

Study on derivation from large-amplitude size dependent internal resonances of homogeneous and FG rod-types

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Abstract. Recently, a lot of research has been done on the analysis of axial vibrations of homogeneous and FG nanotubes (nanorods) with various aspects of vibrations that have been fully mentioned in history. However, there is a lack of investigation of the dynamic internal resonances of FG nanotubes (nanorods) between them. This is one of the essential or substantial characteristics of nonlinear vibration systems that have many applications in various fields of engineering (making actuators, sensors, etc.) and medicine (improving the course of diseases such as cancers, etc.). For this reason, in this study, for the first time, the dynamic internal resonances of FG nanorods in the simultaneous presence of large-amplitude size dependent behaviour, inertial and shear effects are investigated for general state in detail. Such theoretical patterns permit as to carry out various numerical experiments, which is the key point in the expansion of advanced nano-devices in different sciences. This research presents an AFG novel nano resonator model based on the axial vibration of the elastic nanorod system in terms of derivation from large-amplitude size dependent internal modals interactions. The Hamilton's Principle is applied to achieve the basic equations in movement and boundary conditions, and a harmonic deferential quadrature method, and a multiple scale solution technique are employed to determine a semi-analytical solution. The interest of the current solution is seen in its specific procedure that useful for deriving general relationships of internal resonances of FG nanorods. The numerical results predicted by the presented formulation are compared with results already published in the literature to indicate the precision and efficiency of the used theory and method. The influences of gradient index, aspect ratio of FG nanorod, mode number, nonlinear effects, and nonlocal effects variations on the mechanical behavior of FG nanorods are examined and discussed in detail. Also, the inertial and shear traces on the formations of internal resonances of FG nanorods are studied, simultaneously. The obtained valid results of this research can be useful and practical as input data of experimental works and construction of devices related to axial vibrations of FG nanorods.

Keywords: axial theory; FG and homogeneous materials; internal resonances; large-amplitude vibrations; nanoscale rod; size dependent behaviour

1. Introduction

A modern clique of composite material named functionally graded material (FGM) has been expandable stated in the last few years due to their preferable properties. FGMs are the materials that comprising of a mixture of two various-apart ingredients with continuous gradient mechanical and physical specifications. In the general run of things, these features are ordinarily distributed along peculiar directions. Universally, FGMs are compound of metallic and ceramic; metallic barricades material from breaking when subjected to heat stress while the ceramic, with high thermal resistance-low strength, suggests resistance at exceeding temperatures. Plus, these materials can be described by toughness, high strength and resistance to high temperature and corrosion. The main opinion of FGM was proposed to utilize in the industry in 1984 during a space plane project designed in Japan by an association of material scientists. In the traditional

composite materials, the interfacial stresses happen at the interfaces between two various constituent materials due to differences in the material attributes. FGMs prevail over this difficulty making composites materials can be employed for airplanes, vehicles, military and defense projects, space crafts, biomedical field, electronics, energy and engineering constructions. Required to the innumerable supremacies of FGMs, several researches have been accomplished on using FGMs for a number of applications in order to comprehend its behavior under diverse working situations. In the last some years, people have spread their researches regarding static, dynamic, buckling, and vibrations analysis using FGM. As well as, great endeavors to figure out the mechanical performance of FGM have been done, (Sankar 2001, Chakraborty *et al.* 2003, Zhong and Yu 2007, Aydogdu 2009, Li 2008, Kadoli *et al.* 2008, Benatta *et al.* 2008, Shujairi and Mollamahmutoglu 2018, Şimşek 2009, Sina *et al.* 2009, Chen *et al.* 2016, Ke *et al.* 2009, Trinh *et al.* 2016, Thai and Vo 2012, Pradhan and Chakraverty 2013, Mashat *et al.* 2014, Akavci and Tanrikulu 2015, Su and Banerjee 2015, Jing *et al.* 2016, Yang *et al.* 2014, Jamali Shakhilavi *et al.* 2020, Hosseini-Hashemi *et al.* 2014, Gheshlaghi and Hasheminejad 2011, Liu *et al.* 2019, Yadav *et al.* 2019, Alijani and Amabili 2014, Kumar *et al.*

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2016, Arefi and Amabili 2021, Mohammadian and Hosseini 2022).

Since in the current research it is important to study the axial vibrations and compute the internal resonances of FG nanorods considering the effects of size dependent, a weak history in this field is pointed out. So, in the field of longitudinal vibrations analysis (Shakhilavi 2021a, b, 2023a, b, 2020), limited studies have been conducted, including: longitudinal vibration of embedded nanorods under transverse magnetic field effects in terms of nonlocal elastic continuum theory was studied by Murmu *et al.* (2014). Axial nonlinear vibrations investigating of rods is proposed by Fariborz (2012), utilizing local elasticity theory. The axial vibration of nanorods modeled based on the simplest rod theory by using the nonlocal elasticity theory is studied by Aydogdu (2009). Kiani (2010) evaluated the small-scale effect on free axial vibration of nanowires in terms of linearly varied radii. It also employed the simplest rod theory for modelling of the tapered nanorods. A similar examination on the axial vibration of double-nanorod-systems is also found by Murmu and Adhikari (2010). The effect of crack is also studied on the longitudinal vibration of nanobeams by considering nonlocal theory (Hsu *et al.* 2011). The mentioned effect is also investigated on the analysis of the longitudinal wave propagation of coupled nanorod systems, Narendar and Gopalakrishnan (2011). The small-scale effect on the axial vibration of non-uniform nanorods employing boundary characteristic orthogonal polynomials is evaluated by Faroughi and Goushegir (2016). The rod model is used by Aydogdu (2015) for modeling of axial vibration of double-walled carbon nanotubes. In the mentioned research, the Van der Waals forces are weighed in the axial direction and the small-scale effect is investigated on natural axial frequencies of nanotubes. The nonlocal longitudinal vibration of viscoelastic coupled double-nanorod systems is surveyed by (Karličić *et al.* 2015). Finally, there are only a few researches employing the Bishop's model for analyzing the behavior of thick nanoscale rods (Zhu and Li 2017, Yuan *et al.* 2020, Nazemnezhad and Kamali 2018, Li *et al.* 2017, Güven 2014, Mohammadian and Hosseini 2022). These studies have considered a linear homogeneous and a linear functionally graded material for nanostructures. In the mentioned history, more attempts have been made to pay more attention to new researches in the field of axial vibrations with different topics, but in this research, a new approach has been tried to evaluate the relationships of internal resonances of FG nanorods by considering the effects of inertial and shear for the first time.

The above released literature review illustrates that worth research on the free vibration analysis of the functionally graded material nanorod, but most of them are modeled and formulated utilizing the linear beam and without analyzing internal modals interactions, inertial effects and shear model. To best of the authors acknowledge, there is no reported study on the internal resonances analysis of FG nanorod using non classical method similar to what the authors have proposed in this work.

Nanomaterials have emerged as an amazing class of materials that consists of a broad spectrum of examples

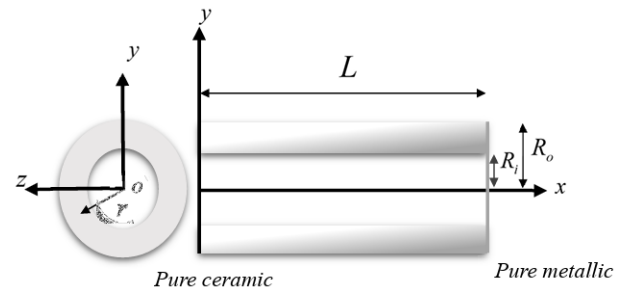


Fig. 1 Geometry of FG-nanorod

with at least one dimension in the range of 1 to 100 nm. The axially functionally graded (AFG) nanorod properties can be tuned as desired via precisely controlling the size, shape, synthesis conditions, and appropriate functionalization. It is a special kind of nonhomogeneous functionally gradient material structure, whose material properties vary continuously along the axial direction of the nanorod by a given distribution form. In fact, one-dimensional nanostructures, such as nanowires or nanorods have attracted remarkable attentions due to their excellent optical or electrical properties for potential applications in nano-electronic, nano-optical and nano-mechanical devices (like: actuators and sensors).

In this manuscript, large-amplitude non-classical free longitudinal vibrations considering inertial traces and shear modeling of axially functionally nanorod are studied employing Bishop axial theory by considering internal resonances as well. It is worthy to note that the governing differential equations and relations of internal resonances are solved and derived using harmonic differential quadrature and multiple scale methods. The material properties of the functionally graded material (ceramic and metal) are assumed to vary through the axial axis of nanorod based on the classical rule method. A number of results of the new analysis are compared with some existing results in the previous researches to acknowledge the validity and the precision of the present formulation. Very good agreement is noticed.

2. Background theory

2.1 Functionally graded materials

The system Taking into account the FG-nanorod with length L where in R_i, R_o, ρ, E, G and A represent the inner radius, the outer radius, the density, elasticity modulus, shear modulus and the cross-sectional area, respectively. The left (Ceramic, $x=0$) and right (metallic, $x=L$) nanorod are composed various materials which as shown in Fig. 1.

In this study, the material properties of the FG nanorod vary continuously across the axial x matching to the power law distribution as follows (Cao *et al.* 2018)

$$\begin{aligned} E(x) &= (E_L - E_R) \left(1 - \frac{x}{L}\right)^m + E_R \\ \rho(x) &= (\rho_L - \rho_R) \left(1 - \frac{x}{L}\right)^m + \rho_R \\ G(x) &= (G_L - G_R) \left(1 - \frac{x}{L}\right)^m + G_R \end{aligned} \quad (1)$$

where m is the power gradient index, and the subscripts of L and R showman left of nanorod and right of nanorod, respectively.

