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# A quasi-3D nonlocal theory for free vibration analysis of functionally graded sandwich nanobeams on elastic foundations

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**Abstract.** The main aims of this study are to develop a new nonlocal quasi-3D theory for the free vibration behaviors of the functionally graded sandwich nanobeams. The sandwich beams consist of a ceramic core and two functionally graded material layers resting on elastic foundations. The two layers, linear spring stiffness and shear layer, are used to model the effects of the elastic foundations. The size-effect is considered using nonlocal elasticity theory. The governing equations of the motion of the functionally graded sandwich nanobeams are obtained via Hamilton's principle in combination with nonlocal elasticity theory. Then the Navier's solution technique is used to solve the governing equations of the motion to achieve the nonlocal free vibration behaviors of the nanobeams. A deep parametric study is also provided to demonstrate the effects of some parameters, such as length-to-height ratio, power-law index, nonlocal parameter, and two parameters of the elastic foundation, on the free vibration behaviors of the functionally graded sandwich nanobeams.
**Keywords:** functionally graded; nanobeams; nonlocal theory; quasi-3D theory; sandwich beam

## 1. Introduction

Normally, micro- and nano-structures are widely used in many fields of electronics, optics, and other high-tech fields, such as Arefi *et al*. (2020), Arefi and Zenkour (2017), Zeng *et al*. (2019), and Ebrahimi *et al*. (2017). Some common nanodevices are nanoelectronics, nano-resonators, nanocomposites, nano-sensors, nanofibers, and so on. Therefore, numerous scientists have focused on the mechanical and thermal analysis of micro- and nano-structures. However, the application of classical elasticity theories to analyze such structures is incompatible, and some non-classical elastic theories have been developed. The most common non-classical elastic theory is nonlocal elasticity theory, which was developed by Eringen (Eringen 1967, Eringen 1983, Eringen and Edelen 1972, Eringen 1972) and applied in numerous research studies. Reddy (2007) studied the bending, buckling, and vibration of the nanobeams using nonlocal elasticity theory. The mechanical behaviors of carbon nanotubes have been studied by Reddy and Pang (2008) using nonlocal continuum beam theory. Aghababaei and Reddy (2009) developed a nonlocal third-order shear deformation theory for the analysis of nanoplates. Aksencer and Aydogdu (2011) applied nonlocal elasticity theory to analyze the buckling and vibration of Levy-type nanoplates. The size-dependent vibration of nanobeams has been analyzed by Eltaher *et al*. (2012) using a nonlocal finite element beam model. Natarajan *et al*. (2012) developed a nonlocal isogeometric-based model for free vibration examination of nanoplates. Thai and Vo (2012) established a nonlocal sinusoidal shear deformation theory for the mechanical analysis of nanobeams. Hosseini-Hashemi *et al*. (2013) studied the mechanical response of circular and annular Mindlin nanoplates using nonlocal elasticity theory. The functionally graded (FG) materials are also applied to make the nano-devices due to their unique blend of mechanical, optical, and electrical properties (Cho 2022, Faraji Oskouie *et al*. 2023, Foroutan and Ahmadi 2022, Ghannadpour and Khajeh 2022, Hosseini *et al*. 2022, İpek *et al*. 2022, Kumar and Kattimani 2022, Wattanasakulpong and Eiadtrong 2023). Nonlinear free vibration of functionally graded (FG) nanobeams was done by Nazemnezhad and Hosseini-Hashemi (2014) via nonlocal elasticity theory. Sobhy (2014) and Sobhy (2015) studied the mechanical behaviors of FG nanoplates resting on elastic foundations. Salehipour *et al*. (2015) developed a modified nonlocal elasticity theory for the analysis of FG nanostructures. In this study, the classical elasticity theory was modified to consider the variation of the nonlocal parameter in FG materials. Larbi *et al*. (2015) studied the bending and buckling analysis of FG size-dependent nanobeams via nonlocal elasticity theory, including the thickness stretching effect. Bensaid *et al*. (2018) studied the vibration behaviors of nanobeams resting on elastic foundations using nonlocal strain gradient theory. Ebrahimi and Barati (2017), Ebrahimi and Fardshad (2018), Ebrahimi *et al*. (2019a), Ebrahimi *et al*. (2019b) and Ebrahimi *et al*. (2019c) developed several nonlocal higher-order shear

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deformation theories and applied them to analyze FG nanobeams with and without elastic foundations. Hadji and Avcar (2021) studied the nonlocal free vibration of FG nanobeams considering the effects of porosity using hyperbolic shear deformation theory. Tlidji *et al*. (2022) examined the free vibration of FG nanobeams with porosity using a state-space approach. Zemri *et al*. (2015) studied the mechanical behavior of FG nanobeams using a refined nonlocal shear deformation theory. Arefi and Zenkour (2016) developed a simplified shear and normal deformation nonlocal theory for the bending analysis of FG piezomagnetic sandwich nanobeams in a magneto-thermo-electric environment. Luat *et al*. (2021) studied the mechanical behaviors of FG sandwich nanobeams using nonlocal elasticity theory. Mama *et al*. (2016) studied the mechanical response of FG nanobeams using nonlocal elasticity theory, including the neutral surface position concept. In this study, the authors show that the neutral surface position plays a major effect on the behaviors of the FG nanobeams. A novel nonlocal zeroth-order shear deformation theory has been developed by Bellifa *et al*. (2017) for nonlinear post-buckling analysis of nanobeams. Yang *et al*. (2018) analyzed the nonlinear bending, buckling, and vibration of bi-directional FG nanobeams. Hana *et al*. (2019) investigated the effects of porosity on the vibration behaviors of FG nanobeams. Aria *et al*. (2019), Aria and Friswell (2019) developed a finite element model based on nonlocal elasticity theory for the analysis of FG nanobeams.

