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Finite element analysis for longitudinal vibration of nanorods based on doublet mechanics

Ufuk Gul* and Metin Aydogdu

Department of Mechanical Engineering, Trakya University, 22030, Edirne, Turkey

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Abstract. In the present study, the axial vibration of the nanorods is investigated in the framework of the doublet mechanics theory. The equations of motion and boundary conditions of nanorods are derived by applying the Hamilton principle. A finite element method is developed to obtain the vibration frequencies of nanorods for different boundary conditions. A two-noded higher order rod finite element is used to solve the vibration problem. The natural frequencies of nanorods obtained with the present finite element analysis are validated by comparing the results of classical doublet mechanics and nonlocal strain gradient theories. The effects of rod length, mode number and boundary conditions on the axial vibration frequencies of nanorods are examined in detail. Mode shapes of the nanorods are presented for the different boundary conditions. It is shown that the doublet mechanics model can be used for the dynamic analysis of nanotubes, and the presented finite element formulation can be used for mechanical problems of rods with unavailable analytical solutions. These new results can also be used as references for the future studies.

Keywords: doublet mechanics; finite element method; mode shape; nanorod; vibration

1. Introduction

With advancement in the nanotechnology, micro/ nanoscale structures have been used in micro/nano scale electro-mechanical structures (MEMS/NMES) such as atomic force microscope, mass sensors, actuators, transistors, gyroscopes, and generators (Hutchinson 2000, Lam *et al*. 2003). A one-dimensional nanorod is also a structural component of these structures. In this manner, a deep understanding has become inevitable to obtain accurate formulations and convenient solution methods for mechanical behaviors of the nanorod structures.

Classical elasticity theory fails to account the scale effect of micro/nanostructures. Thus, same accurate and proper solution techniques have been applied to model of size-dependent structural elements. In general, these solution techniques can be grouped into three groups which are experiments (Anderson and Lakes 1994, Fleck *et al*. 1994), molecular dynamics (MD) simulation (Sears and Batra, 2004) and higher order size-dependent continuum theories. Experimental studies and MD simulation give significant and acceptable results in mechanical analysis of nanostructures. However, experimental approaches have some difficulties related to control and implementation at nano-scale. Also, MD simulations require a great amount of computational load due to its complicated simulation processes while studying the mechanical properties of nano-structures. For this reason, researchers usually focus on size dependent continuum theories to determine the mechanical responses of the micro/nanostructures considering the small size effects which have a direct effect on the stiffness of the structures. Among these size-dependent continuum theories, couple stress theory (Mindlin and Tiersten 1962), nonlocal elasticity theory (Eringen 1972a, b), strain gradient theory (Aifantis 1999) and nonlocal strain gradient models (Zhang *et al*. 2009, Challamel 2013, Lim *et al*. 2015) are mostly used in the open literature.

Nonlocal elasticity theory considers the long-range atomic interaction in the material and it gives fast results of mechanical analysis of size-dependent structures. However, differential form of nonlocal elasticity model could not catch the size effect in nanomaterials based on the analysis for the cantilever beams (Eringen 1972a). The dynamic analyses of one-dimensional nanorods have been performed via nonlocal elasticity theory. In this context, Aydogdu (2009) investigated the longitudinal vibration of nanorods based on nonlocal elasticity theory. Murmu *et al*. (2011) studied the torsional vibration of carbon nanotube-buckyball system using the nonlocal elasticity model. Axial vibration of single-walled carbon nanotube embedded in an elastic matrix has been studied in the framework of the nonlocal elasticity theory by Aydogdu (2012). The effects of nonlocal parameters on the longitudinal vibration of carbon nanotubes were revealed in that paper. In another study, free and forced longitudinal vibration analyses of damped nanorods have been studied based on nonlocal elasticity model (Adhikari *et al*. 2013). A new dynamic finite element method has been proposed for the vibration of damped and undamped nanorod systems. The results of the proposed dynamic finite element method were compared to the conventional finite element results in that paper. Demir and Civalek (2013) investigated the torsional and axial vibration of microtubules using the nonlocal elasticity model. They used finite element method in the solution. Adhikari *et al*.

*Corresponding author, Ph.D.,
E-mail: ufukgul@trakya.edu.tr

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(2014) developed a nonlocal dynamic finite element method for axial vibration of nanorods embedded in an elastic medium. They obtained the natural frequencies for the general boundary conditions of the nanorods. Torsional vibration of nanorods embedded in an elastic medium has been examined by using the nonlocal finite element method (Numanoğlu and Civalek 2019).

In addition to nonlocal elasticity theory, there are also some size-dependent continuum theories to predict dynamic behavior of nanostructures. Kahrobaiyan *et al*. (2013) investigated the axial static and dynamic behaviors of microbars using strain gradient theory. Closed form solutions have been obtained for the static deformation and natural frequencies of microbars. Akgöz and Civalek (2014) investigated the axial vibration of microbars via strain gradient elasticity theory. Additionally, the propagation of axial stress waves was researched using strain gradient elasticity theory by Güven (2014). Love-bishop rod model has been used in the formulation. Mustapha and Ruan (2015) studied the longitudinal vibration of a microbar embedded in magnetic fields with mass-spring attachments based on strain gradient theory. Li *et al*. (2016) examined the longitudinal vibration analysis of small-scaled rods using nonlocal strain gradient theory. The analytical and finite element solutions of vibration problem have been compared for general boundary conditions and hardening or softening material responses occurred depending on the magnitude of the nonlocal parameter in the small-scaled rods. Eghbali *et al*. (2021) studied the free vibration of axially functionally graded nanobeam with an attached mass in the framework of the nonlocal strain gradient theory. The vibration frequencies of nanobeam have been obtained by using the first order of numerical asymptotic development method in that paper. Recently, forced vibration of sandwich beams and free vibration of size-dependent cracked piezoelectric nanobeams have been investigated in Refs. (Eghbali *et al*. 2022a, b).

In the last decades, a size-dependent continuum theory called as doublet mechanics (DM) has been employed to investigate mechanical responses of nanostructures. The DM theory has been developed by Granik (1978) for microstructural mechanics of granular media. DM theory considers the microstrains and microstresses for elongation, shear, and torsional deformations. In DM, material length scale parameter is taken as the bonding length of the atoms within the microstructure of the material, and it appears directly in the DM formulation. At this point, discrete of the considered material would be taken into account in DM theory. The main difference of the DM model from other size-dependent continuum theories mentioned above is the determination of the material length scale parameter of the micro/nanostructures. Determination of the material length scale parameter is still an open issue for couple stress theories (Khorshidi 2018), nonlocal elasticity theory, and strain gradient theories. However, material length scale parameter is taken as bonding length atoms in the considered microstructure in DM theory. This makes DM more physical theory compared to other size-dependent continuum models. Granik and Ferrari (1993) extended DM theory for the mechanics of granular media. Kojic and co-workers (2011) developed a finite element solution to determine the mechanical behavior of the microstructural materials. Vajari and Imam (2016) studied the longitudinal vibration of single-walled carbon nanotubes in the framework of DM theory. Gul *et al*. (2017) studied the longitudinal vibration behavior of nanorods using DM theory. Axial and transverse dynamics of nanorods and nanobeams have been investigated based on DM theory (Gul and Aydogdu 2018). Gul and Aydogdu (2019) investigated the axial vibration of nanorods using DM theory. They derived the governing equation and corresponding boundary conditions for Love rod model in that paper. Moreover, same authors (Aydogdu and Gul 2020) studied the axial vibration of double nanorod systems based on DM theory. Vajari and Azimzadeh (2020) investigated the axial vibration of single-walled carbon nanotubes with damping and elastic boundary conditions. Recently, static and dynamic analyses of the zigzag and armchair nanobeams have been researched by Karamanli (2021) via DM theory. In that paper, using a third shear deformation beam theory, a finite element model has been developed for general boundary conditions.