2.2 The governing equation and boundary conditions

A With the help of Hamiltonian principle (H), the governing equations of the problem and boundary conditions are extracted as below, Rao (2007)

$$\delta H = 0, \quad H = \frac{1}{\tau} \int_0^\tau (T - U + W_{ext}) dt; \quad (2)$$

with $\tau = \frac{2\pi}{\omega} \Rightarrow \omega$: Frequency of FG nanorod

where T , U and W_{ext} respectively showman vibration period, kinetic energy, strain energy and external work caused by an external force. Bishop axial theory defines the displacement field as, Rao (2007)

$$u = u(x, t), \quad v = -vy \frac{\partial u(x, t)}{\partial x}, \quad w = -vz \frac{\partial u(x, t)}{\partial x} \quad (3)$$

where u , v and w respectively are the components of displacement field along x , y and z directions and ν denote the Poisson's ratio of the FG nanorod. The kinetic energy of the FG nanorod is then derived as:

$$T(x) = \frac{1}{2} \int_0^L \rho(x) A \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L \rho(x) v^2 I_p \left(\frac{\partial^2 u}{\partial x \partial t} \right)^2 dx \quad (4)$$

where $I_p = \int_A (y^2 + z^2) dA$: Moment of polar inertia

On the basis of the assumption of large axial displacements, and small strains (ϵ) for a straight Bishop nanorod, the von-Kármán's nonlinear, Amabili (2008), strain-displacement equation is defined as

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \nu^2 (y^2 + z^2) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \\ \epsilon_{yy} &= \epsilon_{zz} = -\nu \frac{\partial u}{\partial x} + \frac{1}{2} \nu^2 \left(\frac{\partial u}{\partial x} \right)^2 \\ \epsilon_{xy} &= \epsilon_{yx} = -\frac{1}{2} \nu y \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \nu^2 y \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \\ \epsilon_{xz} &= \epsilon_{zx} = -\frac{1}{2} \nu z \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \nu^2 z \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \\ \epsilon_{yz} &= \epsilon_{zy} = 0 \end{aligned} \quad (5)$$

And the corresponding stresses (σ)

$$\begin{aligned} \sigma_{xx} &= Z_1(x) \frac{\partial u}{\partial x} + Z_2(x) \left(\frac{\partial u}{\partial x} \right)^2 + Z_{22}(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \\ \sigma_{yy} &= \sigma_{zz} = Z_4(x) \left(\frac{\partial u}{\partial x} \right)^2 + Z_{44}(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \\ \sigma_{xy} &= \sigma_{yx} = -G(x) \nu y \left(\frac{\partial^2 u}{\partial x^2} \right) + G(x) \nu^2 y \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \\ \sigma_{xz} &= \sigma_{zx} = -G(x) \nu z \left(\frac{\partial^2 u}{\partial x^2} \right) + G(x) \nu^2 z \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \\ \sigma_{yz} &= \sigma_{zy} = 0 \end{aligned} \quad (6)$$

The strain energy of the FG nanorod which results from the strains and stresses of the nanorod is specified as

$$\begin{aligned} U(x) &= \int_0^L \int_A \sigma_{ij} \epsilon_{ij} dA dx \Rightarrow \\ \delta U &= \int_V \left(\sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{zz} \delta \epsilon_{zz} + \sigma_{xy} \delta \gamma_{xy} \right) dV \quad (7) \end{aligned}$$

where: $\gamma_{ij} = 2\epsilon_{ij}$: Shear strain, V : Nanorod Volume

In order to consider the vibration to be free, we set the external work due to the applied force to be zero.

$$W_{ext} = 0 \quad (8)$$

By substituting Eqs. (4), (7) and (8) in Eq. (2) and using fundamental lemma of calculus, the first variation of total energy is derived and the local governing equations and local boundary conditions of the FG nanorod can be obtained based on the Bishop axial model as below

Local governing equation:

$$\begin{aligned} & -\frac{\partial}{\partial x} N_{xx} - \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x^2} \left(\nu^2 (y^2 + z^2) N_{xx} \frac{\partial^2 u}{\partial x^2} \right) \\ & + \frac{\partial}{\partial x} (\nu N_{yy}) - \frac{\partial}{\partial x} \left(\nu^2 N_{yy} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} (\nu N_{zz}) \\ & - \frac{\partial}{\partial x} \left(\nu^2 N_{zz} \frac{\partial u}{\partial x} \right) - \frac{\partial^2}{\partial x^2} (\nu M_{xy}) - \frac{\partial}{\partial x} \left(\nu^2 M_{xy} \frac{\partial^2 u}{\partial x^2} \right) \\ & + \frac{\partial^2}{\partial x^2} \left(\nu^2 M_{xy} \frac{\partial u}{\partial x} \right) - \frac{\partial^2}{\partial x^2} (\nu M_{xz}) - \frac{\partial}{\partial x} \left(\nu^2 M_{xz} \frac{\partial^2 u}{\partial x^2} \right) \\ & + \frac{\partial^2}{\partial x^2} \left(\nu^2 M_{xz} \frac{\partial u}{\partial x} \right) + \rho(x) A \frac{\partial^2 u}{\partial t^2} - \nu^2 I_p \frac{\partial \rho(x)}{\partial x} \frac{\partial^3 u}{\partial x \partial t^2} \\ & - \nu^2 I_p \rho(x) \frac{\partial^4 u}{\partial x^2 \partial t^2} = 0 \end{aligned} \quad (9)$$

Local corresponding boundary conditions:

$$\begin{aligned} & \left[\begin{aligned} & N_{xx} + N_{xx} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (\nu^2 (y^2 + z^2) N_{xx}) \frac{\partial^2 u}{\partial x^2} \\ & - \nu N_{yy} + \nu^2 N_{yy} \frac{\partial u}{\partial x} - \nu N_{zz} + \nu^2 N_{zz} \frac{\partial u}{\partial x} \\ & + \frac{\partial}{\partial x} (\nu M_{xy}) + \nu^2 M_{xy} \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left(\nu^2 M_{xy} \frac{\partial u}{\partial x} \right) \\ & + \frac{\partial}{\partial x} \nu M_{xz} + \nu^2 M_{xz} \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} \left(\nu^2 M_{xz} \frac{\partial u}{\partial x} \right) \\ & + \rho(x) \nu^2 I_p \frac{\partial^3 u}{\partial x \partial t^2} \end{aligned} \right]_0^L \delta u = 0 \\ & \left[\begin{aligned} & (\nu^2 (y^2 + z^2) N_{xx}) \frac{\partial^2 u}{\partial x^2} - \nu M_{xy} + \nu^2 M_{xy} \frac{\partial u}{\partial x} \\ & - \nu M_{xz} + \nu^2 M_{xz} \frac{\partial u}{\partial x} \end{aligned} \right]_0^L \delta u' = 0 \end{aligned} \quad (10)$$

2.3 The nonlocal elasticity model for FG nanorod

Based on the nonlocal elasticity theory (Eringen 1972, 1983, Eringen and Edelen 1972) the nonlocal stress-tensor σ_{ij} at point x of the FG and isotropic nano-rod relates to the local stress-tensor t_{ij} as

$$\begin{aligned} [1 - \mu^2 \nabla^2] \sigma_{ij} &= E \epsilon(x) = t_{ij} \\ \Rightarrow \sigma_{xx} - \mu^2 \left(\frac{\partial \sigma_{xx}}{\partial x^2} \right) &= E \epsilon_{xx} = t_{xx} \end{aligned} \quad (11)$$

where t_{xx} and σ_{xx} are the local and nonlocal normal stresses respectively; Here, μ , ϵ_{xx} and E are the nonlocal parameter, local strain, and elastic modulus, respectively.

As a result, the equations of movement FG nanoscale rod in the non-local form are presented as follows

$$\begin{aligned}
 & B'_1(x) \left(\frac{\partial u}{\partial x} \right) + B'_2(x) \frac{\partial^2 u}{\partial x^2} + B'_3(x) \left(\frac{\partial u}{\partial x} \right)^2 \\
 & + B'_4(x) \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) + B'_5(x) \left(\frac{\partial u}{\partial x} \right)^3 + B'_6(x) \frac{\partial^2 u}{\partial x^2} \left(\frac{\partial u}{\partial x} \right)^2 \\
 & + B'_7(x) \frac{\partial^3 u}{\partial x^3} + B'_8(x) \frac{\partial^4 u}{\partial x^4} + B'_9(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \\
 & + B'_{10}(x) \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial x^3} + B'_{11}(x) \frac{\partial^2 u}{\partial x^2} \left(\frac{\partial^3 u}{\partial x^3} \right) + \\
 & B'_{12}(x) \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + B'_{13}(x) \frac{\partial u}{\partial x} \left(\frac{\partial^4 u}{\partial x^4} \right) \\
 & + B'_{14}(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^3 + B'_{15}(x) \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \left(\frac{\partial^3 u}{\partial x^3} \right) \\
 & + B'_{16}(x) \left(\frac{\partial u}{\partial x} \right)^2 \frac{\partial^3 u}{\partial x^3} + B'_{17}(x) \left(\frac{\partial u}{\partial x} \right)^2 \frac{\partial^4 u}{\partial x^4} \\
 & + B'_{18}(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \frac{\partial^3 u}{\partial x^3} + B'_{19}(x) \frac{\partial^2 u}{\partial x^2} \left(\frac{\partial^3 u}{\partial x^3} \right)^2 \\
 & + B'_{20}(x) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \frac{\partial^4 u}{\partial x^4} + B'_{21}(x) \frac{\partial^2 u}{\partial t^2} + B'_{22}(x) \frac{\partial^3 u}{\partial x \partial t^2} \\
 & + B'_{23}(x) \frac{\partial^4 u}{\partial x^2 \partial t^2} + B'_{24}(x) \frac{\partial^5 u}{\partial x^3 \partial t^2} + B'_{25}(x) \frac{\partial^6 u}{\partial x^4 \partial t^2} \\
 & = 0
 \end{aligned} \tag{12}$$

3. Solution procedure

3.1 Linear nonlocal frequencies/Harmonic differential quadrature method

In order to derive the nonlocal movement equation in terms of FG nanorod, Eq. (12), and accordingly to achieve the nonlocal linear axial frequencies of FG nanorod, HDQ method is utilized. The HDQ method is more impressive in comparison with the ordinary differential quadrature (DQ) method in order to solve the mechanical problems especially vibrational systems (Malekzadeh and Karami 2005, Striz *et al.* 1995, Gad-el-Hak 1996). In this method, the partial derivative of a function, with respect to a spatial variable at a given discrete point, approximated by a linear summation of weighted function values at all discrete points chosen in the solution domain of the spatial variable. Assume the amplitude of weighed FG nanorod is $(0 < x < L)$ and being discretized by N points along x coordinate. If $F(x)$ showman either of deformation function (u) within the FG nanorod amplitude, or the derivatives of $F(x)$ given that x at the point x_i can be expressed discretely as

$$\frac{d^n F(x_i)}{dx^n} = \sum_{k=1}^N A_{ik}^n F(x_k); n = 1, \dots, N - 1 \tag{13}$$

where A_{ik}^n is the weighting coefficient in conjunction to the n -pth order derivative of $F(x)$, at the discrete point x_i . The description of HDQ method and how to choose the positions of the nodal points utilizing Chebyshev polynomials were presented by Civalek (2004), "Now, the HDQM can be used to discretize the Eq. (12), governing equation and boundary condition equation. Before do this,

