Molecular dynamics simulations are hard to be applied for investigation nanoscale structures due to extreme high computational requirement. Meanwhile, it is not easy to prepare specimens satisfying with specific dimensional requirement and implement tests at a micro-/nano-scale. Therefore, several high-order continuum models such as strain gradient model (including couple stress model), nonlocal model and so on are developed to capture the size-dependent behaviors of nanoscale structures (Ansari *et al.* 2021, Vaccaro *et al.* 2021, Zhang and Qing 2020).

Among the high-order continuum models, the nonlocal model initiated by Kröner (1967) and developed by Eringen (Eringen 1972, Eringen and Edelen 1972) has been widely used to study the static bending, elastic buckling as well as free and forced vibration of beams, plates, shells and so on. The studies limited to bending analysis of FG nonlocal nanobeams can be classified into two categories:

FG along thickness direction: Based on nonlocal differential model (NDM), Eltaher *et al.* (2013) developed finite element model to study the static bending and elastic buckling of FG Euler-Bernoulli beam. Simsek and Yurtcu (2013) applied Navier solution technique to study the static bending and elastic buckling of FG nonlocal Timoshenko and Euler-Bernoulli nanobeams subjected to simply-supported (SS) boundary conditions. Arefi and Zenkour (2016) applied three-unknown shear and normal deformations nonlocal beam theory to study static bending of a nonlocal sandwich nanobeam with a functionally graded core material and two functionally piezomagnetic layers in magneto-thermo-electric environment. Rahmani *et al.* (2017) studied the effects of length scale parameter, the nonhomogeneous index, and geometric characteristics of nonlocal Timoshenko circular beam with transversely FGM. Based on nonlocal strain gradient differential model (NSGDM) proposed by Lim *et al.* (2015), Li and Hu (2016) developed Navier solution technique to study size-dependent nonlinear bending and free vibration of FG Euler-Bernoulli and Timoshenko naobeams subjected to SS boundary condition. Allam and Radwan (2019) studied the influence of elastic foundation on the bending, buckling, and vibration of a refined three-variable viscoelastic FG curved nanobeam. Simsek (2019) applied Navier solution technique to bending, buckling, free and forced vibration of

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FG Euler-Bernoulli nanobeams with NSGDM.

FG along axial direction: Based on NMD, Nejad and Hadi (2016) modelled the bending behavior of bi-directional FG Euler–Bernoulli nanobeams, Yang *et al*. (2018) modelled the nonlinear bending, buckling and free vibration of bi-directional FG Euler-Bernoulli nanobeams, Sahmani and Safaei (2020) modelled the bi-directional functionally graded micro/nano-beams based on exponential shear deformation beam theory. Li *et al*. (2017) applied NSGDM to formulate the bending, buckling and vibration of axially FG(AFG) Euler-Bernoulli nanobeams. GDQM is widely applied to obtain the numerical results to investigate the influences of power-law variation and material length scale parameter. Rajasekaran and Khaniki(2019) applied NSGDM to model axially functionally graded nonuniform Euler-Bernoulli nanobeams and developed finite element model to study the size-dependent static and dynamic responses.

It should noticed that the mentioned-above studies are based on nonlocal models in differential models. However, It has been reported that NDM results in inconsistent size-dependent tensile response of nanobar under tensile load (Benvenuti and Simone 2013) as well as bending of cantilever beams subjected to point force (Challamel and Wang 2008, Fernandez-Saez *et al*. 2016, Li *et al*. 2015) and clamped-clamped curved beams under uniformly loads (Zhang *et al*. 2019, 2020b). Meanwhile, the results of Li *et al*. (2021), Zhang and Qing (2021) and Bian and Qing (2021) show that NSGDM would lead to inconsistent size-dependent bending responses for straight and curved Euler-Bernoulli beams as well as Timoshenko beam, respectively. Therefore, nonlocal integral model should be applied to model the size-effects of nano-structures. Romano *et al*. (2017) further found that two-phase local/nonlocal integral model should be adopted, since nonlocal integral model would lead to ill-posed mathematical formulation. Koutsoumaris *et al*. (2017) investigated the static responses of Euler-Bernoulli beam within the framework of the modified kernel and the kernel corresponding to the two phase nonlocal integral (TPNI) model. It showed that the two phase nonlocal integral model do not give rise to paradoxes as has been observed in other publications presented in the literature. Later, Koutsoumaris and Eptaimeros(2021) employed the well-posed nonlocal integral model to study the finite element static analysis of nanobeams resting on a Pasternak-type elastic foundation.