It is known that when there is no analytical solution for a mechanics problem, approximate techniques are used such as finite element method. Based on the studies afore-mentioned above, there is no study related to the finite element solution of vibration problem in nanorods based on DM theory. The purpose of this paper is to study the free axial vibration of nanorods based on scale-dependent DM theory using finite element method. Therefore, extensive studies are implemented to reveal the effects of material length scale paramater of the DM theory on dynamic behaviours of nanorods with different rod length, boundary conditions and mode numbers. This paper presents a finite element solution of free axial vibration of carbon nanotubes in the framework of DM for the first time. It is the main contribution and novelty of this paper. In the present study, the numerical results obtained from the DM theory using the finite element solution are compared to classical DM results and nonlocal strain gradient results for validation. It is obtained that the vibration frequencies are predicted by the finite element solution of DM agree well with the analytical solution of DM and classical theory for given boundary conditions.

2. Doublet mechanics equations

Two adjacent atoms (nodes) in the considered material are called as a doublet and the distance between two atoms are defined as doublet distance (η) or material length scale parameter in DM theory. Each node (atom) in α-doublet has longitudinal (p_α), torsional (m_α) and shear (t_α) micro-deformations. It is assumed that only longitudinal micro-stress has been included in the micro deformation relations in the DM model for simplicity. The inclusion of the other two micro-stresses (torsional and shear) in the formulation increases the order of the governing equation and the solution of the governing equation becomes complex. The increment of longitudinal displacement (Δu_α) in scalar form

for each doublet can be defined as (Granik and Ferrari 1993):

$$\Delta u_\alpha = \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^\chi}{\chi!} \tau^o_{\alpha k_1} \dots \tau^o_{\alpha k_\chi} \frac{\partial^\chi u_i}{\partial \chi_{k_1} \dots \partial \chi_{k_\chi}} \tag{1}$$

Here, $k_1, \dots k_\chi$ are equal to 1, 2 and 3 respect to three-dimensional Cartesian coordinate system, η_α is the material length scale parameter (or doublet separation distance) in the considered material, τ^o_α is the unit vector of α-axis, and *M* is the number of terms in Taylor series expansion. Another assumption of the present model is related to the number of the terms in Taylor series expansion while defining the micro-deformations in DM theory. In this study, *M*=3 is taken into account in the Taylor series expansion in DM theory for the present zigzag nanotube model. It is notable that the number of the terms in Taylor series expansion does not represent the accuracy of the approach and it may take different values according to the micro-structure of the considered material. *M*=3 terms in Taylor series expansion give acceptable vibration results for the present nanotube model. It should be noted that the value of Δu_α is smaller than η_α yields $\tau_\alpha = \tau^o_\alpha$. The longitudinal micro-strain (ϵ_α) in terms of the unit vector in α-direction can be written as

$$\epsilon_\alpha = \tau^o_{\alpha i} \sum_{\chi=1}^{M} \frac{(\eta^o_\alpha)^{\chi-1}}{\chi!} \tau^o_{\alpha k_1} \dots \tau^o_{\alpha k_\chi} \frac{\partial^\chi u_i}{\partial \chi_{k_1} \dots \partial \chi_{k_\chi}} \tag{2}$$

If *M*=3, the longitudinal micro-strain is obtained as:

$$\epsilon_\alpha = \tau^o_{\alpha i}\tau^o_{\alpha j}\frac{\partial u_i}{\partial x_j} + \tau^o_{\alpha i}\frac{\eta_\alpha}{2}\tau^o_{\alpha j}\tau^o_{\alpha k}\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \tau^o_{\alpha i}\frac{{\eta_\alpha}^2}{6}\tau^o_{\alpha j}\tau^o_{\alpha k}\tau^o_{\alpha l}\frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} \tag{3}$$

The relation between the longitudinal micro-stress and micro-strain can be written as (Granik and Ferrari 1993):

$$p_\alpha = \sum_\beta B_{\alpha\beta}\, \epsilon_\beta \tag{4}$$

where, $B_{\alpha\beta}$ represents the micro-moduli of the doublet and can be obtained as a constant (B_0) with the assumption of the plane stress condition (Gul and Aydogdu 2018):

$$B_0 = \frac{4}{9}\mu\frac{7\lambda + 10\mu}{\lambda + 2\mu} \tag{5}$$

Here, λ and μ denote the Lamé's constants and B_0 are defined with plane stress assumption as follows (Gul *et al*. 2017):

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad \lambda = 2\mu, \quad B_0 = \frac{8\mu}{3} = E. \tag{6}$$

where, E is the elasticity modulus and ν is the Poisson's ratio. If one selects $\nu = 1/3$, the micro-moduli is obtained as $B_0 = E$. Then, the longitudinal micro-strain for *M*=3 can be written as

$$\epsilon_\alpha = \tau^o_{\alpha i}\tau^o_{\alpha j}\frac{\partial u_i}{\partial x_j} + \tau^o_{\alpha i}\frac{\eta_\alpha}{2}\tau^o_{\alpha j}\tau^o_{\alpha k}\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \tau^o_{\alpha i}\frac{{\eta_\alpha}^2}{6}\tau^o_{\alpha j}\tau^o_{\alpha k}\tau^o_{\alpha l}\frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} \tag{7}$$

Similarly, the longitudinal micro-stress for *M*=3 can be written as

$$p_\alpha = B_0\tau^0_{\alpha m}\tau^0_{\alpha n}\left(\varepsilon_{mn} + \frac{1}{2}\eta_\alpha\tau^0_{\alpha s}\frac{\partial \varepsilon_{mn}}{\partial x_s} + \frac{1}{6}\eta_\alpha^2\tau^0_{\alpha t}\tau^0_{\alpha s}\frac{\partial^2 \varepsilon_{mn}}{\partial x_t \partial x_s}\right) \tag{8}$$

With the assumption of the plane stress condition, the axial macro-stress can be expressed as follows (Gul and Aydogdu 2018):

$$\sigma_{k_1 i}^{(M)} = \sum_{\alpha=1}^{M} \tau^o_{\alpha k_1} \sum_{\chi=1}^{M} (-1)^{\chi+1}\left[\frac{(\eta_\alpha)^{\chi-1}}{\chi!}\tau^o_{\alpha k_2} \dots \tau^o_{\alpha k_\chi}\frac{\partial^{\chi-1} p_{\alpha i}}{\partial \chi_{k_2} \dots \partial \chi_{k_\chi}}\right] \tag{9}$$

The axial macro-stress of DM theory in Eq. (9) incorporates the material length scale parameter η_α is equal to 0.1421 nm in carbon nanotubes (CNTs). Eq. (9) can be used for the three dimensional formulations. For the plane stress problem, the relation between micro-stress and macro-stress can be expressed as (Granik and Ferrari, 1993):

$$\{\sigma\} = [H]\{p\} \tag{10}$$

where $[H]$ is a transformation matrix. For plane stress problem, macro-stress tensor, micro-stress tensor and transformation matrix can be expressed as

$$\{\sigma\} = \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}, \quad [H] = \begin{bmatrix}(\tau^0_{11})^2 & (\tau^0_{21})^2 & (\tau^0_{31})^2\\ (\tau^0_{12})^2 & (\tau^0_{22})^2 & (\tau^0_{32})^2\\ \tau^0_{11}\tau^0_{12} & \tau^0_{21}\tau^0_{22} & \tau^0_{31}\tau^0_{32}\end{bmatrix}, \quad \{p\} = \begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}. \tag{11}$$

Fig. 1 represents the nanotube consisting of three doublets with the same length and in this figure, the unit vectors $\vec{\tau}^0_{ij}$ which the cosines of the angles between the micro-stresses and Cartesian coordinates can be expressed as follows:

$$\begin{aligned}&\tau^0_{11} = cos\theta, \quad \tau^0_{12} = cos90, \quad \tau^0_{13} = sin\theta,\\ &\tau^0_{21} = -\sin(30-\theta), \quad \tau^0_{22} = cos90,\\ &\tau^0_{23} = -\cos(30-\theta), \quad \tau^0_{31} = -\cos(60-\theta),\\ &\tau^0_{32} = cos90, \quad \tau^0_{33} = \sin(60-\theta).\end{aligned} \tag{12}$$

In the present study, we consider the zigzag nanotube model for axial vibration problem. Accordingly, if one sets $\theta = 0°$ in Eq. (12), the unit vectors take the following form for the zigzag nanotube model:

$$\begin{aligned}&\tau^0_{11} = 1,\ \tau^0_{12} = 0,\ \tau^0_{13} = 0,\ \tau^0_{21} = -\tfrac{1}{2},\ \tau^0_{22} = 0,\\ &\tau^0_{23} = -\sqrt{3}/2,\ \tau^0_{31} = -1/2,\ \tau^0_{32} = 0,\ \tau^0_{33} = \sqrt{3}/2\end{aligned} \tag{13}$$