$X = \frac{x}{L}$ and $\bar{U} = \frac{u}{L}$ are used to achieve the non-dimensional form of Eq. (12) as bellow

$$\begin{aligned}
 & \text{Nonlocal - Linear Equation} \\
 & \frac{1}{L} \frac{d}{dX} \left(E(X) A \frac{d\bar{U}(X)}{dX} \right) - \frac{1}{L^3} \frac{d^2}{dX^2} \left(v^2 G(X) I_p \frac{d^2 \bar{U}(X)}{dX^2} \right) \\
 & + \rho A \omega^2 L \bar{U}(X) - \frac{\mu \omega^2}{L} \frac{d^2}{dX^2} (\rho(X) A \bar{U}(X)) - \\
 & \frac{1}{L} \frac{d}{dX} \left(\rho(X) v^2 I_p \omega^2 \frac{d\bar{U}(X)}{dX} \right) + \frac{\mu}{L^3} \frac{d^3}{dX^3} \left(\frac{\rho(X) v^2 *}{I_p \omega^2} \frac{d\bar{U}(X)}{dX} \right) = 0 \\
 & \text{Nonlocal - Linear Boundary Conditions} \\
 & \left[\begin{aligned} & \left(E(X) A \frac{d\bar{U}(X)}{dX} - \frac{1}{L^2} \frac{d}{dX} \left(v^2 G(X) I_p \frac{d^2 \bar{U}(X)}{dX^2} \right) \right) \\ & - \rho(X) v^2 I_p \omega^2 \frac{d\bar{U}(X)}{dX} + \\ & \left(\frac{\mu}{L^2} \frac{d^2}{dX^2} \left(\rho(X) v^2 I_p \omega^2 \frac{d\bar{U}(X)}{dX} \right) \right) \\ & + \left(\frac{v^2 G(X) I_p}{L} \frac{d^2 \bar{U}(X)}{dX^2} \right) \end{aligned} \right]_{\delta \bar{U}} \Bigg|_0^L = 0 \tag{14}
 \end{aligned}$$

Consequently, by performing HDQM, the discretized forms of governing and boundary condition equations at $X_i = \frac{x_i}{L}$ are derived as

$$\begin{aligned}
 & \frac{1}{L} \sum_{k=1}^N \left(A(i, k) E(X) A \sum_{j=1}^N (A(k, j) \bar{U}(X_j)) \right) - \\
 & \frac{1}{L^3} \sum_{k=1}^N \left(B(i, k) v^2 G(X) I_p \sum_{j=1}^N (B(k, j) \bar{U}(X_j)) \right) \\
 & + \rho(X) A \omega^2 L \bar{U}(X_i) - \frac{\mu \omega^2}{L} \sum_{k=1}^N (B(i, k) \rho(X) A \bar{U}(X_k)) \\
 & - \frac{1}{L} \sum_{k=1}^N \left(A(i, k) \rho(X) v^2 I_p \omega^2 \sum_{j=1}^N (A(k, j) \bar{U}(X_j)) \right) \\
 & + \frac{\mu}{L^3} \sum_{k=1}^N \left(C(i, k) \rho(X) v^2 I_p \omega^2 \sum_{j=1}^N (A(k, j) \bar{U}(X_j)) \right) = 0 \tag{15}
 \end{aligned}$$

$$\left[\begin{aligned} & \left(E(X) A \sum_{k=1}^N (A(i, k) \bar{U}(X_k)) - \right. \\ & \frac{1}{L^2} \sum_{k=1}^N A(i, k) v^2 G(X) I_p \sum_{j=1}^N (B(k, j) \bar{U}(X_j)) \\ & - \rho(X) v^2 I_p \omega^2 \sum_{k=1}^N (A(i, k) \bar{U}(X_k)) + \\ & \left. \left(\frac{\mu}{L^2} \sum_{k=1}^N \left(\sum_{j=1}^N (A(k, j) \bar{U}(X_j)) \right) \right) \right) \\ & + \left(\frac{v^2 G(X) I_p}{L} \sum_{k=1}^N (B(i, k) \bar{U}(X_k)) \right) \end{aligned} \right]_{\delta(\sum_{k=1}^N (A(i, k) \bar{U}(X_k)))} \Bigg|_0^L = 0 \tag{16}$$

The discretized forms of BCs, Eq. (16) are employed into the discretized forms of governing equation, Eq. (15), and by separating domain and boundary degrees of freedom (DOF), the bellowing assembled matrix equations are derived.

$$\begin{aligned}
& \begin{bmatrix} [K_{bb}] & [K_{bd}] \\ [K_{db}] & [K_{dd}] \end{bmatrix} \begin{Bmatrix} \{\bar{U}_b\} \\ \{\bar{U}_d\} \end{Bmatrix} \\
& = \omega^2 \begin{bmatrix} [0] & [0] \\ [M_{db}] & [M_{dd}] \end{bmatrix} \begin{Bmatrix} 0 \\ \{\bar{U}_d\} \end{Bmatrix}
\end{aligned} \quad (17)$$

where \bar{U}_b and \bar{U}_d showman the boundary conditions and amplitude DOF, respectively, as follow

$$\begin{aligned}
\{\bar{U}_b\} &= \{\bar{U}(X_1), \bar{U}(X_2), \bar{U}(X_{N-1}), \bar{U}(X_N)\} \\
\{\bar{U}_d\} &= \{\bar{U}(X_2), \bar{U}(X_3), \dots, \bar{U}(X_{N-3}), \bar{U}(X_{N-2})\}
\end{aligned} \quad (18)$$

Some mathematical reductions on Eq. (17) are done, and then the natural linear frequencies of the FG nanorod can be computed by solving the following equation

$$\begin{aligned}
& \left[[M_{dd}] - [M_{db}] [K_{bb}]^{-1} [K_{bd}] \right]^{-1} * \\
& \left[[K_{dd}] - [K_{db}] [K_{bb}]^{-1} [K_{bd}] \right] \{\bar{U}_d\} = \omega^2 \{\bar{U}_d\}
\end{aligned} \quad (19)$$

According to the above outlined formulations and with the help of the MATLAB program solver a self-developed computer program is written by which the natural linear frequencies of the FG nanorod can be derived.

3.2 Perturbation analysis /Nonlinear nonlocal frequencies

For nonlinear axial free vibration analysis of FG nanorods incorporating the size dependent effect, the longitudinal displacement of the n^{th} mode of the nanorod can be determined as Eq. (20) or Eq. (21) to obtain an ordinary differential equation.

$$u_n(x, t) = \psi_n(x)Y(t) \quad (20)$$

$$u_n(x, t) = \sum_{i=1}^N \psi_i(x)Y_i(t) \quad (21)$$

where, $Y(t)$ is a time dependent function to be defined and $\psi_n(x)$ is the normalized linear mode shapes of the FG nanorod which can be obtained from Eq. (19), Eq. (20) and Eq. (21) are the single term Galerkin procedure and the multi-term Galerkin procedure, respectively.

In the following, the multi-term Galerkin procedure is used to convert the nonlinear partial differential equation of motion to an ordinary differential equation. Substituting Eq. (21) into Eq. (12) results in the following ordinary differential Eq. (22). Then, utilizing the non-dimensional definitions given as Eq. (23), multiplying Eq. (22) by the normalized linear mode shape $\bar{\psi}_m(X)$, and integrating from $X=0$ to $X=1$ result in Eqs. (23) and (24), Nayfeh and Nayfeh (1994).

3.2.1 Method of multiple scale

In general, there are various methods for solving nonlinear motion equations (Hernández-Acosta *et al.* 2018, Mohammadian and Hosseini 2022), but because the main focus of this research is on evaluating the potential internal resonances occurring in the structure's response, the authors used the method of multiple scale to extract neat relationships between two engaged modes. To the best of author's knowledge, this approach is one of the excellent

candidates to obtain analytical expression governing the internal resonance characteristics of vibrational structures. It is also worth mentioning that this method can provide us with sufficiently accurate results in cases where the vibration amplitude is relatively high, (Jamali Shakhilavi *et al.* 2022a, b, Yapanmiş and Bağdatlı 2022, Noroozi and Ghadiri 2021, Qing and Wei 2022). Hence, to solve the nonlinear equation, Eq. (24), the method of multiple scales which is a semi-analytical method is adopted. To this end, a small dimensionless parameter ε as a bookkeeping device is introduced and Eq. (24) is rewritten as, Nayfeh and Nayfeh (1994)

$$\begin{aligned}
& \sum_{i=1}^N Y_i \begin{bmatrix} B'_1(x) \frac{d\psi_i(x)}{dx} \\ +B'_2(x) \frac{d^2\psi_i(x)}{dx^2} \\ +B'_7(x) \frac{d^3\psi_i(x)}{dx^3} \\ +B'_8(x) \frac{d^4\psi_i(x)}{dx^4} \end{bmatrix} \\
& + \sum_{i=1}^N \sum_{j=1}^N Y_i Y_j \begin{bmatrix} B'_3(x) \frac{d\psi_i(x)}{dx} \frac{d\psi_j(x)}{dx} \\ +B'_4(x) \frac{d\psi_i(x)}{dx} \frac{d^2\psi_j(x)}{dx^2} \\ +B'_9(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^2\psi_j(x)}{dx^2} \\ +B'_{10}(x) \frac{d\psi_i(x)}{dx} \frac{d^3\psi_j(x)}{dx^3} \\ +B'_{11}(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^3\psi_j(x)}{dx^3} \\ +B'_{13}(x) \frac{d\psi_i(x)}{dx} \frac{d^4\psi_j(x)}{dx^4} \end{bmatrix} + \\
& \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N Y_i Y_j Y_k \begin{bmatrix} B'_5(x) \frac{d\psi_i(x)}{dx} \frac{d\psi_j(x)}{dx} \frac{d\psi_k(x)}{dx} \\ +B'_6(x) \frac{d\psi_i(x)}{dx} \frac{d\psi_j(x)}{dx} \frac{d^2\psi_k(x)}{dx^2} \\ +B'_{12}(x) \frac{d\psi_i(x)}{dx} \frac{d^2\psi_j(x)}{dx^2} \frac{d^2\psi_k(x)}{dx^2} \\ +B'_{14}(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^2\psi_j(x)}{dx^2} \frac{d^2\psi_k(x)}{dx^2} \\ +B'_{15}(x) \frac{d\psi_i(x)}{dx} \frac{d^2\psi_j(x)}{dx^2} \frac{d^3\psi_k(x)}{dx^3} \\ +B'_{16}(x) \frac{d\psi_i(x)}{dx} \frac{d\psi_j(x)}{dx} \frac{d^3\psi_k(x)}{dx^3} \\ +B'_{17}(x) \frac{d\psi_i(x)}{dx} \frac{d\psi_j(x)}{dx} \frac{d^4\psi_k(x)}{dx^4} \\ +B'_{18}(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^2\psi_j(x)}{dx^2} \frac{d^3\psi_k(x)}{dx^3} \\ +B'_{19}(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^3\psi_j(x)}{dx^3} \frac{d^3\psi_k(x)}{dx^3} \\ +B'_{20}(x) \frac{d^2\psi_i(x)}{dx^2} \frac{d^2\psi_j(x)}{dx^2} \frac{d^4\psi_k(x)}{dx^4} \end{bmatrix} \\
& + \sum_{i=1}^N \ddot{Y}_i \begin{bmatrix} B'_{21}(x) \psi_i(x) \\ +B'_{22}(x) \frac{d\psi_i(x)}{dx} \\ +B'_{23}(x) \frac{d^2\psi_i(x)}{dx^2} \\ +B'_{24}(x) \frac{d^3\psi_i(x)}{dx^3} \\ +B'_{25}(x) \frac{d^4\psi_i(x)}{dx^4} \end{bmatrix} = 0
\end{aligned} \quad (22)$$