To the best of the authors' knowledge, there are no studies performed in the literature about the static bending of AFG nanobeam based on two-phase local/nonlocal integral model. Meanwhile, Batra (2021) pointed out recently that nonlocal differential model is in fact local stress-driven nonlocal model, which is valid only for *homogeneous* materials or *nonhomogeneous* materials independent on length direction, and strain-driven nonlocal model should be used to model AFG structures. However, most researchers commonly used the local stress(L)-driven two-phase local/nonlocal integral models (TPNIMs) to study the mechanical responses of AFG nanobeams. In view of this oversight, the aim of the present study is to investigate size-dependent bending of AFG Euler–Bernoulli nanobeams based on the local stress (L)- and strain-driven (D) two-phase local/nonlocal integral models (TPNIMs). The integral relation between strain and nonlocal stress components based on local stress- and strain-driven two-phase local/nonlocal integral models can be transformed unitedly and equivalently into differential form with constitutive boundary conditions. GDQM is applied to obtain the numerical results, and a compare study of the influence of nonlocal parameters and the AFG index on bending deflection of nanobeams subjected to different boundary and loading conditions is demonstrated with numerical results. Moreover, a comparison is made between the bending deflection of the L - and D-TPNIMs to investigate the influence of nonlocal parameters and AFG index on the distinction of the two models in detail.

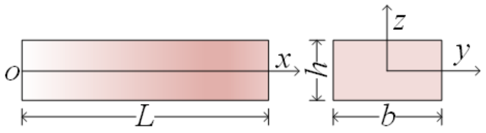

Fig. 1 Scematic diagram of AFG beam

2. Mathematical formulation

2.1 Governing Eqs. of AFG nanobeam

All of the materials (exluding nS) used in the studies came from the area of Polish. Nanosilica was imported from South Korea. The constituting materials of concrete used in the present study are:

Fig. 1 shows a nanobeam with length L, thickness h and width b. It is assumed that material properties are non-uniform and continuously graded in beam axial dimension, which follows exponential law as

$$E = E_0 e^{\beta \frac{x}{L}} \tag{1}$$

in which, β is the AFG index, E_0 is Young's modulus at left end of beam. According to classic local elasticity, the relation between local stress and strain can be expressed as

$$\tau_x = E \varepsilon_x \tag{2}$$

The nonzero strain component based on Euler-Bernoulli beam theory can be calculated as

$$\varepsilon_x = -z \partial_x^2 w \tag{3}$$

Based on Zhang *et al*. (2020a), the differential governing Eq. and standard boundary conditions can be expressed as

$$\partial_x^2 M + q = 0 \tag{4}$$

$$\begin{cases} M\delta(\partial_x w)|_0^L = 0 \\ (\partial_x M + \tilde{V})\delta w\Big|_0^L = 0 \end{cases} \tag{5}$$

where, w is bending deflection, q and $\tilde{V}$ are distributed load and point force at beam ends, and M is bending moment which is defined as

$$M = \int_A z\sigma_x \, dA \tag{6}$$

In order to simplify the expressions, the following nominal variables are introduced

$$\begin{gathered} x = L\eta, w(x) = LW(\eta), \\ \kappa = \lambda L, \widehat{M} = LM/(E_0 I_2) \end{gathered} \tag{7}$$

in which, $I_2 = b\int_{-h/2}^{h/2} z^2\, dz$. The differential governing Eq. and standard boundary conditions in nominal form can be expressed as

$$\widehat{M}'' + \hat{q} = 0 \tag{8}$$

$$\begin{cases} \widehat{M}\delta W'|_0^1 = 0 \\ (\widehat{M}' + \hat{\tilde{V}})\delta W|_0^1 = 0 \end{cases} \tag{9}$$

in which, $Y' = dY/d\eta$ (Y can be W and $\widehat{M}$), and

$$\hat{q} = qL^3/(E_0 I_2), \hat{\tilde{V}} = L^2\tilde{V}/(E_0 I_2) \tag{10}$$

2.2 Two-phase local/nonlocal integral model

Taking into account Eq. (2), the relation between nonlocal stress and strain components based on D-TPNIM (Eringen 1987) can be expressed as

$$\sigma_x = E[\xi\varepsilon_x + \frac{1-\xi}{2\kappa}\int_0^L e^{-\frac{|x-s|}{\kappa}}\varepsilon_x(s)ds] \tag{11}$$

in which, κ is nonlocal length parameter, and ξ is local volume fraction varying from 0 to 1. When $\xi = 0$the above model degenerates into the pure nonlocal integral model, $\xi = 1$ the model will become a classic local theory. Combining Eqs. (1) and (3) with (6) and (11), and taking into account Eqs. (7), one gets

$$-\widehat{M}e^{-\beta\eta} = \xi W'' + \frac{1-\xi}{2\lambda}\int_0^1 e^{-\frac{|\eta-s|}{\lambda}}W''(s)ds \tag{12}$$