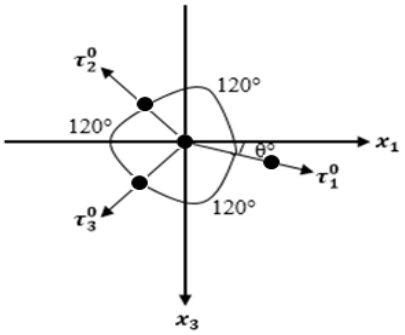

Fig. 1 Configuration of doublet nodes in a microstructure

Using Eqs. (8) and (9) and neglecting the terms with higher than η_α^2, Eq. (9) becomes

$$\sigma_{ij} = \sum_{\alpha=1}^{M} B_0 \tau_{\alpha i}^o \tau_{\alpha j}^o \tau_{\alpha m}^o \tau_{\alpha n}^o \left[\varepsilon_{mn} + \frac{\eta_\alpha^2}{12} \tau_{\alpha t}^o \tau_{\alpha s}^o \frac{\partial^2 \varepsilon_{mn}}{\partial x_t \partial x_t} \right] \quad (14)$$

Then, by using Eqs. (13) and (14), the axial macro-stress in the DM model can be obtained as

$$\sigma_{xx} = B_0 \left(\varepsilon_{xx} + \frac{\eta_\alpha^2}{12} \frac{\partial^2 \varepsilon_{xx}}{\partial x^2} + \frac{\eta_\alpha^2}{32} \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \right) \quad (15)$$

Considering only the axial strain in Eq. (15) and assuming $B_0 = E$ for plane stress condition, following axial macro-stress equation for the zigzag nanotube model can be obtained as follows:

$$\sigma_{xx} = E \left(\varepsilon_{xx} + \frac{\eta^2}{12} \frac{\partial^2 \varepsilon_{xx}}{\partial x^2} \right) \quad (16)$$

3. Variational formulation

The equations of motion of any continuum can be obtained using Newton's second law or Hamilton's principle. The latter generally preferred since it also leads to boundary conditions. The general form of Hamilton's principle can be written as (Reddy 2002):

$$\int_{t_0}^{t_1} \delta(U - T - W) dt = 0 \quad (17)$$

Here, U denotes the strain energy of the nanorod, T denotes the kinetic energy of the nanorod, and W denotes the total work done by external loads. The strain energy can be given by

$$U = \frac{1}{2} EA \int_0^L \left[\left(\frac{\partial u}{\partial x} \right)^2 - \frac{\eta^2}{12} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 \right] dx \quad (18)$$

where, A is the cross-sectional area of the rod, u is the axial displacement, L is the rod length and η is the material length scale parameter in DM theory. Considering the axial motion, the kinetic energy of the rod (T) can be expressed as

$$T = \frac{1}{2} \int_0^L \rho A \left(\frac{\partial u}{\partial t} \right)^2 dx \quad (19)$$

where ρ is the density of the rod and t is the time. The virtual total work done by external loads can be defined as

$$\int_{t_0}^{t_1} \delta W dt = \int_{t_0}^{t_1} \int_0^L f(x,t)\, \delta u dx dt + \int_{t_0}^{t_1} \{ [P \delta u]_0^L + [R \delta u']_0^L \} dt \quad (20)$$

Here, concentrated, double and distributed forces are denoted by P, R and f, respectively. Finally, Eq. (17) can be obtained as

$$\begin{aligned} &\int_{t_0}^{t_1} \int_0^L \left[AE \left(\frac{\partial^2 u}{\partial x^2} + \frac{\eta^2}{12} \frac{\partial^4 u}{\partial x^4} \right) + f - \rho A \frac{\partial^2 u}{\partial t^2} \right] \delta u dx dt \\ &+ \int_{t_0}^{t_1} \left[\left\{ P - AE \left(\frac{\partial u}{\partial x} + \frac{\eta^2}{12} \frac{\partial^3 u}{\partial x^3} \right) \right\} \delta u \right]_0^L dt \\ &+ \int_{t_0}^{t_1} \left[\left\{ R - AE \left(-\frac{\eta^2}{12} \frac{\partial^2 u}{\partial x^2} \right) \right\} \delta \frac{\partial u}{\partial x} \right]_0^L dt + \int_0^L \left[\rho A \frac{\partial u}{\partial t} \delta u \right]_{t_0}^{t_1} dx \\ &= 0 \end{aligned} \quad (21)$$

By equating to zero each bracket in Eq. (21), the equation of motion of the nanorod is obtained as

$$AE \left[\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\eta^2}{12} \frac{\partial^4 u(x,t)}{\partial x^4} \right] + f = \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \quad (22)$$

with classical and non-classical boundary conditions (at x=0 or x=L):

$$\begin{aligned} &\left[P(L,t) - AE \left[\frac{\partial u(L,t)}{\partial x} + \frac{\eta^2}{12} \frac{\partial^3 u(L,t)}{\partial x^3} \right] \right] \delta u(L,t) \\ &- \left[P(0,t) - AE \left[\frac{\partial u(0,t)}{\partial x} + \frac{\eta^2}{12} \frac{\partial^3 u(0,t)}{\partial x^3} \right] \right] \delta u(0,t) = 0, \\ &\left[R(L,t) - AE \left[-\frac{\eta^2}{12} \frac{\partial^2 u(L,t)}{\partial x^2} \right] \right] \delta \frac{\partial u(L,t)}{\partial x} \\ &- \left[R(0,t) - AE \left[-\frac{\eta^2}{12} \frac{\partial^2 u(0,t)}{\partial x^2} \right] \right] \delta \frac{\partial u(0,t)}{\partial x} = 0. \end{aligned} \quad (23)$$

It is noted that axial displacement, u and the axial force, $P = AE \left(\frac{\partial u}{\partial x} + \frac{\eta^2}{12} \frac{\partial^3 u}{\partial x^3} \right)$ are considered as the classical boundary conditions and the boundary strain u' with the double force, $R = AE \frac{\eta^2}{12} \frac{\partial^2 u}{\partial x^2}$ are considered as the non-classical boundary conditions of the nanorod in DM. These non-classical boundary conditions can be considered as change of the classical boundary conditions at the edge of the rod. The non-classical boundary conditions u' and R always be satisfied when the length scale parameter equal to zero. It is noted that the governing equation (22) is a fourth-order differential equation. It may be the reason why the non-classical boundary conditions are presented and must be used to solve the vibration behavior of the rod based on DM theory. When the length scale parameter η is ignored in Eqs. (22) and (23), classical rod theory can be obtained with classical boundary conditions.

4. Analytical solution of longitudinal vibration problem of nanorods

Assuming the harmonic vibration motion and using the separation of variables method, longitudinal displacement can be written as

$$u(x,t) = \varphi(x) e^{i\omega t} \quad (24)$$

where $i = \sqrt{-1}$, ω denotes the natural frequency. Inserting Eq. (24) into Eq. (22) yields

$$AE\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\eta^2}{12}\frac{\partial^4\varphi}{\partial x^4}\right] + \rho A\omega^2 = 0 \tag{25}$$

The dimensionless form of the previous equation can be determined as

$$\frac{\partial^2\varphi}{\partial\bar{x}^2} + \beta^2\frac{\partial^4\varphi}{\partial\bar{x}^4} + \lambda^2 U = 0 \tag{26}$$

where dimensionless material length scale parameter (β^2), dimensionless coordinate axis ($\bar{x}$) and dimensionless axial frequency parameter (λ^2) are

$$\beta^2 = \frac{\eta^2}{12L^2}, \quad \bar{x} = \frac{x}{L}, \quad \lambda^2 = \frac{\rho\omega^2 L^2}{E}. \tag{27}$$

The analytical solution of the Eq. (26) can be obtained as

$$\varphi(\bar{x}) = D_1 \sin\gamma_1\bar{x} + D_2 cos\gamma_1\bar{x} + D_3 sin\gamma_2\bar{x} + D_4 cos\gamma_2\bar{x} \tag{28}$$

To determine a non-trivial solution, the determinant of the coefficient matrix of Eq. (32) must vanish. Thus, we obtain

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -{\gamma_1}^2 & 0 & -{\gamma_2}^2 \\ \sin(\gamma_1) & \cos(\gamma_1) & \sin(\gamma_2) & \cos(\gamma_2) \\ -{\gamma_1}^2\sin(\gamma_1) & -{\gamma_1}^2\cos(\gamma_1) & -{\gamma_2}^2\sin(\gamma_2) & -{\gamma_2}^2\cos(\gamma_2) \end{vmatrix} = 0 \tag{33}$$

The solution of the determinant of the previous equation leads to the longitudinal dimensionless frequency parameters of the C-C nanorods based on the DM theory.