$$X = \frac{x}{L}; \quad \bar{\psi} = \frac{\psi}{L}; \quad \bar{Y} = \frac{Y}{Y_{\max}} \quad (23)$$

$$\ddot{\bar{Y}}_m + \frac{1}{\Pi} \bar{Y}_m + \frac{Y_{\max}}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \beta_{mij} \bar{Y}_i \bar{Y}_j + \frac{Y_{\max}^2}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \alpha_{mijk} \bar{Y}_i \bar{Y}_j \bar{Y}_k = 0 \quad (24)$$

$$\text{where } \Rightarrow \frac{1}{\Pi} = \omega_m^2$$

$$\ddot{\bar{Y}}_m + \omega_m^2 \bar{Y}_m + \varepsilon \left\{ \frac{Y_{\max}}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \beta_{mij} \bar{Y}_i \bar{Y}_j \right\} + \varepsilon^2 \left\{ \frac{Y_{\max}^2}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \alpha_{mijk} \bar{Y}_i \bar{Y}_j \bar{Y}_k \right\} = 0 \quad (25)$$

In using the method of multiple scales, a second-order uniform expansion of the solution of Eq. (25) is desired as

$$\bar{Y}_m(t_i; \varepsilon) = \bar{Y}_{m0}(t_0, t_1, t_2) + \varepsilon \bar{Y}_{m1}(t_0, t_1, t_2) + \varepsilon^2 \bar{Y}_{m2}(t_0, t_1, t_2) + O(\varepsilon^3(t)) \quad (26)$$

where $t_0 = t$ is the fast time scale denoting the main oscillatory behavior, and $t_n = \varepsilon^n t; n \geq 1$ is the slow time scale implying the amplitude and phase modulation. Substitution of Eq. (26) into Eq. (25) and equating coefficients of like powers of ε to zero result in hierarchy of linear differential equations which need to be solved successively. These equations are, Nayfeh and Nayfeh (1994)

$$\begin{aligned} O(\varepsilon^0): D_0^2 \bar{Y}_{m0} + \omega_m^2 \bar{Y}_{m0} &= 0 \\ O(\varepsilon^1): D_0^2 \bar{Y}_{m1} + \omega_m^2 \bar{Y}_{m1} &= \left\{ -2D_0 D_1 \bar{Y}_{m0} - \frac{Y_{\max}}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \beta_{mij} \bar{Y}_{i0} \bar{Y}_{j0} \right\} \\ O(\varepsilon^2): D_0^2 \bar{Y}_{m2} + \omega_m^2 \bar{Y}_{m2} &= \left\{ \begin{aligned} &-(D_1^2 + 2D_0 D_2) \bar{Y}_{m0} - 2D_0 D_1 \bar{Y}_{m1} - \\ &\frac{Y_{\max}}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \beta_{mij} (\bar{Y}_{i0} \bar{Y}_{j1} + \bar{Y}_{i1} \bar{Y}_{j0}) \\ &-\frac{Y_{\max}^2}{\Pi} \sum_{i=1}^N \sum_{j=1}^N \sum_{p=1}^N \alpha_{mijp} \bar{Y}_{i0} \bar{Y}_{j0} \bar{Y}_{p0} \end{aligned} \right\} \quad (27) \end{aligned}$$

where,

$$\begin{aligned} D_n &= \frac{\partial}{\partial T_n}, \quad n = 0, 1, 2, \dots \\ D_0 &= \frac{\partial}{\partial T_0}, \quad D_1 = \frac{\partial}{\partial T_1}, \quad D_2 = \frac{\partial}{\partial T_2} \end{aligned} \quad (28)$$

The first equation of Eq. (28), is extracted as following

$$\bar{Y}_{m0}(T_0, T_1, T_2) = A_m(T_1, T_2) e^{i\omega_m T_0} + \bar{A}_m(T_1, T_2) e^{-i\omega_m T_0} \quad (29)$$

where $A_m(T_1, T_2)$ and $\bar{A}_m(T_1, T_2)$ are the complex functions in terms of T_1 and T_2 also, i is called as the square root of -1 . To construct the non-linear normal mode that reduces to the k^{th} linear mode as ($\varepsilon \rightarrow 0$ i.e., the non-linearity vanishes), we take the solutions of Eqs. (29), as follows, Nayfeh and Nayfeh (1994)

$$\begin{aligned} \bar{Y}_{k0}(T_0, T_1, T_2) &= A_k(T_1, T_2) e^{i\omega_k T_0} + \bar{A}_k(T_1, T_2) e^{-i\omega_k T_0} \\ &\rightarrow \text{for } k \bar{Y}_{m0} = 0 \rightarrow \text{for } m \neq k \end{aligned} \quad (30)$$

Replacing Eq. (30) into the second equation of Eq. (27) gives

$$\begin{aligned} D_0^2 \bar{Y}_{k1} + \omega_k^2 \bar{Y}_{k1} &= \left[\begin{aligned} &-2D_1 (A_k i\omega_k e^{i\omega_k T_0} - \bar{A}_k i\omega_k e^{-i\omega_k T_0}) \\ &-\frac{Y_{\max}}{\Pi} \beta_{kkk} (A_k^2 e^{2i\omega_k T_0} + 2A_k \bar{A}_k + \bar{A}_k^2 e^{-2i\omega_k T_0}) \end{aligned} \right] \\ D_0^2 \bar{Y}_{m1} + \omega_m^2 \bar{Y}_{m1} &= -\frac{Y_{\max}}{\Pi} \beta_{kkk} (A_k^2 e^{2i\omega_k T_0} + 2A_k \bar{A}_k + \bar{A}_k^2 e^{-2i\omega_k T_0}) \end{aligned} \quad (31)$$

The above equation has some secular terms. They provide non-periodic responses that are inconvenient with oscillatory suppositions. Therefore, the secular terms must be vanished Nayfeh and Nayfeh (1994). Eliminating secular terms in Eq. (31) yields

$$\begin{aligned} \frac{\partial A_k}{\partial T_1}, \frac{\partial \bar{A}_k}{\partial T_1} &= 0 \rightarrow A_k(T_2), \bar{A}_k(T_2) \\ \bar{Y}_{k1} &= \frac{Y_{\max}}{\Pi} \frac{\beta_{kkk} A_k^2}{3\omega_k^2} e^{2i\omega_k T_0} - \frac{2Y_{\max}}{\Pi} \frac{\beta_{kkk} A_k \bar{A}_k}{\omega_k^2}; \text{ for } k \\ \bar{Y}_{m1} &= \frac{Y_{\max}}{\Pi} \frac{\beta_{kkk} A_k^2}{(4\omega_k^2 - \omega_m^2)} e^{2i\omega_k T_0} - \frac{Y_{\max}}{\Pi} \frac{\beta_{kkk} A_k \bar{A}_k}{\omega_m^2}; \text{ for } m \neq k \end{aligned} \quad (32)$$

Replacing \bar{Y}_{m0} , \bar{Y}_{k0} and \bar{Y}_{m1} , \bar{Y}_{k1} from Eqs. (30), and (32) into the third equation of Eq. (27) renders

$$\begin{aligned} D_0^2 \bar{Y}_{k2} + \omega_k^2 \bar{Y}_{k2} &= \left[\begin{aligned} &-2 \frac{\partial A_k}{\partial T_2} i\omega_k + 4 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^2(T_2) \bar{A}_k(T_2)}{\omega_k^2} \\ &-2 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^2(T_2) \bar{A}_k(T_2)}{3\omega_k^2} \\ &-3 \frac{Y_{\max}^2}{\Pi} \alpha_{kkkk} A_k^2(T_2) \bar{A}_k(T_2) \end{aligned} \right] e^{i\omega_k T_0} \\ &- \left[2 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^3(T_2)}{3\omega_k^2} + \frac{Y_{\max}^2}{\Pi} \alpha_{kkkk} A_k^3(T_2) \right] e^{3i\omega_k T_0} + cc; \text{ for } k \\ D_0^2 \bar{Y}_{m2} + \omega_m^2 \bar{Y}_{m2} &= \left[\begin{aligned} &2 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^3(T_2)}{3\omega_k^2} \\ &+\frac{Y_{\max}^2}{\Pi} \alpha_{kkkk} A_k^3(T_2) \end{aligned} \right] e^{3i\omega_k T_0} + cc; \text{ for } m \neq k \end{aligned} \quad (33)$$

By removing the secular terms of Eq. (33), one renders

$$\begin{aligned} &-2 \frac{\partial A_k}{\partial T_2} i\omega_k + 4 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^2(T_2) \bar{A}_k(T_2)}{\omega_k^2} \\ &-2 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^2(T_2) \bar{A}_k(T_2)}{3\omega_k^2} \\ &-3 \frac{Y_{\max}^2}{\Pi} \alpha_{kkkk} A_k^2(T_2) \bar{A}_k(T_2) = 0 \end{aligned} \quad (34)$$

It is supposed that, $A_k(T_2)$ function can be defined as, Nayfeh and Nayfeh (1994)

$$\begin{aligned}
A_k(T_2) &= \frac{1}{2} a e^{i\beta} \\
\text{complex function} \\
&\rightarrow a, \beta \text{ are real functions of } T_2; \\
\text{So, } A_k(T_2) &= \frac{1}{2} a(T_2) e^{i\beta(T_2)}
\end{aligned} \quad (35)$$