According to Wang *et al*. (2016) and Romano *et al*. (2017), integral Eq. (12) can be equivalently transformed into differential form as

$$\begin{gathered} \lambda^2(\widehat{M}'' - 2\beta\widehat{M}' + \beta^2\widehat{M}) - \\ \widehat{M} + e^{\beta\eta}(\lambda^2\xi W''{}''{}'' \end{gathered} \tag{13}$$

with constitutive boundary conditions

$$\begin{cases} \lambda(\widehat{M}'(0) - \beta\widehat{M}(0)) - \widehat{M}(0) + \\ \xi(\lambda W'''(0) - W''(0)) = 0 \\ \lambda(\widehat{M}'(1) - \beta\widehat{M}(1)) + \widehat{M}(1) + \\ \xi e^{\beta}(\lambda W'''(1) + W''(1)) = 0 \end{cases} \tag{14}$$

Similarly, the relation between nonlocal and local stress components based on local stress-driven two-phase local/ nonlocal integral model (L D-TPNIM) (Eringen 1987) can be expressed as

$$\sigma_x = \xi\tau_x + \frac{1-\xi}{2\kappa}\int_0^L e^{-\frac{|x-s|}{\kappa}}\tau_x(s)ds \tag{15}$$

Combining Eqs. (1), (3) and (6) with (15) and taking into the nominal variables (7), one gets

$$-\widehat{M} = \xi e^{\beta\eta}W'' + \frac{1-\xi}{2\lambda}\int_0^1 e^{-\frac{|\eta-s|}{\lambda}}e^{\beta s}W''(s)ds \tag{16}$$

Integral Eq. (16) can be converted into differential form as

$$\lambda^2\widehat{M}''(\eta) - \widehat{M}(\eta) \ +e^{\beta\eta}[\lambda^2\xi(W''{}''''{}^{2}{}'''' \tag{17}$$

with constitutive boundary conditions

$$\begin{cases} \lambda\widehat{M}'(0) - \widehat{M}(0) + \\ \xi[\lambda(W'''(0) + \beta W''(0)) - W''(0)] = 0 \\ \lambda\widehat{M}'(1) + \widehat{M}(1) + \\ \xi e^{\beta}[\lambda(W'''(1) + \beta W''(1)) + W''(1)] = 0 \end{cases} \tag{18}$$

The differential constitutive Eqs. and constitutive boundary conditions for D- and L D-TPNIMs can be unitedly expressed as

$$\begin{gathered} \lambda^2(\widehat{M}'' - 2\Omega_1\widehat{M}' + \Omega_2\widehat{M}) - \widehat{M} + \\ e^{\beta\eta}[\lambda^2\xi(W''{}^{''}_{3\,4}{}^{'''}{}^{''''} \end{gathered} \tag{19}$$

$$\begin{cases} \lambda(\widehat{M}'(0) - \Omega_1\widehat{M}(0)) - \widehat{M}(0) + \\ \xi[\lambda(W'''(0) + \Omega_3 W''(0)) - W''(0)] = 0 \\ \lambda(\widehat{M}'(1) - \Omega_1\widehat{M}(1)) + \widehat{M}(1) + \\ \xi e^{\beta}[\lambda(W'''(1) + \Omega_3 W''(1)) + W''(1)] = 0 \end{cases} \tag{20}$$

in which,

$$\begin{aligned} &[\Omega_1, \Omega_2, \Omega_3, \Omega_4] \\ &= \begin{cases} [\beta, \beta^2, 0, 0] & \text{for } \varepsilon\text{D-TPNIM} \\ [0, 0, \beta, \beta^2] & \text{for L}\sigma\text{D-TPNIM} \end{cases} \end{aligned} \tag{21}$$

Remark I: For two-phase local/nonlocal integral model ($\xi > 0$), one can obtain two differential governing Eqs. (8) and (19) about second-order $\widehat{M}$ and fourth-order W, whose total differential order is six. Meanwhile, there are total six boundary conditions(four standard boundary conditions and two constitutive boundary conditions). That's to say, the total differential order equals to the number of boundary conditions, which indicates the mathematical formulation for static bending of AFG Euler-Bernoulli beam is well-posed.