Similarly, the same procedure can be applied to obtain the natural frequencies of the nanorods with clamped-free (C-F) ends. According to Eq. (23), the boundary conditions for C-F nanorods in dimensionless form can be defined as

$$\frac{\partial\varphi(\bar{x})}{\partial\bar{x}} + \beta^2\frac{\partial^3\varphi(\bar{x})}{\partial\bar{x}^3} = 0, \quad \frac{\partial^2\varphi(\bar{x})}{\partial\bar{x}^2} = 0 \quad (\bar{x} = 0,1) \tag{34}$$

Using the Eqs. (28) and (34), the analytical solution of C-F nanorods can be expressed in the following matrix formula:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -{\gamma_1}^2 & 0 & -{\gamma_2}^2 \\ \mathrm{Cs}(\gamma_1)\gamma_1 - \beta^2{\gamma_1}^3\mathrm{Cs}(\gamma_1) & -\mathrm{S}(\gamma_1)\gamma_1 + \beta^2{\gamma_1}^3\mathrm{S}(\gamma_1) & \mathrm{Cs}(\gamma_2)\gamma_2 - \beta^2{\gamma_2}^3\mathrm{Cs}(\gamma_2) & -\mathrm{S}(\gamma_2)\gamma_2 + \beta^2{\gamma_2}^3\mathrm{S}(\gamma_2) \\ -{\gamma_1}^2\mathrm{S}(\gamma_1) & -{\gamma_1}^2\mathrm{Cs}(\gamma_1) & -{\gamma_2}^2\mathrm{S}(\gamma_2) & -{\gamma_2}^2\mathrm{Cs}(\gamma_2) \end{bmatrix} \times \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{35}$$

where D_i $(i = 1,2,3,4)$ are arbitrary constants. γ_1 and γ_2 are

$$\gamma_1 = \sqrt{\frac{1+\sqrt{1-4\beta^2\lambda^2}}{2\beta^2}} \tag{29}$$

Here, Cs and S represent the abbreviation of Cosine and Sine terms, respectively. Then, the determinant of the coefficient matrix of Eq. (35) must vanish to obtain a non-trivial solution. Thus, we obtain

The solution of the determinant of the previous equation leads to the longitudinal dimensionless frequency parameters of the C-F nanorods based on the DM theory.

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -{\gamma_1}^2 & 0 & -{\gamma_2}^2 \\ Cs(\gamma_1)\gamma_1 - \beta^2{\gamma_1}^3 Cs(\gamma_1) & -S(\gamma_1)\gamma_1 + \beta^2{\gamma_1}^3 S(\gamma_1) & Cs(\gamma_2)\gamma_2 - \beta^2{\gamma_2}^3 Cs(\gamma_2) & -S(\gamma_2)\gamma_2 + \beta^2{\gamma_2}^3 S(\gamma_2) \\ -{\gamma_1}^2 S(\gamma_1) & -{\gamma_1}^2 Cs(\gamma_1) & -{\gamma_2}^2 S(\gamma_2) & -{\gamma_2}^2 Cs(\gamma_2) \end{vmatrix} = 0 \tag{36}$$

$$\gamma_2 = \sqrt{\frac{1-\sqrt{1-4\beta^2\lambda^2}}{2\beta^2}} \tag{30}$$

According to Eq. (23), the boundary conditions for clamped-clamped (C-C) rods in dimensionless form can be defined as

$$\varphi(\bar{x}) = 0, \quad \frac{\partial^2\varphi(\bar{x})}{\partial\bar{x}^2} = 0 \quad (\bar{x} = 0,1). \tag{31}$$

Using the Eqs. (28) and (31), the analytical solution of C-C nanorods can be expressed in the following matrix formula:

5. Finite element solution of longitudinal vibration problem of nanorods

To the best of the authors' knowledge, there is currently no publication available that explains the axial vibration responses of nanorods with various boundary conditions using a finite element model based on DM theory. In addition to analytical solution of axial vibration of nanorods, a finite element method (FEM) has been developed to solve the axial free vibration problem of nanorods using DM theory. Using the analytical

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -{\gamma_1}^2 & 0 & -{\gamma_2}^2 \\ \sin(\gamma_1) & \cos(\gamma_1) & \sin(\gamma_2) & \cos(\gamma_2) \\ -{\gamma_1}^2\sin(\gamma_1) & -{\gamma_1}^2\cos(\gamma_1) & -{\gamma_2}^2\sin(\gamma_2) & -{\gamma_2}^2\cos(\gamma_2) \end{bmatrix} \times \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

methods in modeling of nano-scaled systems may be impossible in some cases like special loading cases, specific types of boundary conditions and variable material or geometrical properties in some directions (Sharma *et al*. 2015). Therefore, a finite-element formulation for the axial vibration of the nanorods is also developed in this article and the developed formulation is useful for mechanical analyses. The developed finite-element formulation can be also used for nanorods with variable cross-section or variable material properties in axial direction for the future studies.

FEM can be considered as a classical Rayleigh Ritz method. In the FEM, an admissible function is chosen along an element. Similar to other approximate method FEM give approximate results to a given problem. The accuracy of FEM can be increased either increasing the number of elements used or by increasing the order of polynomial representing the displacement. The accuracy of the FEM depends on the satisfaction of geometric boundary conditions and the equations of motion. The other possibility is to increase the number of elements. This operation is limited due to computational errors (ill-conditioning problems in the solution of the eigenvalue problem).

As it is seen in Eq. (23), there are two essential classical and non-classical boundary conditions. For this reason, two variables which are axial displacement and strain can be considered for each node elements in the finite element solution. This leads to four degrees of freedom (DOF) for each element. By using a two-node higher order rod element in DM formulation, the displacement is expressed as the following form in the framework of the cubic Hermite interpolation polynomial:

$$u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \tag{37}$$

where, c_i $(i = 0,1,2,3)$ denote the arbitrary constants. According to Eq. (38), the strain can be obtained as

$$\frac{du(x)}{dx} = c_1 + 2c_2 x + 3c_3 x^2 \tag{38}$$

Since the sum of the degrees of freedom at both ends of the element is four, the displacement function of the rod ($u(x)$) must be represented by a cubic function in terms of four constants (Eq. 37). Here, to find the constants c_0, c_1, c_2, c_3 in Hermite third-order polynomial, $u_i = u|_{x=0}, u_j = u|_{x=L}$, $\frac{du_i}{dx} = \frac{du}{dx}\Big|_{x=0}$, $\frac{du_j}{dx} = \frac{du}{dx}\Big|_{x=L}$ conditions are written. This also ensures that the continuity condition between the elements is directly provided. Inserting the displacements and strain of two nodes which are the *i* and *j* nodes at the two ends of the rod element into Eqs. (37) and (38) leads to

$$\begin{bmatrix} u_i & \frac{du_i}{dx} & u_j & \frac{du_j}{dx} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} [c_0 \quad c_1 \quad c_2 \quad c_3]^T \tag{39}$$

where l denotes the length of an element. Then, longitudinal displacement u can be re-expressed in terms of nodal displacement as follows

$$u = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{bmatrix} u_i & \frac{du_i}{dx} & u_j & \frac{du_j}{dx} \end{bmatrix}^T \tag{40}$$

The previous equation can be re-written in terms of associated shape function vector ($\boldsymbol{\phi}(x)$) in the following form

$$u = \boldsymbol{\phi}(x) \begin{bmatrix} u_i & \frac{du_i}{dx} & u_j & \frac{du_j}{dx} \end{bmatrix}^T \tag{41}$$