By replacing Eq. (35) into the Eq. (34) gives

$$\begin{aligned}
&\left[-ia'\omega_k + a\beta'\omega_k + \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^3}{2\omega_k^2} \right. \\
&\left. - \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^3}{12\omega_k^2} - \frac{3}{8} \left(\frac{Y_{\max}}{\Pi}\right)^2 \alpha_{kkkk} a^3 \right] = 0 \\
&\rightarrow \left\{ \begin{array}{l} \text{Real part} = 0 \\ \text{Imaginary part} = 0 \end{array} \right\}, \text{So:} \\
&\left\{ \begin{array}{l} a\beta'\omega_k + 5 \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^3}{12\omega_k^2} - \frac{3}{8} \left(\frac{Y_{\max}}{\Pi}\right)^2 \alpha_{kkkk} a^3 = 0 \\ -a'\omega_k = 0 \end{array} \right. \\
&\left\{ \begin{array}{l} \beta' = -5 \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^2}{12\omega_k^3} + \frac{3}{8\omega_k} \left(\frac{Y_{\max}}{\Pi}\right)^2 \alpha_{kkkk} a^2 \\ a' = 0 \end{array} \right. \quad (36) \\
&\left\{ \begin{array}{l} \frac{d\beta}{dT_2} = -5 \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^2}{12\omega_k^3} + \frac{3}{8\omega_k} \left(\frac{Y_{\max}}{\Pi}\right)^2 \alpha_{kkkk} a^2 \\ \frac{da}{dT_2} = 0 \end{array} \right. \\
&\beta(T_2) = \\
&\left\{ -5 \left(\frac{Y_{\max}}{\Pi}\right)^2 \frac{\beta_{kkk}^2 a^2}{12\omega_k^3} + \frac{3}{8\omega_k} \left(\frac{Y_{\max}}{\Pi}\right)^2 \alpha_{kkkk} a^2 \right\} T_2 + \beta_0 \\
&a(T_2) = a = \text{constan t}
\end{aligned}$$

$$\text{finally } \rightarrow A_k(T_2) = \frac{1}{2} a e^{i \left(\left(\frac{3}{8\omega_k} \left(\frac{Y_{\max}}{\Pi} \right)^2 \alpha_{kkkk} a^2 - 5 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 a^2}{12\omega_k^3} \right) T_2 + \beta_0 \right)}$$

And for solving third section of Eq. (27) can be written

$$\begin{aligned}
\bar{Y}_{k2}(T_0, T_1, T_2) &= \\
&\left(\left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^3}{12\omega_k^4} + \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\alpha_{kkkk} A_k^3}{8\omega_k^2} \right) e^{3i\omega_k T_0}; \text{ for } k \\
\bar{Y}_{m2}(T_0, T_1, T_2) &= \\
&\left(2 \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2 A_k^3}{3\omega_k^2} + \left(\frac{Y_{\max}}{\Pi} \right)^2 \alpha_{kkkk} A_k^3 \right) \frac{e^{3i\omega_k T_0}}{9\omega_k^2 - \omega_m^2}; \text{ for } m \neq k
\end{aligned} \quad (37)$$

Now, by putting Eq. (36) into the Eq. (30), the non-linear size dependent frequencies of FG nano-rods are extracted as follows

$$\omega_{\text{Non-Linear}}^{\text{non-local}} \Big|_k = \left(\frac{3}{8} \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\alpha_{kkkk}}{\omega_k} - \frac{5}{12} \left(\frac{Y_{\max}}{\Pi} \right)^2 \frac{\beta_{kkk}^2}{\omega_k^3} \right) + \omega_k \quad (38)$$

Since the time responses of the FG nanorod frequency are computed based on the multi-Galerkin method for two general modes m and k , the relationships between the internal resonances of the FG nanorod are determined by considering different effects such as inertial, shear, nonlocal and nonlinear as follows

$$\begin{aligned}
\omega_k = \omega_m &\Rightarrow \text{one - to - one} \\
2\omega_k = \omega_m &\Rightarrow \text{two - to - one} \\
3\omega_k = \omega_m &\Rightarrow \text{three - to - one}
\end{aligned} \quad (39)$$

4. Results and discussion

In the numerical results, free axial nonlinear nonlocal vibration frequencies of FG nanorod are computed for various values of the nonlocal parameter, gradient index, mode number, and nonlinear amplitude. In order to examine the reliability of the presented formulation, a number of examples are now provided. Hence, the first third natural frequencies of a clamped-clamped and a clamped-free nanorod with constant and FG material properties are evaluated based on the classical rod theory and compared with the results of (Kiani 2010, Nazemnezhad and Kamali 2018, Fernandes *et al.* 2017) in Tables 1 and 2.

In this section, the numerical results of the proposed vibration system behavior based on the non-classical nonlinear's Bishop Theory are examined. Therefore, the FG nanorod composed of pure ceramic-pure metallic with a high thickness is considered. In order to better analyze the results, the aspect ratio of the FG nanorod and the non-local coefficient are specified in ranges $2 \leq \frac{L}{d} \leq 4$ and $0 \leq \mu \leq 1$ (nm^2), respectively. The mechanical properties of the FG nanorod are presented in Table 3. Both clamped-clamped and clamped-free boundary conditions have been evaluated in vibrational behavior analysis. In addition to the effects of inertia, shear effects have also been considered in order to study nonlocal nonlinear frequencies.

As the first factor, the effects of gradient index and mode number of homogeneous and FG nanorod on the nonlinear frequencies for different values of FG and homogeneous nanorod length and nonlinear amplitude for both clamped-clamped and clamped-free boundary conditions have been investigated. The result of this study can be seen in Figs. 2. Fig. 2 shows the nonlinear frequencies changes for the clamped-clamped and clamped-free boundary conditions in terms of nonlocal case.

According to Fig 2, it can be seen that the changes in nonlocal nonlinear axial frequencies with increasing nonlinear amplitude of vibrations for both homogeneous and FG thick nanorods in the first frequency mode are very small and uniform. But in higher frequency modes, its values gradually significant increase. The values of nonlocal nonlinear axial frequencies for homogeneous nanorods are lower than FG nanorods. In general, using the FG structure, we can achieve the desired values of frequency and corresponding nonlinear amplitudes depending on the application of nanorods, which is one of the important advantages of these materials in engineering (making actuators, sensors, etc.) and medical (diagnosis and treat diseases like cancer, etc.) sciences. In Tables 4 and 5,

Table 1 Comparison study of the present work with Ref: (Fernandes *et al.* 2017) in terms of the first third dimensionless natural frequencies for FG nanorod, considering: $q_{\max}=0.1$, $0 < L < 1$, $E_R/E_L = 1$

Mode number	Clamped-Free BCs.		Clamped- Clamped BCs.	
	Present work	(Fernandes <i>et al.</i> 2017)	Present work	(Fernandes <i>et al.</i> 2017)
1	1.5708	1.5708	3.1416	3.1416
2	4.7124	4.7124	6.2832	6.2832
3	7.8540	7.8540	9.4248	9.4248

Table 2 Comparison study of the present work with Refs: Kiani (2010) and Nazemnezhad and Kamali (2018) in terms of the natural frequencies of a clamped-clamped nanorod by considering: ($E=70$ GPa, $\rho=2370$ kg.m⁻³, $L=20$ nm, $R=0.5$ nm)

Mode number	Nonlocal parameter	Present work	Kiani (2010)	Nazemnezhad and Kamali (2018)
1	0	135.868	135.868	135.856
2		271.735	271.735	271.643
3		407.603	407.603	407.292
1	1 (nm ²)	134.220	134.220	134.210
2		259.241	259.241	259.155
3		368.710	368.710	368.432

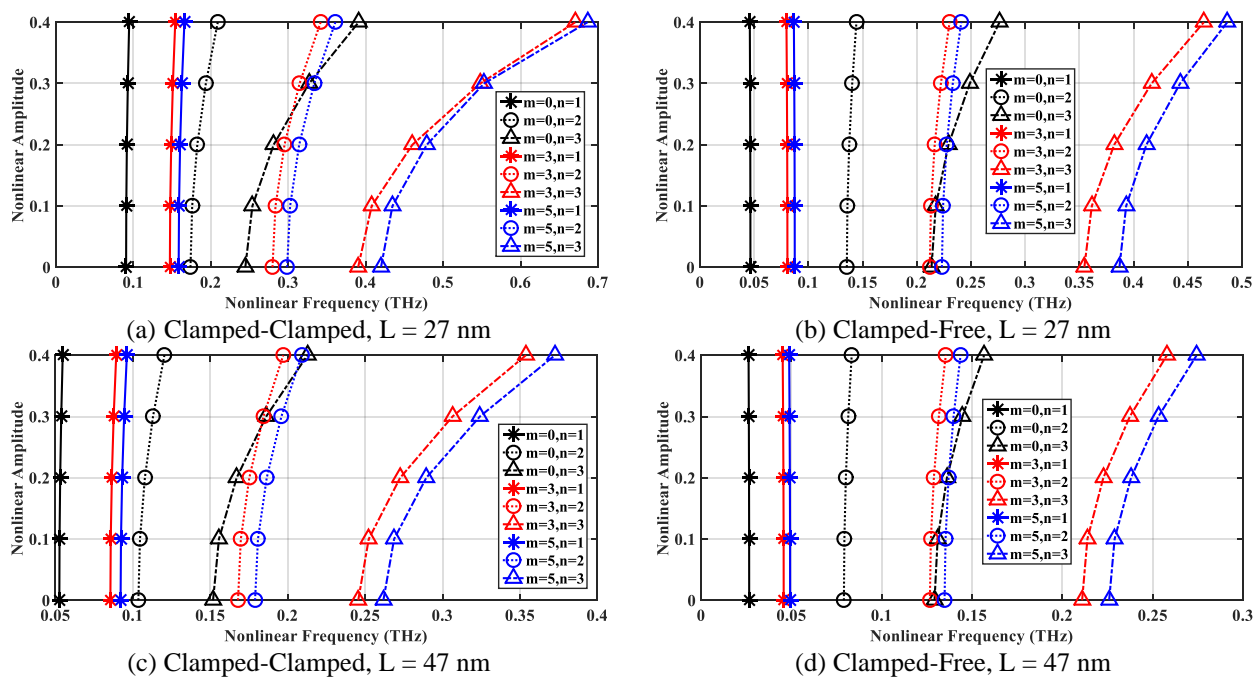


Fig. 2 Variations of nonlinear frequencies versus various values of nonlinear amplitudes by considering $\mu=1$ (nm²), $d_0=12$ (nm), $d_i=0$ in terms of clamped-clamped and clamped-free FG nanorods

the precision values of nonlinear frequencies are indicated for the first three frequencies of homogeneous and FG nanorods in terms of local and nonlocal states for clamped-clamped and clamped-free boundary conditions, respectively.