Remark II: For purely nonlocal integral model($\xi = 0$), combining Eqs. (8) and (19) and after a lengthy but straight simplification, one can obtain a single differential governing Eq.

$$\begin{gathered} W^{(4)} + 2\beta W^{(3)} + \beta^2 W'' - \\ e^{-\beta\eta}[(1 - \lambda^2\Omega_2)\hat{q} + 2\lambda^2\Omega_1\hat{q}' - \lambda^2\hat{q}''] = 0 \end{gathered} \tag{22}$$

Clearly, the governing Eq. is a fourth-order differential equation. Therefore, the number of total boundary conditions is larger than that of differential order. In other words, the mathematical formulation is ill-posed.

Remark III: For nonlocal differential model (purely nonlocal integral model but neglecting the constitutive boundary conditions), the mathematical formulation is well-posed. However, there is no size-dependent response for a AFG clamped-clamped nanobeam under concentrated force at free end because that both governing Eq. and boundary conditions are independent on nonlocal parameter λ. In addition, for L D-NDM, nonlocal parameter plays no role in the static bending of AFG clamped-clamped nanobeam subjected to uniformly and lineally distributed loads as well as AFG clamped-guided nanobeam subjected to uniformly distributed load. In other words, nonlocal differential model would lead to inconsistent size-dependent response.

3. Numerical solution by GDQM

The differential Eqs. can be solved through the general differential quadrature method (GDQM). According to Chebyshev–Gauss–Lobatto rule, the beam domain can be discretized with N grid nodes as

$$\eta_i = [1 - cos(\,(i-1)\pi/(N-1))]/2, \quad i = 1,2,\dots,N \tag{23}$$

Following the standard GDQM procedure, the second-order and fourth-order differential variables can be respectively approximated as

$$\hat{M}(\eta) = \sum_{i=1}^{N} L_i(\eta)\,\hat{M}(\eta_i) \tag{24}$$

$$W(\eta) = \sum_{i=1}^{N} \varphi_i(\eta)\,W(\eta_i) + \tilde{\varphi}_1(\eta)W'(\eta_1) + \tilde{\varphi}_N(\eta)W'(\eta_N) \tag{25}$$

in which, $L_i(\eta) = \prod_{k=1,\,k\neq i}^{N} \frac{\eta-\eta_k}{\eta_i-\eta_k}$ is the Lagrange interpolation function, and

$$\tilde{\varphi}_j(\eta) = L_j(\eta)(\eta-\eta_j)(\eta-\eta_{N-j+1})/(\eta_j-\eta_{N-j+1})^2$$

$$\varphi_i(\eta) = \begin{cases} \dfrac{\eta-\eta_{N-i+1}}{\eta_i-\eta_{N-i+1}}L_i(\eta) - \left(L_i'(\eta_i) + \dfrac{1}{\eta_i-\eta_{N-i+1}}\right)\tilde{\varphi}_i(\eta) & (i = J) \\ (\eta-\eta_1)(\eta-\eta_N)L_i(\eta)/[(\eta_i-\eta_1)(\eta_i-\eta_N)] & \text{(others)} \end{cases} \tag{26}$$

where, $J = 1, N$.

Performing derivative respect to η on Eqs. (24) and (25), one obtains

$$\hat{M}^{(i)} = \boldsymbol{X}^{(i)}\boldsymbol{\Delta}^{\hat{M}} \tag{27}$$

$$W^{(i)} = \boldsymbol{Y}^{(i)}\boldsymbol{\Delta}^{W} \tag{28}$$

in which, $\boldsymbol{X}^{(i)}$ and $\boldsymbol{Y}^{(i)}$ are the weighting coefficients of the i-th-order derivative that are defined explicitly in (Bellman *et al*. 1972, Wu and Liu 2001), and

$$\boldsymbol{\Delta}^{\hat{M}} = [\hat{M}(\eta_1)\;\hat{M}(\eta_2)\cdots\hat{M}(\eta_N)]^T$$
$$\boldsymbol{\Delta}^{W} = \begin{bmatrix} W(\eta_1)\;W(\eta_2)\cdots W(\eta_N) \\ W'(\eta_1)\;W'(\eta_N) \end{bmatrix}^T \tag{29}$$