Here, the shape function vector is provided by:

$$\boldsymbol{\phi}(x) = [\phi_1(x) \quad \phi_2(x) \quad \phi_3(x) \quad \phi_4(x)] \tag{42}$$

where the components of the shape function vector are

$$\begin{aligned} \phi_1(x) &= 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}, \\ \phi_2(x) &= x - 2\frac{x^2}{l} + \frac{x^3}{l^2}, \\ \phi_3(x) &= 3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}, \\ \phi_4(x) &= -\frac{x^2}{l} + \frac{x^3}{l^2} \end{aligned} \tag{43}$$

A variational equation can be obtained using Eqs. (21) and (24) as follows

$$\begin{aligned} &-AE\int_0^l \left(\frac{\partial u}{\partial x}\right)\delta\frac{\partial u}{\partial x}dx - AE\frac{\eta^2}{12}\int_0^l \left(\frac{\partial^3 u}{\partial x^3}\right)\delta\frac{\partial u}{\partial x}dx \\ &+\rho A\omega^2\int_0^l u\delta u dx - \left[AE\frac{\eta^2}{12}\delta\frac{\partial u}{\partial x}\right]_0^l = 0 \end{aligned} \tag{44}$$

By considering the Eq. (44) and Galerkin approach, the elemental stiffness matrix ($\mathbf{K}^{\mathbf{e}}$) and mass matrix ($\mathbf{M}^{\mathbf{e}}$) can be obtained in the following form

$$\mathbf{K}^{\mathbf{e}} = AE\int_0^l \frac{\partial \boldsymbol{\phi}^{\mathrm{T}}}{\partial x}\frac{\partial \boldsymbol{\phi}}{\partial x}dx + AE\frac{\eta^2}{12}\int_0^l \frac{\partial \boldsymbol{\phi}^{\mathrm{T}}}{\partial x}\frac{\partial^3 \boldsymbol{\phi}}{\partial x^3}dx \tag{45}$$

$$\mathbf{M}^{\mathbf{e}} = \rho A\int_0^l \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}dx \tag{46}$$

Then, substituting the Eq. (42) into Eqs. (45) and (46), expression of the elemental stiffness and mass matrices can be found as

$$\begin{aligned} \mathbf{K}^{\mathbf{e}} &= \frac{AE}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \\ &+\frac{6AE\eta^2}{12l^3}\begin{bmatrix} -2 & -l & 2 & -l \\ 0 & 0 & 0 & 0 \\ 2 & l & -2 & l \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{47}$$

Table 1 Comparison of the dimensionless frequency parameter (DFP) of C-C and C-F nanotubes (L=10 nm, η=l_s=0.1421 nm)

Mode number (n)	DM (FEM)	DM (Analytical)	CT	e_0a=0 (Li *et al.* 2016) (FEM)	e_0a=0.4 nm (Li *et al.* 2016) (FEM)	e_0a=0.5 nm (Li *et al.* 2016) (FEM)	e_0a=0.6 nm (Li *et al.* 2016) (FEM)
C-C							
1	3.11376	3.14075	3.14159	3.34403	3.31658	3.30144	3.28319
2	6.22632	6.27671	6.28318	6.70210	6.48960	6.37844	6.24986
3	9.33648	9.41073	9.42477	10.08551	9.40397	9.07530	8.71588
C-F							
1	1.56382	1.57041	1.57079	1.61960	1.61633	1.61450	1.61227
2	4.69079	4.70822	4.71238	4.86720	4.78077	4.73411	4.67887
3	7.81569	7.84845	7.85938	8.13852	7.75464	7.56099	7.34270

$$\mathbf{M}^{\mathrm{e}} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{48}$$

It should be noted that the last term of the elemental stiffness matrix in Eq. (47) contains the scale effect of the nanorod. It is also noted that the material length scale parameter of DM (η) must be less than the element length (l). The global stiffness matrix ($\mathbf{K}^{\mathrm{G}}$) and global mass matrix ($\mathbf{M}^{\mathrm{G}}$) can be defined for overall structure as follows (Li *et al.* 2016):

$$\mathbf{K}^{\mathrm{G}} = \underset{i=1}{\overset{m}{F}} \mathbf{K}_i^e, \qquad \mathbf{M}^{\mathrm{G}} = \underset{i=1}{\overset{m}{F}} \mathbf{K}_i^e. \tag{49}$$

Here, $\underset{i=1}{\overset{m}{F}}$ denotes the conventional matrix operator, m denotes the total number of the elements, $\mathbf{M}_i^e$ and $\mathbf{K}_i^e$ denote the ith elemental mass and stiffness matrices, respectively. Finally, finite element solution for longitudinal vibration of nanorods can be obtained from the following eigenvalue problem:

$$(\mathbf{K}^{\mathrm{G}} - \omega^2 \mathbf{M}^{\mathrm{G}})\varphi = 0 \tag{50}$$

where ω is the natural frequency (eigenvalues) and φ denotes the eigenvectors which give mode shapes.

6. Numerical results

In this section, numerical examples of longitudinal vibration of nanorods are presented. The results with various theories such as DM, nonlocal strain grain gradient theory and classical rod theory. In Table 1, the comparison of the first three dimensionless frequency parameters ($\lambda^2 = \rho\omega^2 L^2/E$) of the nanorod is presented for the analytical solution of DM, finite element solution of DM, nonlocal strain gradient elasticity theory (Li *et al.* 2016) and classical theory (CT). It is noted that 100 finite rod elements are used in FEM solution. Here, the material length scale parameter (η) of DM and material length scale parameter (l_s) of strain gradient theory (Li *et al.* 2016) is taken as 0.1421 nm. This is because, nanorods are modelled as a carbon nanotube that its material length scale parameter is considered as the bonding length of carbon atoms is equal to 0.1421 nm. It is seen that the results are predicted by the finite element solution of DM agree well with the analytical solution of DM and CT theory for both boundary conditions. The highest difference between DM results with finite element solution and DM results with analytical solution is 0.866% and 0.421% for C-C and C-F boundary conditions, respectively. It is clear that the DM results obtained by FEM are lower than CT for the given rod length (L=10 nm). The softening material behavior is observed in DM theory with respect to CT for C-C and C-F boundary conditions. It is also seen that the frequency results are compared with those from Li *et al.* (2016), who derived finite element solutions using nonlocal strain gradient theory. DM results with FEM are in a good agreement with the nonlocal strain gradient results for e_0a = 0.6 nm. However, the difference between DM and nonlocal strain gradient results increases with increasing the nonlocal scale parameter e_0a. The possible reasons of the differences between the two theories can be definition of the material length scale parameter of the nonlocal elasticity theory (e_0a). Unlike the DM theory, the material length scale parameter of the nonlocal elasticity theory (e_0a) is not a constant and it can take different values. By considering the different material length scale parameters in nonlocal elasticity theory, different frequency results can be obtained for the considered micro/nanostructure. There is not any formula to match the material length scale parameter of DM and material length scale parameter of the nonlocal elasticity theory. As a result, some variations in frequency results between the DM and nonlocal strain gradient theory can be observed in Table 1.