4.1 Internal resonance analysis of homogeneous and FG nanorods

In this section, the most important subject, namely the extraction of the internal resonances of homogeneous and

FG nanorods are discussed in presence of inertial and shear effects, simultaneously. In this regard, the values of gradient index equal to 0 and 5, aspect ratio of homogeneous and FG nanorods equal to $L/d = 3$ and non-local parameter values equal to 0, 0.05 and 0.1 (nm²) have been selected. By utilizing both Multiple Scale and Multi Galerkin methods simultaneously, the analytical relationships of internal resonances for the general state of this type of problems are extracted. As these general relationships of internal resonances are applicable for all rod-types, including quadratic and cubic stiffness nonlinear grades.

Table 3 Mechanical properties of FG nanorod Yang and Shen (2002)

Material	Bulk elastic modulus (GPa)	Shear modulus (GPa)	Mass density (kg/m ³)
Ceramic (SUS304)	E _L = 201.04	G _L = 75.80	ρ _L = 8166
Metallic	E _R = 349.55	G _R = 140.95	ρ _R = 3800

Table 4 The variations of nonlinear frequencies (THz) versus values of nonlinear amplitude for **clamped-clamped** BCs

Mode number	μ = 0				μ = 1 (nm ²)			
	Homogeneous nanorod		FG nanorod (m = 5)		Homogeneous nanorod		FG nanorod (m = 5)	
	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1
n = 1	0.066770	0.066946	0.117041	0.117370	0.066530	0.066705	0.116570	0.116897
n = 2	0.131918	0.133359	0.228007	0.230607	0.130055	0.131475	0.224285	0.226834
n = 3	0.194165	0.199762	0.335459	0.345679	0.188149	0.193572	0.323644	0.333602

Table 5 The variations of nonlinear frequencies (THz) versus values of nonlinear amplitude for clamped-free BCs

Mode number	μ = 0				μ = 1 (nm ²)			
	Homogeneous nanorod		FG nanorod (m = 5)		Homogeneous nanorod		FG nanorod (m = 5)	
	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1	Y _{Max} = 0	Y _{Max} = 0.1
n = 1	0.033894	0.033868	0.062975	0.062935	0.033862	0.033836	0.063050	0.063010
n = 2	0.100819	0.101183	0.171981	0.172730	0.099974	0.100334	0.169054	0.169796
n = 3	0.165345	0.167798	0.288499	0.292848	0.161583	0.163978	0.284401	0.288630

Table 6 The list of internal resonances (IR) of homogeneous and FG nanorods for clamped-clamped BCs

With Inertia and shear effects											
Homogeneous nanorod, Gradient index (m) = 0											
Two-to-one						Three-to-one					
μ = 0		μ = 0.05 (nm ²)		μ = 0.1 (nm ²)		μ = 0		μ = 0.05 (nm ²)		μ = 0.1 (nm ²)	
IR	Y _{Max}	IR	Y _{Max}	IR	Y _{Max}	IR	Y _{Max}	IR	Y _{Max}	IR	Y _{Max}
ω ₂ =2ω ₁	0.124408	ω ₂ =2ω ₁	0.156341	ω ₂ =2ω ₁	0.182475	ω ₂ =3ω ₁	0.860490	ω ₂ =3ω ₁	0.871517	ω ₂ =3ω ₁	0.882312
ω ₃ =2ω ₂	0.500769	ω ₃ =2ω ₂	0.515297	ω ₃ =2ω ₂	0.529077	ω ₃ =3ω ₂	1.231476	ω ₃ =3ω ₂	1.271353	ω ₃ =3ω ₂	1.311177
ω ₄ =2ω ₃	0.525552	ω ₄ =2ω ₃	0.548378	ω ₄ =2ω ₃	0.569769	ω ₃ =3ω ₁	0.111707	ω ₃ =3ω ₁	0.141701	ω ₃ =3ω ₁	0.165814
ω ₅ =2ω ₄	0.565222	ω ₅ =2ω ₄	0.613927	ω ₅ =2ω ₄	0.661681	ω ₄ =3ω ₂	0.346666	ω ₄ =3ω ₂	0.362047	ω ₄ =3ω ₂	0.376163
ω ₄ =2ω ₂	0.092880	ω ₄ =2ω ₂	0.119885	ω ₄ =2ω ₂	0.140780	ω ₅ =3ω ₃	0.356071	ω ₅ =3ω ₃	0.375408	ω ₅ =3ω ₃	0.392730
ω ₅ =2ω ₃	0.172266	ω ₅ =2ω ₃	0.188429	ω ₅ =2ω ₃	0.202088	ω ₅ =3ω ₂	0.160870	ω ₅ =3ω ₂	0.178334	ω ₅ =3ω ₂	0.193177
FG nanorod, Gradient index (m) = 5											
ω ₂ =2ω ₁	0.178014	ω ₂ =2ω ₁	0.205620	ω ₂ =2ω ₁	0.229199	ω ₂ =3ω ₁	0.868030	ω ₂ =3ω ₁	0.882763	ω ₂ =3ω ₁	0.896591
ω ₃ =2ω ₂	0.484035	ω ₃ =2ω ₂	0.496175	ω ₃ =2ω ₂	0.507088	ω ₃ =3ω ₂	1.174035	ω ₃ =3ω ₂	1.195051	ω ₃ =3ω ₂	1.216085
ω ₄ =2ω ₃	0.497453	ω ₄ =2ω ₃	0.488846	ω ₄ =2ω ₃	0.417658	non	-	non	-	ω ₄ =3ω ₃	1.233300
ω ₅ =2ω ₄	0.519327	ω ₅ =2ω ₄	0.412262	ω ₅ =2ω ₄	0.083496	non	-	non	-	ω ₅ =3ω ₄	0.129810
ω ₄ =2ω ₂	0.090780	ω ₄ =2ω ₂	0.117797	ω ₄ =2ω ₂	0.126680	ω ₃ =3ω ₁	0.131174	ω ₃ =3ω ₁	0.159707	ω ₃ =3ω ₁	0.181967
ω ₅ =2ω ₃	0.163884	ω ₅ =2ω ₃	0.162986	ω ₅ =2ω ₃	0.056263	ω ₄ =3ω ₂	0.332890	ω ₄ =3ω ₂	0.339325	ω ₄ =3ω ₂	0.321054
non	-	non	-	non	-	ω ₅ =3ω ₃	0.336442	ω ₅ =3ω ₃	0.312006	ω ₅ =3ω ₃	0.098083
non	-	non	-	non	-	ω ₅ =3ω ₂	0.153758	ω ₅ =3ω ₂	0.158261	ω ₅ =3ω ₂	0.060531

Internal resonances are one of the main characteristics of nonlinear systems, the study of which is always important in the behavior of a vibrational system. Therefore, in this research, the internal resonances of thick

homogeneous and FG nanorods and the exact amplitude of their occurrence with and without inertial and shear effects have been calculated. For this aim, both clamped-clamped and clamped-free boundary conditions are considered.

Table 7 The list of internal resonances (IR) of homogeneous and FG nanorods for clamped-clamped BCs

Without Inertia and shear effects											
Homogeneous nanorod, Gradient index (m) = 0											
Two-to-one						Three-to-one					
$\mu = 0$		$\mu = 0.05 \text{ (nm}^2\text{)}$		$\mu = 0.1 \text{ (nm}^2\text{)}$		$\mu = 0$		$\mu = 0.05 \text{ (nm}^2\text{)}$		$\mu = 0.1 \text{ (nm}^2\text{)}$	
IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}
$\omega_2=2\omega_1$	0.000207	$\omega_2=2\omega_1$	0.070145	$\omega_2=2\omega_1$	0.098920	$\omega_2=3\omega_1$	0.644227	$\omega_2=3\omega_1$	0.653020	$\omega_2=3\omega_1$	0.661640
$\omega_3=2\omega_2$	0.445168	$\omega_3=2\omega_2$	0.460332	$\omega_3=2\omega_2$	0.474855	$\omega_3=3\omega_2$	1.835902	$\omega_3=3\omega_2$	2.037362	$\omega_3=3\omega_2$	2.292620
$\omega_4=2\omega_3$	0.701948	$\omega_4=2\omega_3$	0.761279	$\omega_4=2\omega_3$	0.823641	$\omega_3=3\omega_1$	0.000204	$\omega_3=3\omega_1$	0.070998	$\omega_3=3\omega_1$	0.099838
$\omega_4=2\omega_2$	0.000203	$\omega_4=2\omega_2$	0.071778	$\omega_4=2\omega_2$	0.100387	$\omega_4=3\omega_2$	0.330526	$\omega_4=3\omega_2$	0.347833	$\omega_4=3\omega_2$	0.363887
$\omega_5=2\omega_3$	0.176052	$\omega_5=2\omega_3$	0.197730	$\omega_5=2\omega_3$	0.216182	$\omega_5=3\omega_3$	0.459971	$\omega_5=3\omega_3$	0.498811	$\omega_5=3\omega_3$	0.536323
non	-	non	-	non	-	$\omega_5=3\omega_2$	0.146628	$\omega_5=3\omega_2$	0.168142	$\omega_5=3\omega_2$	0.186173
FG nanorod, Gradient index (m) = 5											
$\omega_2=2\omega_1$	0.094417	$\omega_2=2\omega_1$	0.118910	$\omega_2=2\omega_1$	0.138503	$\omega_2=3\omega_1$	0.647804	$\omega_2=3\omega_1$	0.652424	$\omega_2=3\omega_1$	0.656213
$\omega_3=2\omega_2$	0.441503	$\omega_3=2\omega_2$	0.449713	$\omega_3=2\omega_2$	0.453717	$\omega_3=3\omega_2$	2.004696	$\omega_3=3\omega_2$	2.060582	$\omega_3=3\omega_2$	2.010723
$\omega_4=2\omega_3$	0.703720	$\omega_4=2\omega_3$	0.716802	$\omega_4=2\omega_3$	0.681885	$\omega_3=3\omega_1$	0.059176	$\omega_3=3\omega_1$	0.094256	$\omega_3=3\omega_1$	0.117834
non	non	$\omega_4=2\omega_2$	0.073480	$\omega_4=2\omega_2$	0.100079	$\omega_4=3\omega_2$	0.327073	$\omega_4=3\omega_2$	0.337115	$\omega_4=3\omega_2$	0.339457
$\omega_5=2\omega_3$	0.172461	$\omega_5=2\omega_3$	0.187893	$\omega_5=2\omega_3$	0.188907	$\omega_5=3\omega_3$	0.451582	$\omega_5=3\omega_3$	0.460349	$\omega_5=3\omega_3$	0.426465
non	-	non	-	non	-	$\omega_5=3\omega_2$	0.143901	$\omega_5=3\omega_2$	0.161929	$\omega_5=3\omega_2$	0.168895