The differential governing Eqs. (8) and (19) can be respectively discreted as

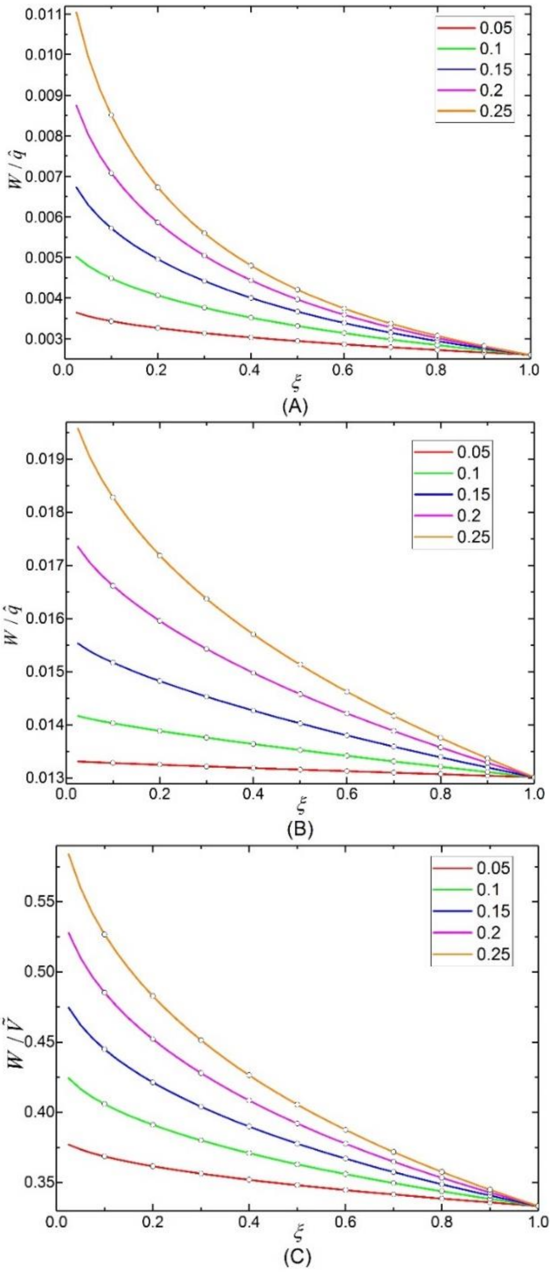

Fig. 1 The effect of λ on the curves of nominal deflections versus ξ for homogeneous (A) CCU and (B) SSU and (C) CFC nanobeams (solid lines and scatters indicate the results from current study and Wang *et al.* (2016), respectively)

$$\sum_{k=1}^{N} X_{jk}^{(2)}\Delta_k^{\hat{M}} + \hat{q}(\eta_j) = 0$$
$$\sum_{k=1}^{N}[\lambda^2(X_{jk}^{(2)} - 2\Omega_1X_{jk}^{(1)} + \Omega_2X_{jk}^{(0)}) - X_{jk}^{(0)}]\,\Delta_k^{\hat{M}} + e^{\beta\eta_j}\sum_{k=1}^{N+2}\begin{bmatrix}\lambda^2\xi(Y_{jk}^{(4)} + 2\Omega_3Y_{jk}^{(3)} \\ +\Omega_4Y_{jk}^{(2)}) - Y_{jk}^{(2)}\end{bmatrix}\Delta_k^{W} = 0 \tag{30}$$

in which, $j = 2,3,\dots,N-2$.

The constitutive boundary conditions (Eq. (20)) can be approximately expressed as