After verification of the developed FEM model based on the DM theory, the variation of longitudinal vibration frequencies of CNTs with respect to nanotube length and mode number is illustrated for C-C and C-F boundary conditions in Figs. 2 and 3. Dimensionless frequencies depend on nanorod length in DM theory. As the nanotube length increases, the longitudinal frequencies of DM model increases, whereas the frequencies of CT are not affected by the rod length, and they remain constant at all lengths. The length scale effect is lost after a certain rod length (L>30 nm). Rod length is more effective for clamped boundary conditions compared to free boundary conditions for DM theory. It is seen that there is a small difference between DM results with FEM and analytical results of DM and this is more pronounced for C-F boundary condition compared to C-C boundary condition. This confirms that the developed FEM

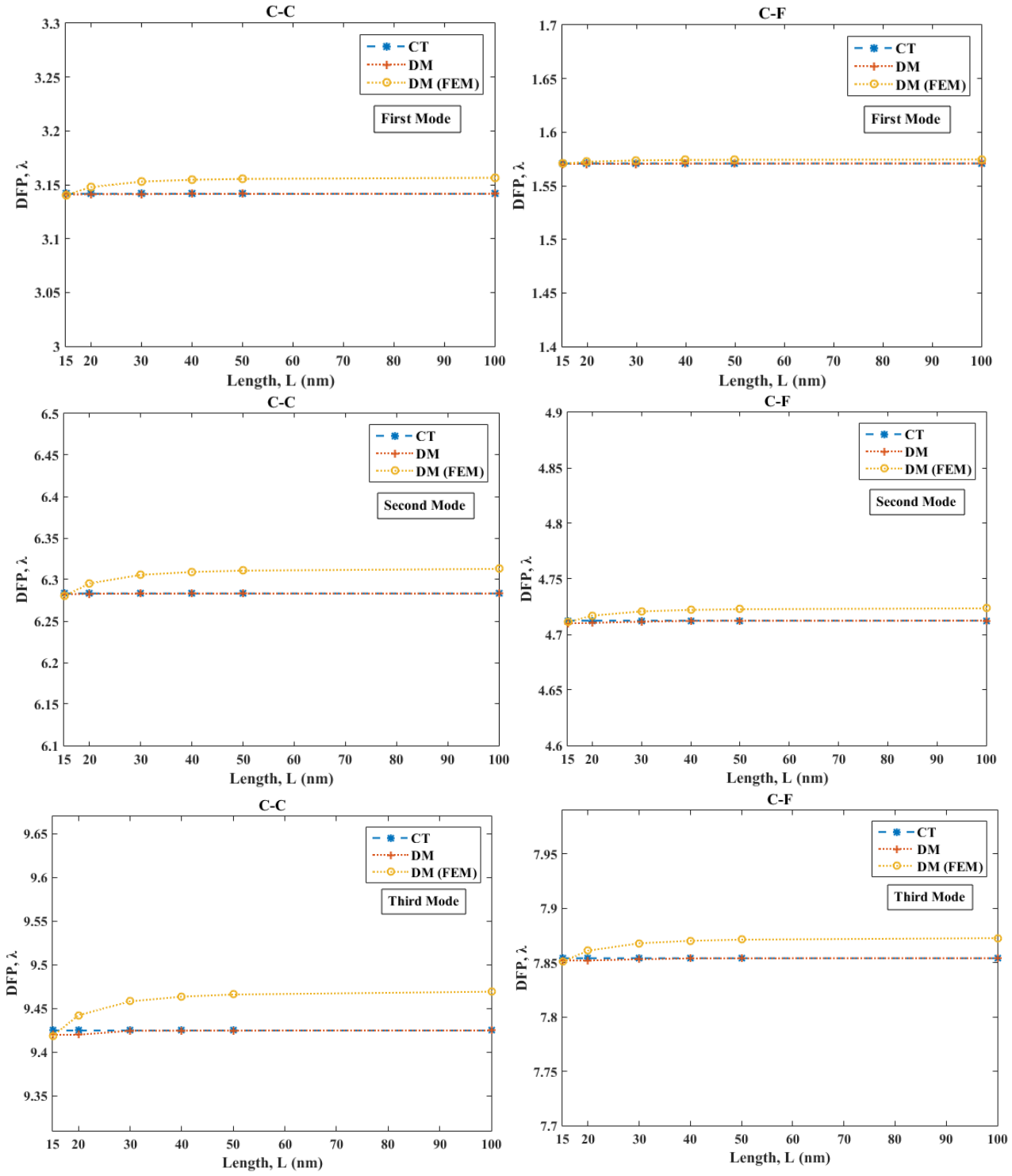

Fig. 2 Variation of DFP with nanotube length (CT: Classical rod theory, DM: Doublet mechanics with analytical solution, η=0.1421 nm)

approach can be used to analyze the vibration of CNTs. DM theory with analytical solution always predicts softening material responses for all nanotube length. The difference of material behaviour in DM theory originates from the solution method (analytical or FEM) of DM theory. Fig. 3 illustrates the variation of natural frequencies with respect to mode number. It should be noticed that DM results with FEM and analytical solutions are in excellent agreement with CT results. The size effect is more apparent for higher modes (n>6) and the softening effect is observed in DM theory for the given length of CNT. With increasing mode number, half wave number becomes shorter and comparable to intrinsic length and scale effect is more pronounced for higher modes. Vibration mode shapes of CNTs are plotted based on DM with analytical solution, DM with FEM solution and CT in Figs. 4 and 5. It is seen that even though their frequencies in these cases are different, the corresponding mode

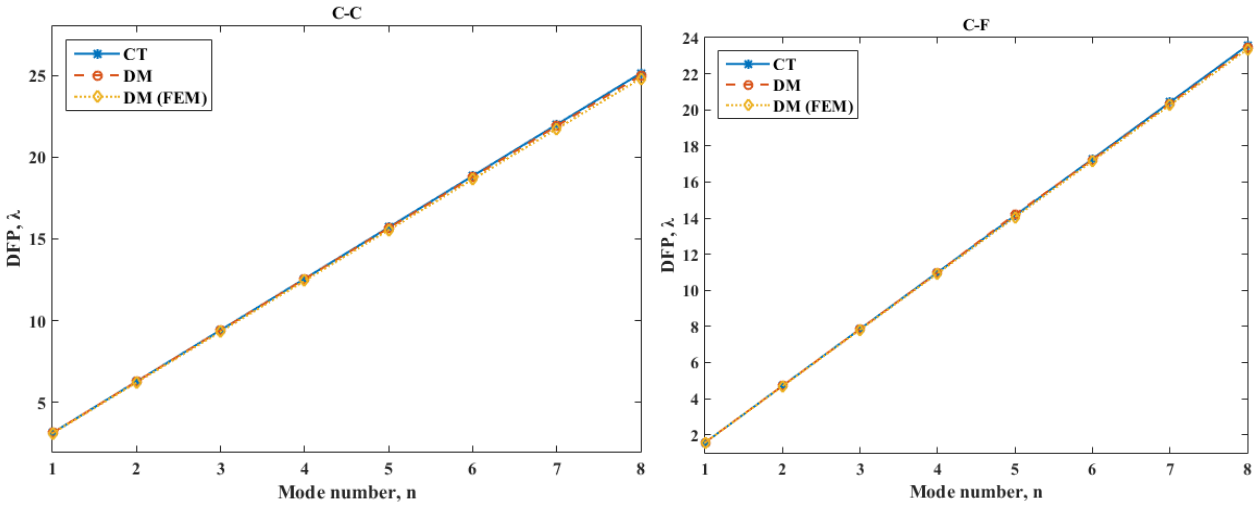

Fig. 3 Variation of DFP with mode number (CT: Classical rod theory, DM: Doublet mechanics with analytical solution, L=10 nm, η=0.1421 nm)