Table 8 The list of internal resonances (IR) of homogeneous and FG nanorods for clamped-free BCs

With Inertia and shear effects											
Homogeneous nanorod, Gradient index (m) = 0											
Two-to-one						Three-to-one					
$\mu = 0$		$\mu = 0.05 \text{ (nm}^2\text{)}$		$\mu = 0.1 \text{ (nm}^2\text{)}$		$\mu = 0$		$\mu = 0.05 \text{ (nm}^2\text{)}$		$\mu = 0.1 \text{ (nm}^2\text{)}$	
IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}
$\omega_3=2\omega_2$	0.455817	$\omega_3=2\omega_2$	0.469593	$\omega_3=2\omega_2$	0.482707	$\omega_2=3\omega_1$	0.143048	$\omega_2=3\omega_1$	0.179506	$\omega_2=3\omega_1$	0.209476
$\omega_4=2\omega_3$	0.525525	$\omega_4=2\omega_3$	0.542944	$\omega_4=2\omega_3$	0.559319	$\omega_3=3\omega_2$	0.995049	$\omega_3=3\omega_2$	1.010570	$\omega_3=3\omega_2$	1.025699
$\omega_5=2\omega_4$	0.539550	$\omega_5=2\omega_4$	0.568257	$\omega_5=2\omega_4$	0.595093	$\omega_4=3\omega_3$	1.443702	$\omega_4=3\omega_3$	1.528478	$\omega_4=3\omega_3$	1.617901
$\omega_5=2\omega_3$	0.161200	$\omega_5=2\omega_3$	0.179977	$\omega_5=2\omega_3$	0.195808	$\omega_4=3\omega_2$	0.316052	$\omega_4=3\omega_2$	0.332010	$\omega_4=3\omega_2$	0.346717
non	-	non	-	non	-	$\omega_5=3\omega_3$	0.366337	$\omega_5=3\omega_3$	0.383371	$\omega_5=3\omega_3$	0.398802
non	-	non	-	non	-	$\omega_5=3\omega_2$	0.094202	$\omega_5=3\omega_2$	0.122355	$\omega_5=3\omega_2$	0.143933
FG nanorod, Gradient index (m) = 5											
$\omega_3=2\omega_2$	0.437420	$\omega_3=2\omega_2$	0.437869	$\omega_3=2\omega_2$	0.437297	$\omega_2=3\omega_1$	0.442121	$\omega_2=3\omega_1$	0.471889	$\omega_2=3\omega_1$	0.499908
$\omega_4=2\omega_3$	0.493456	$\omega_4=2\omega_3$	0.503290	$\omega_4=2\omega_3$	0.498748	$\omega_3=3\omega_2$	1.028936	$\omega_3=3\omega_2$	1.037843	$\omega_3=3\omega_2$	1.050087
$\omega_5=2\omega_4$	0.528951	$\omega_5=2\omega_4$	0.487909	$\omega_5=2\omega_4$	0.230290	$\omega_4=3\omega_3$	1.187923	$\omega_4=3\omega_3$	1.151483	$\omega_4=3\omega_3$	1.054691
$\omega_5=2\omega_3$	0.156941	$\omega_5=2\omega_3$	0.166823	$\omega_5=2\omega_3$	0.119469	$\omega_4=3\omega_2$	0.299933	$\omega_4=3\omega_2$	0.313360	$\omega_5=3\omega_4$	0.468476
non	-	non	-	non	-	non	-	non	-	$\omega_4=3\omega_2$	0.321041
non	-	non	-	non	-	$\omega_5=3\omega_3$	0.352236	$\omega_5=3\omega_3$	0.349809	$\omega_5=3\omega_3$	0.241955
non	-	non	-	non	-	$\omega_5=3\omega_2$	0.076749	$\omega_5=3\omega_2$	0.095245	$\omega_5=3\omega_2$	0.069234

The effects of non-local parameters on the internal resonances of homogeneous and FG nanorods in both cases with and without inertial and shear effects have been fully investigated. The results of this study are shown in Tables 6-9. Table 6 shows a list of internal resonances generated (3:1 (three-to-one) and 2:1 (two-to-one)) in homogeneous and FG nanorods for clamped-clamped boundary conditions in the presence of inertial and shear effects for different

non-local factor values.

The number of internal resonances occurred for the 2:1 state in both homogeneous and FG nanorods is the same for all non-local factors. On the other hand, the number of internal resonances generated in the 3:1 state is the same for both homogeneous and FG nanorods in terms of non-local factors equal to 0 and 0.05 (nm²). While for the non-local factor equal to 0.1 (nm²) in this ratio (3:1) we see an

Table 9 The list of internal resonances (IR) of homogeneous and FG nanorods for clamped-free BCs

Without Inertia and shear effects											
Homogeneous nanorod, Gradient index (m) = 0											
Two-to-one						Three-to-one					
$\mu = 0$		$\mu = 0.05$ (nm ²)		$\mu = 0.1$ (nm ²)		$\mu = 0$		$\mu = 0.05$ (nm ²)		$\mu = 0.1$ (nm ²)	
IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}	IR	Y_{Max}
$\omega_3=2\omega_2$	0.349374	$\omega_3=2\omega_2$	0.360483	$\omega_3=2\omega_2$	0.371060	non	-	$\omega_2=3\omega_1$	0.073338	$\omega_2=3\omega_1$	0.103535
$\omega_4=2\omega_3$	0.523617	$\omega_4=2\omega_3$	0.545456	$\omega_4=2\omega_3$	0.566382	$\omega_3=3\omega_2$	0.847452	$\omega_3=3\omega_2$	0.862532	$\omega_3=3\omega_2$	0.877293
$\omega_5=2\omega_4$	0.956914	$\omega_5=2\omega_4$	1.136316	$\omega_5=2\omega_4$	1.386393	$\omega_4=3\omega_2$	0.258411	$\omega_4=3\omega_2$	0.272871	$\omega_4=3\omega_2$	0.286175
$\omega_5=2\omega_3$	0.135793	$\omega_5=2\omega_3$	0.157791	$\omega_5=2\omega_3$	0.176040	$\omega_5=3\omega_3$	0.383564	$\omega_5=3\omega_3$	0.405624	$\omega_5=3\omega_3$	0.426023
non	-	non	-	non	-	non	-	$\omega_5=3\omega_2$	0.072673	$\omega_5=3\omega_2$	0.101466
FG nanorod, Gradient index (m) = 5											
$\omega_3=2\omega_2$	0.357291	$\omega_3=2\omega_2$	0.367417	$\omega_3=2\omega_2$	0.376112	$\omega_2=3\omega_1$	0.166136	$\omega_2=3\omega_1$	0.180979	$\omega_2=3\omega_1$	0.194303
$\omega_4=2\omega_3$	0.507426	$\omega_4=2\omega_3$	0.507319	$\omega_4=2\omega_3$	0.496947	$\omega_3=3\omega_2$	0.891402	$\omega_3=3\omega_2$	0.903695	$\omega_3=3\omega_2$	0.912570
$\omega_5=2\omega_4$	1.141740	$\omega_5=2\omega_4$	1.309946	$\omega_5=2\omega_4$	1.093135	$\omega_4=3\omega_2$	0.257458	$\omega_4=3\omega_2$	0.267791	$\omega_4=3\omega_2$	0.274385
$\omega_5=2\omega_3$	0.134401	$\omega_5=2\omega_3$	0.154367	$\omega_5=2\omega_3$	0.165126	$\omega_5=3\omega_3$	0.382323	$\omega_5=3\omega_3$	0.391641	$\omega_5=3\omega_3$	0.384769
non	-	non	-	non	-	$\omega_5=3\omega_2$	0.019225	$\omega_5=3\omega_2$	0.077885	$\omega_5=3\omega_2$	0.103443

increase in internal resonances created in FG nanorods relative to homogeneity. The list of internal resonances generated in homogeneous and FG clamped-clamped nanorods without the presence of inertial and shear effects for different values of nonlocal factors is shown in Table 7. The number of internal resonances generated in 2:1 mode for the local state without the presence of inertial and shear effects in homogeneous nanorods is higher than FG. For the 3:1 state of internal resonances, we see the same results for both homogeneous and FG clamped-clamped nanorods without the presence of inertial and shear effects.

Finally, the internal resonances of homogeneous and FG nanorods for clamped-free boundary conditions have also been analyzed. Tables 8 and 9 are provided for this purpose. The number of internal resonances generated for homogeneous and FG clamped-free nanorods in the 2:1 mode is the same in the presence of inertial and shear effects for various non-local factors. The number of internal resonances generated for homogeneous and FG clamped-free nanorods in the 3:1 mode with the presence of inertial and shear effects is the same for the non-local factors of 0 and 0.05, while for the non-local factor is equal to 0.1, The number of internal resonances generated for the FG nanorod is greater than the homogeneity. The same results are re-extracted without the presence of inertial and shear effects.

The number of local and non-local internal resonances generated for the 2:1 mode is the same for homogeneous and FG nanorods. But the results for the 3:1 mode are such that the number of locally generated internal resonances is different for both homogeneous and FG nanorods, while for the non-local state the results are similar.

Existence of inertial and shear effects in the analysis of nonlinear nonlocal axial vibrations of homogeneous and FG nanorods has increased the number of internal modals interactions, which is important depending on the application of the nanorod. These results are true for both clamped-clamped and clamped-free boundary conditions.

5. Conclusions

In this research, a new theoretical-analytical modeling of FG nanorods is performed by considering large-amplitudes for deriving analytical relationships of internal resonances in the presence of inertial and shear effects in non-local space. These analytical relationships are computed for the public state in terms of rod-type, so can be generalized to almost all homogeneous and FG nanorods exposed to axial vibrations in order to extract internal resonances. The linear governing equations of FG nanorod are discretized by HDQ method. And also, the nonlinear frequencies and internal resonances of the FG nanorod have been computed employing simultaneous Multi Galerkin (MG) and Multiple Scale (MS) methods. The internal resonances of homogeneous and FG nanorods exposed to nonlinear axial vibrations are comprehensively investigated and its results are fully presented in terms of clamped-clamped and clamped-free boundary conditions. In addition, the effects of important factors such as nonlocal coefficients, inertial and shear effects on the internal modal interactions of homogeneous and FG nanorods for various states are evaluated and compared. The results obtained in this paper can be used to fabricate nanostructures of rod-types exposed to the axial vibrations and are also applicable to the analysis of forced vibrations and experimental works.

Data availability statement

The raw and processed data required to reproduce these findings cannot be shared at this time due to technical or time limitations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have

appeared to influence the work reported in this paper.