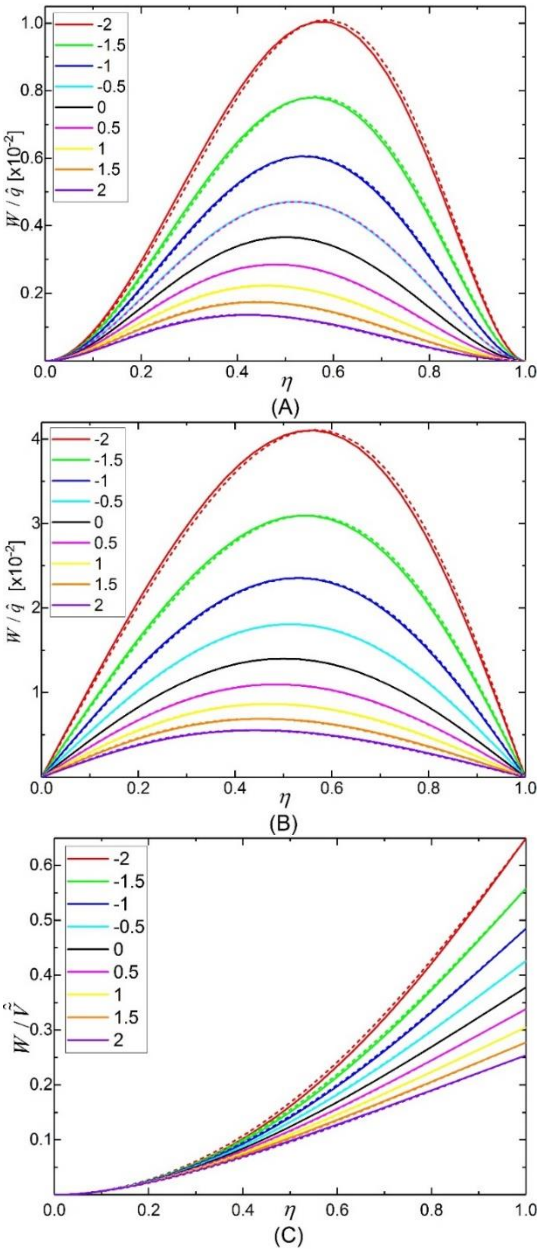

Fig. 2 The influence of β on the bending deflections of (A) CCU, (B) SSU and (C) CFC AFG-nanobeams for $\lambda = 0.15$ and $\xi = 0.5$

$$\begin{cases} \sum_{k=1}^{N}[\lambda(X_{1k}^{(1)} - \Omega_1 X_{1k}^{(0)}) - X_{1k}^{(0)}]\,\Delta_k^{\widehat{M}} + \\ \xi\sum_{k=1}^{N+2}[\lambda(Y_{1k}^{(3)} + \Omega_3 Y_{1k}^{(2)}) - Y_{1k}^{(2)}]\,\Delta_k^{W} = 0 \\ \sum_{k=1}^{N}[\lambda(X_{Nk}^{(1)} - \Omega_1 X_{Nk}^{(0)}) + X_{Nk}^{(0)}]\,\Delta_k^{\widehat{M}} + \\ e^{\beta}\xi\sum_{k=1}^{N+2}[\lambda(Y_{Nk}^{(3)} + \Omega_3 Y_{Nk}^{(2)}) + Y_{Nk}^{(2)}]\,\Delta_k^{W} = 0 \end{cases} \tag{31}$$

Similarly, the standard boundary conditions can be also discreted as following:

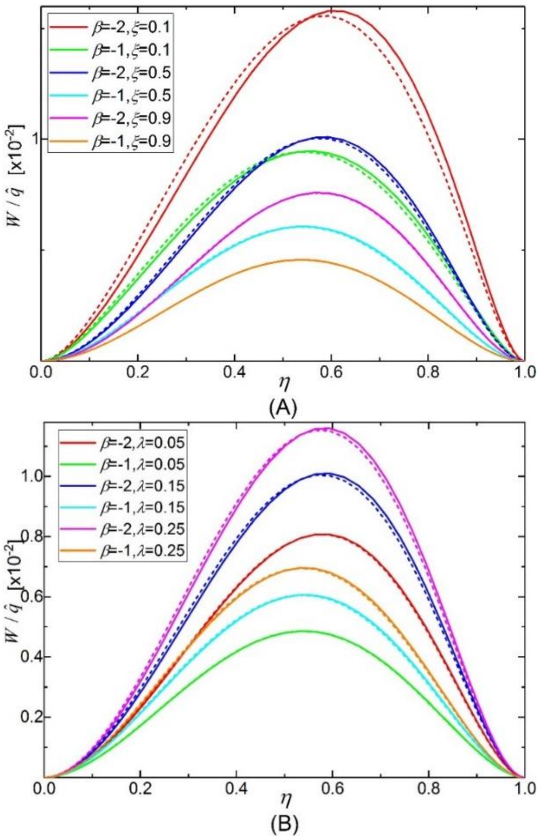

Fig. 3 The influence of (A) ξ and β for $\lambda = 0.15$ and (B) λ and β for $\xi = 0.5$ on bending deflection of CCU AFG-nanobeams