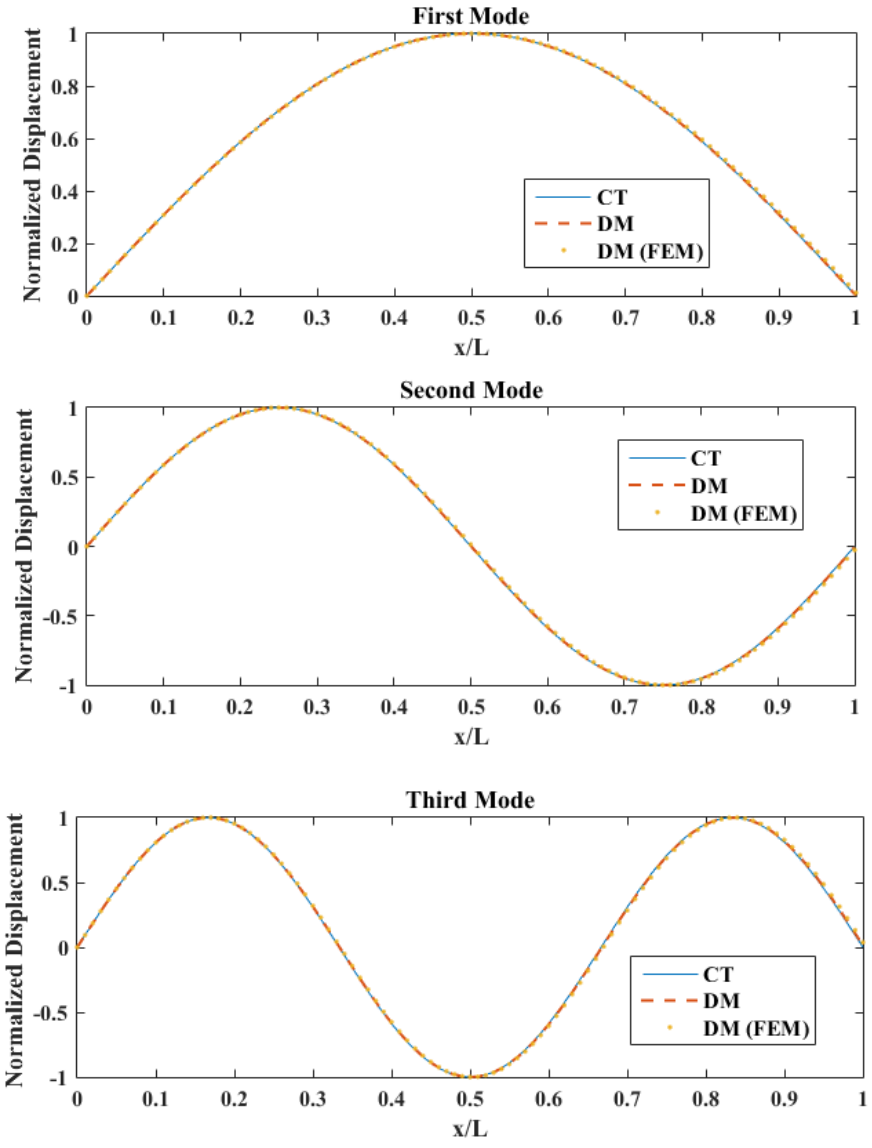

Fig. 4 The first three vibration mode shapes of C-C nanotubes (CT: Classical rod theory, DM: Doublet mechanics with analytical solution, L=10 nm, η=0.1421 nm)

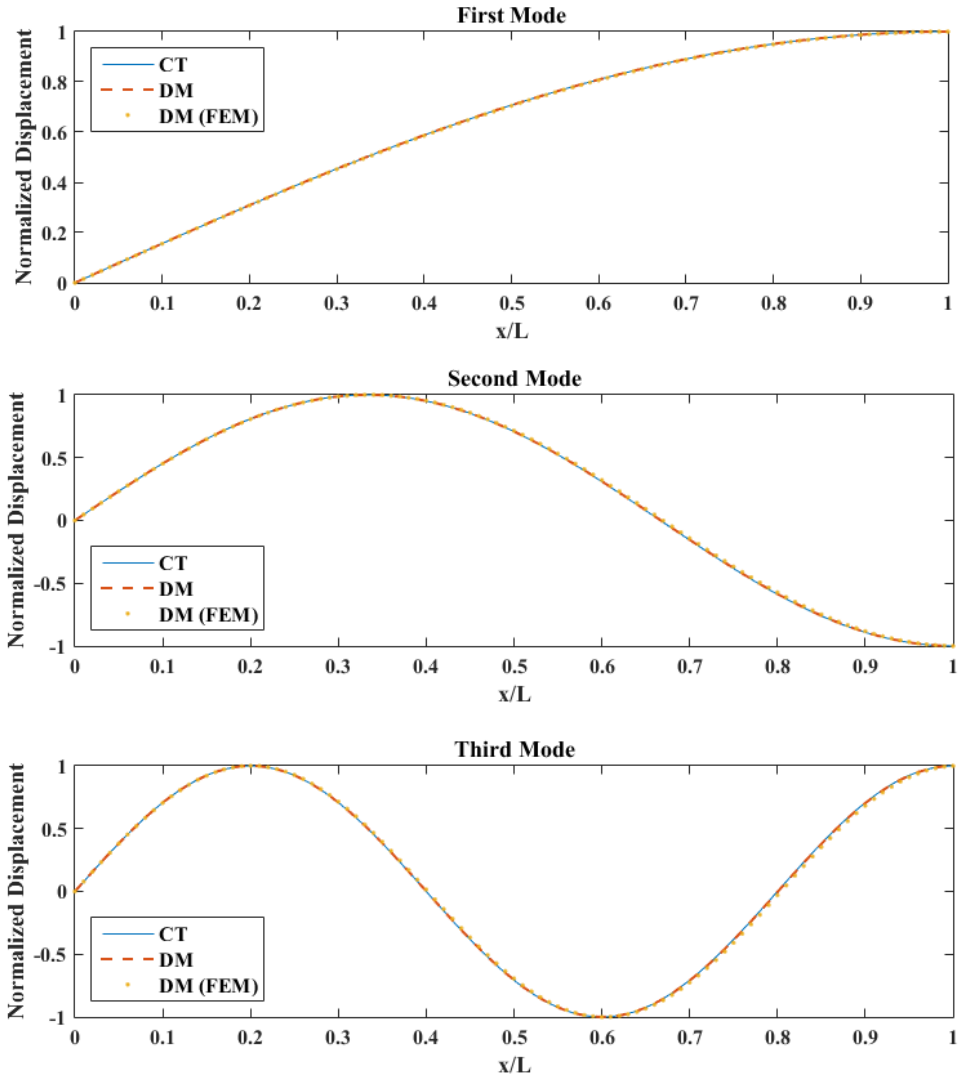

Fig. 5 The first three vibration mode shapes of C-F nanotubes (CT: Classical rod theory, DM: Doublet mechanics with analytical solution, L=10 nm, η=0.1421 nm)

shapes are almost identical for the first three modes of vibration. The maximum longitudinal displacements are always observed at free end in C-F CNTs, and the maximum displacements are always observed in the middle of the nanotube in C-C boundary conditions which have symmetric according to the center of the nanotube.

7. Conclusions

Based on the DM theory and FEM formulation, the longitudinal vibration analysis of CNTs is investigated for the first time in this paper. The vibration of CNTs is analyzed for general boundary conditions such as C-C and C-F ends. The results predicted by the present model are found to be in excellent agreement with those from the analytical solution of DM and CT. New results for the natural frequencies of CNTs based on DM and FEM formulation are compared to the natural frequencies of nonlocal strain gradient theory with FEM solution for validation. The comparison showed that the proposed finite element model agrees well with analytical solution of DM model. This behavior is more apparent for C-F boundary condition and the first mode of vibration. The effect of material length scale parameter of the DM is more apparent for C-C boundary condition compared to C-F boundary condition. The softening material behaviour is observed in the analytical solution of DM compared to CT. The mode shapes of the CNTs are derived for the DM with FEM solution, DM with analytical solution, and CT. These mode shapes are almost identical for all models due to the small differences of natural frequencies in DM and CT. The present approach can be used to predict the accuracy of the responses of CNTs.

References

Adhikari, S., Murmu, T. and McCarthy, M.A. (2013), "Dynamic finite element analysis of axially vibrating nonlocal rods", *Finite Elem. Anal. Des.*, **63**, 42-50. https://doi.org/10.1016/j.finel.2012.08.001.

Adhikari, S., Murmu, T. and McCarthy, M.A. (2014), "Frequency domain analysis of nonlocal rods embedded in an elastic medium", *Physica E*, **59**, 33-40.