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Appendix

The variables of Eq. (6) are defined as follows, Eq. (a):

where:

$$\begin{cases} Z_1(x) = \frac{E(x)v}{(1+v)} + 2G(x) \\ Z_2(x) = \frac{E(x)v}{(1+v)(1-2v)} \left(\frac{1}{2} + v^2 \right) + G(x) \\ Z_{22}(x) = v^2(y^2 + z^2) \left(\frac{E(x)v}{2(1+v)(1-2v)} + G(x) \right) \\ Z_4(x) = \frac{E(x)v}{(1+v)(1-2v)} \left(\frac{1}{2} + v^2 \right) + v^2G(x) \\ Z_{44}(x) = v^2(y^2 + z^2) \left(\frac{E(x)v}{2(1+v)(1-2v)} \right) \end{cases} \quad (a)$$

The assumptions of Eqs. (9) and (10) are as follows, Eq.

(b)

$$\begin{aligned} N_{xx} &= \int_A \sigma_{xx} dA \\ &= AE(x) \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} v^2 (y^2 + z^2) \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right] \\ N_{yy} = N_{zz} &= \int_A \sigma_{yy} dA = \int_A \sigma_{zz} dA \\ &= AE(x) \left[-v \frac{\partial u}{\partial x} + \frac{1}{2} v^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] \\ M_{xy} &= \int_A \sigma_{xy} y dA \\ &= AE(x) y \left[-\frac{1}{2} v y \left(\frac{\partial^2 u}{\partial x^2} \right) + \frac{1}{2} v^2 y \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \right] \\ M_{xz} &= \int_A \sigma_{xz} z dA \\ &= AE(x) z \left[-\frac{1}{2} v z \left(\frac{\partial^2 u}{\partial x^2} \right) + \frac{1}{2} v^2 z \frac{\partial u}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} \right) \right] \end{aligned} \quad (b)$$

All the variables of Eq. (23) are determined as follows,

Eq. (c).

$$\begin{aligned} B'_1(x) &= -A \frac{\partial Z_1(x)}{\partial x}; B'_2(x) = v^2 I_p \frac{\partial^2 G(x)}{\partial x^2} - AZ_1(x) \\ B'_3(x) &= 2vA \frac{\partial Z_4(x)}{\partial x} - A \frac{\partial Z_2(x)}{\partial x} - A \frac{\partial Z_1(x)}{\partial x} \\ B'_4(x) &= \left[\begin{aligned} &4vAZ_4(x) - 2AZ_2(x) - 2AZ_1(x) - \\ &2v^3 I_p \frac{\partial^2 G(x)}{\partial x^2} + v^2 I_p \frac{\partial^2 Z_1(x)}{\partial x^2} \end{aligned} \right] \\ B'_5(x) &= -A \frac{\partial Z_2(x)}{\partial x} - 2Av^2 \frac{\partial Z_4(x)}{\partial x} \\ B'_6(x) &= \left[\begin{aligned} &v^4 I_p \frac{\partial^2 G(x)}{\partial x^2} - 3AZ_2(x) - 6Av^2 Z_4(x) \\ &+ v^2 I_p \frac{\partial^2 Z_2(x)}{\partial x^2} \end{aligned} \right] \\ B'_7(x) &= 2v^2 I_p \frac{\partial G(x)}{\partial x}; B'_8(x) = v^2 I_p G(x) \\ B'_9(x) &= \left[\begin{aligned} &-3v^3 I_p \frac{\partial G(x)}{\partial x} + 2v^2 I_p \frac{\partial Z_1(x)}{\partial x} - A \frac{\partial Z_{22}(x)}{\partial x} \\ &+ 2vA \frac{\partial Z_{44}(x)}{\partial x} \end{aligned} \right] \\ B'_{10}(x) &= -4v^3 I_p \frac{\partial G(x)}{\partial x} + 2v^2 I_p \frac{\partial Z_1(x)}{\partial x} \\ B'_{11}(x) &= \left[\begin{aligned} &-4v^3 I_p G(x) + 3v^2 I_p Z_1(x) - 2AZ_{22}(x) \\ &+ 4vAZ_{44}(x) \end{aligned} \right] \\ B'_{12}(x) &= \left[\begin{aligned} &3v^4 I_p \frac{\partial G(x)}{\partial x} + 4v^2 I_p \frac{\partial Z_2(x)}{\partial x} - A \frac{\partial Z_{22}(x)}{\partial x} \\ &- 2v^2 A \frac{\partial Z_{44}(x)}{\partial x} \end{aligned} \right] \\ B'_{13}(x) &= -2v^3 I_p G(x) + v^2 I_p Z_1(x) \\ B'_{14}(x) &= \left[\begin{aligned} &v^4 I_p G(x) + v^2 I_p \frac{\partial^2 Z_{22}(x)}{\partial x^2} - AZ_{22}(x) - \\ &2v^2 AZ_{44}(x) + 2v^2 I_p Z_2(x) \end{aligned} \right] \\ B'_{15}(x) &= \left[\begin{aligned} &4v^4 I_p G(x) + 6v^2 I_p Z_2(x) - 2AZ_{22}(x) \\ &- 4v^2 AZ_{44}(x) \end{aligned} \right] \\ B'_{16}(x) &= 2v^4 I_p \frac{\partial G(x)}{\partial x} + 2v^2 I_p \frac{\partial Z_2(x)}{\partial x} \\ B'_{17}(x) &= v^4 I_p G(x) + v^2 I_p Z_2(x) \\ B'_{18}(x) &= 6v^2 I_p \frac{\partial Z_{22}(x)}{\partial x}; B'_{19}(x) = 6v^2 I_p \frac{\partial Z_{22}(x)}{\partial x} \\ B'_{20}(x) &= 3v^2 I_p Z_{22}(x); B'_{21}(x) = \rho(x)A - \mu A \frac{\partial^2 \rho(x)}{\partial x^2} \\ B'_{22}(x) &= \mu v^2 I_p \frac{\partial^3 \rho(x)}{\partial x^3} - v^2 I_p \frac{\partial \rho(x)}{\partial x} - 2\mu A \frac{\partial \rho(x)}{\partial x} \\ B'_{23}(x) &= 3\mu v^2 I_p \frac{\partial^2 \rho(x)}{\partial x^2} - v^2 I_p \rho(x) - \mu A \rho(x) \\ B'_{24}(x) &= 3\mu v^2 I_p \frac{\partial \rho(x)}{\partial x}; B'_{25}(x) = \mu v^2 I_p \rho(x) \end{aligned} \quad (c)$$

In Eq. (25), Y_{max} is a maximum amplitude corresponding to the time dependent function $Y(t)$ and the following relations are implemented (Eqs. (d1-d4))

$$I = \left[\begin{aligned} &\int_0^1 B'_1(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} dX \\ &+ \int_0^1 B'_2(X) \bar{\psi}_m(X) \frac{d^2 \bar{\psi}_i(X)}{dX^2} dX \\ &+ \int_0^1 B'_7(X) \bar{\psi}_m(X) \frac{d^3 \bar{\psi}_i(X)}{dX^3} dX \\ &+ \int_0^1 B'_8(X) \bar{\psi}_m(X) \frac{d^4 \bar{\psi}_i(X)}{dX^4} dX \end{aligned} \right] \quad (d1)$$

$$\Pi = \left[\begin{aligned} & \int_0^1 B'_{21}(X) \bar{\psi}_m(X) \bar{\psi}_i(X) dX \\ & + \int_0^1 B'_{22}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} dX \\ & + \int_0^1 B'_{23}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} dX \\ & + \int_0^1 B'_{24}(X) \bar{\psi}_m(X) \frac{d^3\bar{\psi}_i(X)}{dX^3} dX \\ & + \int_0^1 B'_{25}(X) \bar{\psi}_m(X) \frac{d^4\bar{\psi}_i(X)}{dX^4} dX \end{aligned} \right] \quad (d2)$$

$$\beta_{mij} = \left[\begin{aligned} & \int_0^1 B'_3(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d\bar{\psi}_j(X)}{dX} dX \\ & + \int_0^1 B'_4(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d^2\bar{\psi}_j(X)}{dX^2} dX \\ & + \int_0^1 B'_9(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^2\bar{\psi}_j(X)}{dX^2} dX \\ & + \int_0^1 B'_{10}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d^3\bar{\psi}_j(X)}{dX^3} dX \\ & + \int_0^1 B'_{11}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^3\bar{\psi}_j(X)}{dX^3} dX \\ & + \int_0^1 B'_{13}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d^4\bar{\psi}_j(X)}{dX^4} dX \end{aligned} \right] \quad (d3)$$

$$\alpha_{mijk} = \left[\begin{aligned} & \int_0^1 B'_5(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d\bar{\psi}_j(X)}{dX} \frac{d\bar{\psi}_k(X)}{dX} dX \\ & + \int_0^1 B'_6(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d\bar{\psi}_j(X)}{dX} \frac{d^2\bar{\psi}_k(X)}{dX^2} dX \\ & + \int_0^1 B'_{12}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d^2\bar{\psi}_j(X)}{dX^2} \frac{d^2\bar{\psi}_k(X)}{dX^2} dX \\ & + \int_0^1 B'_{14}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^2\bar{\psi}_j(X)}{dX^2} \frac{d^2\bar{\psi}_k(X)}{dX^2} dX \\ & + \int_0^1 B'_{15}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d^2\bar{\psi}_j(X)}{dX^2} \frac{d^3\bar{\psi}_k(X)}{dX^3} dX \\ & + \int_0^1 B'_{16}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d\bar{\psi}_j(X)}{dX} \frac{d^3\bar{\psi}_k(X)}{dX^3} dX \\ & + \int_0^1 B'_{17}(X) \bar{\psi}_m(X) \frac{d\bar{\psi}_i(X)}{dX} \frac{d\bar{\psi}_j(X)}{dX} \frac{d^4\bar{\psi}_k(X)}{dX^4} dX \\ & + \int_0^1 B'_{18}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^2\bar{\psi}_j(X)}{dX^2} \frac{d^3\bar{\psi}_k(X)}{dX^3} dX \\ & + \int_0^1 B'_{19}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^3\bar{\psi}_j(X)}{dX^3} \frac{d^3\bar{\psi}_k(X)}{dX^3} dX \\ & + \int_0^1 B'_{20}(X) \bar{\psi}_m(X) \frac{d^2\bar{\psi}_i(X)}{dX^2} \frac{d^2\bar{\psi}_j(X)}{dX^2} \frac{d^4\bar{\psi}_k(X)}{dX^4} dX \end{aligned} \right] \quad (d4)$$