- Clamped end (C)

$$W(\eta_J) = W'(\eta_J) = 0 \tag{32}$$

- Simply supported end (S)

$$W(\eta_J) = \widehat{M}(\eta_J) = 0 \tag{33}$$

- Free end (F)

$$\widehat{M}(\eta_J) = \sum_{k=1}^{N} X_{Jk}^{(1)}\Delta_k^{\widehat{M}} + \widehat{\hat{V}}(\eta_J) = 0 \tag{34}$$

Clearly, one would obtain a set of $2N + 2$ linear Eqs. about $\Delta_k^{\widehat{M}}$ and Δ_k^{W} for static bending of AFG nonlocal Euler-Bernoulli nanobeam under different boundary and loading conditions.

4. Numerical results and discussions

In this section, the influence of nonlocal parameters and the AFG index on the static bending of AFG nonlocal Euler-Bernoulli nanobeam is investigated numerically for L=20h. Numerical results show that D- and L D-TPNIMs lead to same bending response for homogeneous nonlocal Euler-Bernoulli nanobeam. Fig. 2 shows the influence of nonlocal parameters on the nominal maximum bending deflections of homogeneous CCU, SSU and CFC nanobeams, in which CCU and SSU indicate clamped-clamped and simply-

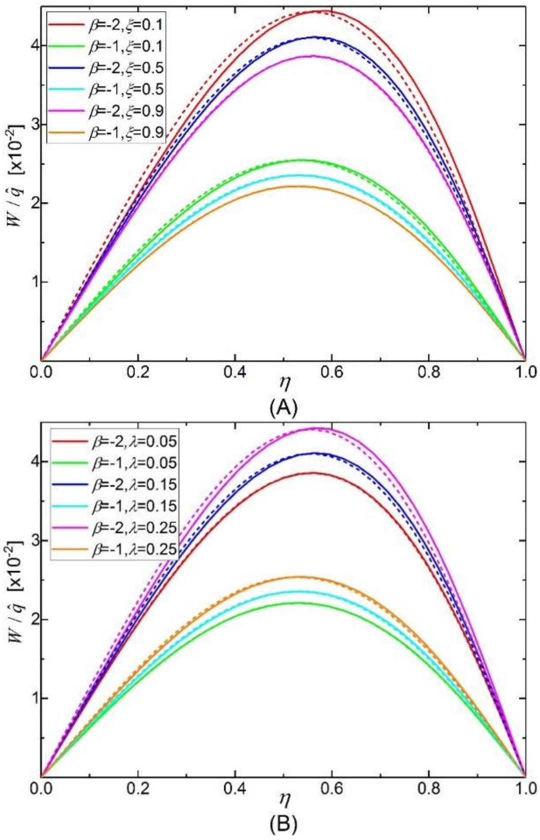

Fig. 4 The influence of (A) ξ and β for $\lambda = 0.15$ and (B) λ and β for $\xi = 0.5$ on bending deflection of SSU AFG-nanobeams

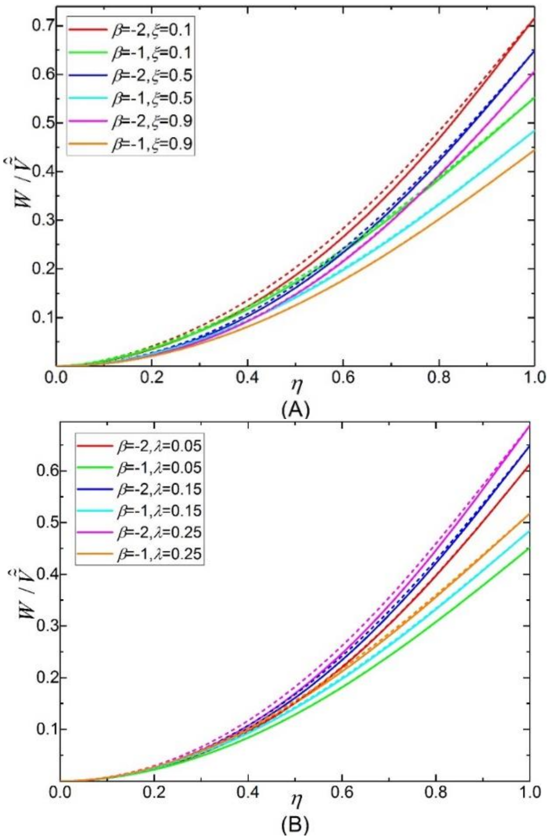

Fig. 5 The influence of (A) ξ and β for $\lambda = 0.15$ and (B) λ and β for $\xi = 0.5$ on bending deflection of CFC AFG-nanobeams