https://doi.org/10.1016/j.physe.2013.11.001.
Aifantis, E.C. (1999), "Gradient deformation models at nano, micro, and macro scales", *ASME J. Eng. Mater. Technol.*, **121**(2), 189-202. https://doi.org/10.1115/1.2812366.
Akgöz, B. and Civalek, Ö. (2014), "Longitudinal vibration analysis for microbars based on strain gradient elasticity theory", *J. Vib. Control*, **20**(4), 606-616. https://doi.org/10.1177/1077546312463752.
Anderson, W. and Lakes, R. (1994), "Size effects due to Cosserat elasticity and surface damage in closed-cell polymethacrylimide foam", *J. Mater. Sci.*, **29**, 6413-6419. https://doi.org/10.1007/BF00353997.
Aydogdu, M. (2009), "Axial vibration of the nanorods with the nonlocal continuum rod model", *Physica E*, **41**(5), 861-864. https://doi.org/10.1016/j.physe.2009.01.007.
Aydogdu, M. (2012), "Axial vibration analysis of nanorods (carbon nanotubes) embedded in an elastic medium using nonlocal elasticity", *Mech. Res. Commun.*, **43**, 34-40. https://doi.org/10.1016/j.mechrescom.2012.02.001.
Aydogdu, M. and Gul, U. (2020), "Longitudinal vibration of double nanorod systems using doublet mechanics theory", *Struct. Eng. Mech.*, **73**(1), 37-52. http://doi.org/10.12989/sem.2020.73.1.037.
Challamel, N. (2013), "Variational formulation of gradient or/and nonlocal higher-order shear elasticity beams", *Compos. Struct.*, **105**, 351-368. https://doi.org/10.1016/j.compstruct.2013.05.026.
Demir, Ç. and Civalek, Ö. (2013), "Torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models", *Appl. Math. Model.*, **37**(22), 9355-9367. https://doi.org/10.1016/j.apm.2013.04.050.
Eghbali, M., Hosseini, S.A. and Rahmani, O. (2021), "Free vibration of axially functionally graded nanobeam with an attached mass based on nonlocal strain gradient theory via new ADM numerical method", *Amirkabir J. Mech. Eng.*, **53**(2), 275-276. https://doi.org/10.22060/mej.2020.17013.6495.
Eghbali, M., Hosseini, S.A. and Pourseifi, M. (2022a), "Free transverse vibrations analysis of size-dependent cracked piezoelectric nano-beam based on the strain gradient theory under mechanic-electro forces", *Eng. Anal. Bound. Elem.*, **143**, 606-612. http://doi.org/10.1016/j.enganabound.2022.07.006.
Eghbali, M., Hosseini, S.A. and Hamidi, B.A. (2022b), "Mantari's higher-order shear deformation theory of sandwich beam with CNTRC face layers with porous core under thermal loading", *Int. J. Struct. Stab. Dyn.*, **22**(16), 2250181. https://doi.org/10.1142/S0219455422501814.
Eringen, A.C. (1972a), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16. https://doi.org/10.1016/0020-7225(72)90070-5.
Eringen, A.C. (1972b), "Linear theory of nonlocal elasticity and dispersion of plane waves", *Int. J. Eng. Sci.*, **10**(5), 425-435. https://doi.org/10.1016/0020-7225(72)90050-X.
Fatahi-Vajari, A. and Azimzadeh, Z. (2020), "Axial vibration of single-walled carbon nanotubes with fractional damping using doublet mechanics", *Indian J. Phys.*, **94**, 975-986. https://doi.org/10.1007/s12648-019-01547-y.
Fleck, N., Muller, G., Ashby, M. and Hutchinson, J. (1994), "Strain gradient plasticity: theory and experiment", *Acta Metall. Mater.*, **42**(2), 475-487. https://doi.org/10.1016/0956-7151(94)90502-9.
Granik, V. (1978), "Microstructural mechanics of granular media", Technique Report IM/MGU Institute of Mechanics of Moscow State University,78-241.
Granik, V.T. and Ferrari, M. (1993), "Microstructural mechanics of granular media", *Mech. Mater.*, **15**(4), 301-322. https://doi.org/10.1016/0167-6636(93)90005-C.
Gul, U., Aydogdu, M. and Gaygusuzoglu, G. (2017), "Axial dynamics of a nanorod embedded in an elastic medium using doublet mechanics", *Compos. Struct.*, **160**, 1268-1278. https://doi.org/10.1016/j.compstruct.2016.11.023.
Gul, U. and Aydogdu, M. (2018), "Structural modelling of nanorods and nanobeams using doublet mechanics theory", *Int. J. Mech. Mater. Des.*, **14**, 195-212. https://doi.org/10.1007/s10999-017-9371-8.
Gul, U. and Aydogdu, M. (2019), "Vibration analysis of Love nanorods using doublet mechanics theory", *J. Braz. Soc. Mech. Sci. Eng.*, **41**, 351. https://doi.org/10.1007/s40430-019-1849-x.
Güven, U. (2014), "Love–Bishop rod solution based on strain gradient elasticity theory", *Comptes Rendus Mécanique*, **342**(1), 8-16. https://doi.org/10.1016/j.crme.2013.10.011.
Hutchinson, J.W. (2000), "Plasticity at the micron scale", *Int. J. Solid. Struct.*, **37**(1-2), 225-238. https://doi.org/10.1016/S0020-7683(99)00090-6.
Kahrobaiyan, M.H., Asghari, M. and Ahmadian, M.T. (2013), "Longitudinal behavior of strain gradient bars", *Int. J. Eng. Sci.*, **66-67**, 44-59. https://doi.org/10.1016/j.ijengsci.2013.02.005.
Karamanli, A. (2021), "Structural behaviours of zigzag and armchair nanobeams using finite element doublet mechanics", *Europ. J. Mech. A Solid.*, **89**, 104287. https://doi.org/10.1016/j.euromechsol.2021.104287.
Khorshidi, M.A. (2018), "The material length scale parameter used in couple stress theories is not a material constant", *Int. J. Eng. Sci.*, **133**, 15-25. https://doi.org/10.1016/j.ijengsci.2018.08.005.
Kojic, M., Vlastelica, I., Decuzzi, P., Granik, V.T. and Ferrari, M. (2011), "A finite element formulation for the doublet mechanics modeling of microstructural materials", *Comput. Meth. Appl. Mech. Eng.*, **200**(13-16), 1446-1454. https://doi.org/10.1016/j.cma.2011.01.001.
Lam, D., Yang, F., Chong, A., Wang, J. and Tong, P. (2003), "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solid.*, **51**(8), 1477-1508. https://doi.org/10.1016/S0022-5096(03)00053-X.
Li, L., Hu, Y. and Li, X. (2016), "Longitudinal vibration of size-dependent rods via nonlocal strain gradient theory", *Int. J. Mech. Sci.*, **115-116**, 135-144. https://doi.org/10.1016/j.ijmecsci.2016.06.011.
Lim, C.W., Zhang, G. and Reddy, J.N. (2015), "A higher-order nonlocal elasticity and strain gradient theory and its application in wave propagation", *J. Mech. Phys. Solid.*, **78**, 298-313. https://doi.org/10.1016/j.jmps.2015.02.001.
Mindlin, R. and Tiersten, H. (1962), "Effects of couple-stresses in linear elasticity", Columbia University, New York., U.S.A.
Murmu, T., Adhikari, S. and Wang, C.Y. (2011), "Torsional vibration of carbon nanotube-buckyball systems based on nonlocal elasticity theory", *Physica E*, **43**(6), 1276-1280. https://doi.org/10.1016/j.physe.2011.02.017.
Mustapha, K.B. and Ruan, D. (2015), "Size-dependent axial dynamics of magnetically-sensitive strain gradient microbars with end attachments", *Int. J. Mech. Sci.*, **94-95**, 96-110. https://doi.org/10.1016/j.ijmecsci.2015.02.010.
Numanoğlu, H.M. and Civalek, Ö. (2019), "On the torsional vibration of nanorods surrounded by elastic matrix via nonlocal FEM", *Int. J. Mech. Sci.*, **161-162**, 105076. https://doi.org/10.1016/j.ijmecsci.2019.105076.
Reddy J.N. (2002), "Energy Principles and Variational Methods in Applied Mechanics", John Wiley & Sons Inc., Hoboken.
Sharma, R., Jadon, V.K. and Singh, B. (2015), "A review on the finite element methods for heat conduction in functionally graded materials", *J. Inst. Eng. India Ser. C*, **96**, 73-81. https://doi.org/10.1007/s40032-014-0125-1.
Sears, A. and Batra, R.C. (2004), "Macroscopic properties of carbon nanotubes from molecular-mechanics simulations", *Phys. Rev. B.*, **69**, 235406.

https://doi.org/10.1103/PhysRevB.69.235406.
Vajari, A.F. and Imam, A. (2016), "Axial vibration of single-walled carbon nanotubes using doublet mechanics", *Indian J. Phys.*, **90**, 447-455. https://doi.org/10.1007/s12648-015-0775-8.
Zhang, Y.Y., Wang, C.M. and Challamel, N. (2009), "Bending, buckling, and vibration of micro/nanobeams by hybrid nonlocal beam model", *J. Eng. Mech.*, **136**(5), 562-574. https://doi.org/10.1061/(ASCE)EM.1943-7889.0000107.

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