supported nanobeams under uniformly distributed loads, and CFC indicates clamped-free nanobeam under concentrated force at free end. Meanwhile, the result from Wang *et al*. (2016) are plotted as scatters in Fig. 2 for comparison. It can be seen from Fig. 2 that a good agreement is obtained.

Fig. 3 illustrates the influence of β on the curves between nominal bending deflections versus position of CCU, SSU and CFC AFG-nanobeams for $\lambda = 0.15$and $\xi = 0.5$, in which and hereafter solid and dash lines represent data for D- and L D-TPNIMs, respectively. It clearly shows that maximum bending deflections increase with the decrease of β because the stiffness of nanobeam increases with the increase of β. Meanwhile, for CCU, SSU and CFC nanobeams, the distinction of bending deflections based D- and L D-TPNIMs increases with the decrease of β for $\beta < 0$and the increase of β for $\beta > 0$. Fig. 3(A-B) show that, for CCU and SSU AFG-nanobeams, bending deflections in left regions based on D-TPNIM is larger and less than those based on L D-TPNIM for $\beta < 0$ and $\beta > 0$, respectively. However, bending deflections of CCU and SSU AFG-nanobeams in right regions based on D-TPNIM is less and larger than those based on L D-TPNIM for $\beta < 0$ and $\beta > 0$, respectively. The positions of maximum deflections of CCU and SSU nanobeams are $\eta < 0.5$ and $\eta > 0.5$ for $\beta > 0$ and $\beta < 0$, respectively. It is interesting to find from Fig. 3(C) that the bending deflections at free end of CFC nanobeams based on D- and L D-TPNIMs are same for different β, through bending deflections between endpoints based on D-TPNIM are larger and less than those based on L D-TPNIM for $\beta > 0$ and $\beta < 0$, respectively.

Figs. 4-6 illustrates the influence of λ and ξ on the curves between nominal bending deflections versus position of CCU, SSU and CFC AFG-nanobeams for different$P$. It can be seen from Figs. 4-6 that the distinction of bending deflections based D- and L D-TPNIMs increases with the decrease of ξ for $\lambda = 0.15$and the increase of λ for $\xi = 0.5$.

5. Conclusions

In this paper, the bending response of axially functionally graded nanobeams is studied and compared on the basis of local stress- and strain-driven two-phase local/nonlocal integral models. The integral constitutive relations are transformed unitedly and equivalently into differential from with constitutive boundary conditions. The study results show that mathematical formulation for static bending based εD- and LσD-TPNIMs is well-posed. Purely εD- and LσD-NIMs would lead to ill-posed mathematical formulation, and Purely εD- and LσD-NDMs would result in inconsistent size-dependent bending responses. Numerical results show that εD- and LσD-TPNIMs lead to same bending response for homogeneous nonlocal Euler-

Bernoulli nanobeam. The bending deflection increases with the increase of nonlocal parameter λ or the decrease of local volume fraction ξ, and it indicated that the nonlocal parameters exert consistent stiffness-softening effects for εD- and LσD-TPNIMs. The maximum bending deflections increase with the decrease of β because the stiffness of nanobeam increases with the increase of β. The distinction of bending deflections based εD- and LσD-TPNIMs increases with the decrease of β for $\beta < 0$ and ξ as well as the increase of β for $\beta > 0$ and λ. The positions of maximum deflections of CCU and SSU nanobeams are $\eta < 0.5$ and $\eta > 0.5$ for $\beta > 0$ and $\beta < 0$, respectively. Furthermore, bending deflections of CCU and SSU nanobeams in left regions based on εD-TPNIM is larger and less than those based on LσD-TPNIM for $\beta < 0$ and $\beta > 0$, respectively. However, the opposite trend is observed in right regions of bending deflection curves. The bending deflections at free end of CFC nanobeams based on εD- and LσD-TPNIMs have same value, through those between endpoints based on εD-TPNIM are larger and less than those based on LσD-TPNIM for $\beta > 0$ and $\beta < 0$, respectively.

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