A sandwich nanobeam is a specialized nanostructure composed of three distinct layers with varying properties, typically consisting of two outer layers separated by a central core layer. These layers can be engineered to possess different material compositions, thicknesses, and mechanical properties, resulting in a versatile platform for tailoring specific functionalities. Sandwich nanobeams are synthesized through two primary methods. One approach employs advanced lithography techniques, such as electron-beam lithography (EBL) or focused ion beam (FIB) lithography, to pattern the desired structure. Subsequently, deposition techniques like chemical vapor deposition (CVD) or molecular beam epitaxy (MBE) are used to grow the layers (Chandel *et al*. 2020). Alternatively, bottom-up methods, such as atomic layer deposition (ALD) and layer-by-layer assembly, enable precise control over layer composition and thickness, facilitating the creation of intricate sandwich structures.

Sandwich nanobeams with FG layers find applications in sensitive mechanical sensors, photonic devices, and energy harvesting systems. They are also valuable in flexible electronics, materials testing, and aerospace components due to their unique blend of mechanical, optical, and electrical properties. The ceramic core provides excellent mechanical stability and thermal insulation, making it suitable for structural support and minimizing heat transfer. The FG outer layers allow precise control over material properties, enabling tailored characteristics like electrical conductivity, optical transparency, or thermal conductivity. This combination maximizes the performance and functionality of the nanobeam in different applications by harnessing the specific strengths of each material (Chandel and Talha 2022a, Chandel and Talha 2022b, Chandel and Talha 2023). Besides, the sandwich nanobeams usually lying substrate or support which can be modeled by elastic foundation (Arefi *et al*. 2020, Arefi and Zenkour 2017). One specific application area where the inclusion of an elastic foundation plays a significant role is in the development of advanced nanomechanical resonators and sensors. For instance, in nanomechanical resonators, the foundation can represent the interaction with a biological sample, enabling the detection of specific biomolecules or pathogens. In nanoscale sensors, it can mimic the response of the nanobeam to external forces, such as those encountered in environmental monitoring or industrial quality control (Ebrahimi and Barati 2017, Ebrahimi *et al*. 2017).

While previous research has extensively examined the mechanical properties of FG and sandwich FG nanobeams, there remains a significant gap in our understanding regarding the vibration characteristics of sandwich FG nanobeams specifically featuring a ceramic core and FG layers, especially when resting on elastic foundations. In this study, we introduce a novel quasi-3D nonlocal theory uniquely tailored to analyze the vibration behaviors of these functionally graded sandwich (FGS) nanobeams. Our approach incorporates nonlocal elasticity theory to account for small-scale effects at the nanoscale. Through an extensive parametric study, we unveil previously undiscovered insights into the influence of various parameters on the free vibration response of FGS nanobeams. These novel findings hold immense promise for advancing the design, testing, and analysis of nano-structures, pushing the boundaries of our understanding in this field.

## 2. Theoretical formulation

### 2.1 Functionally graded sandwich nanobeams

Fig. 1 presents the FGS nanobeam which is considered in this study. The beam has the heigh $h$ and the length $L$, is resting on elastic foundations with two parameters $k_w$ and $k_s$ for spring and shear layers, respectively. It is assumed that the delamination between the core and face-sheets does not appear when the nanobeams vibrates.

The following formulation is used to evaluate the volume fraction of the ceramic component through the thickness of the FGS beams.

$$\begin{cases} V_c^{\text{bottom}} = \left(\dfrac{z - z_0}{z_1 - z_0}\right)^k & z_0 \le z \le z_1 \\ V_c^{\text{core}} = 1 & z_1 < z < z_2 \\ V_c^{\text{top}} = \left(\dfrac{z - z_3}{z_2 - z_3}\right)^k & z_2 \le z \le z_3 \end{cases} \tag{1}$$

The effective material properties of the FGS nanobeams are estimated as follows

$$\begin{aligned} E(z) &= E_c V_c + E_m (1 - V_c) \\ \rho(z) &= \rho_c V_c + \rho_m (1 - V_c) \\ \nu(z) &= \nu_c V_c + \nu_m (1 - V_c) \end{aligned} \tag{2}$$

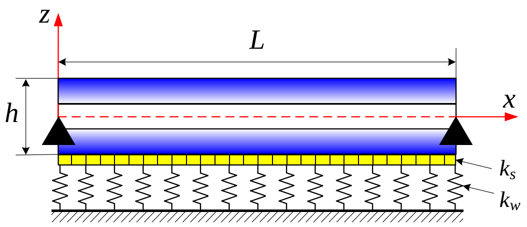
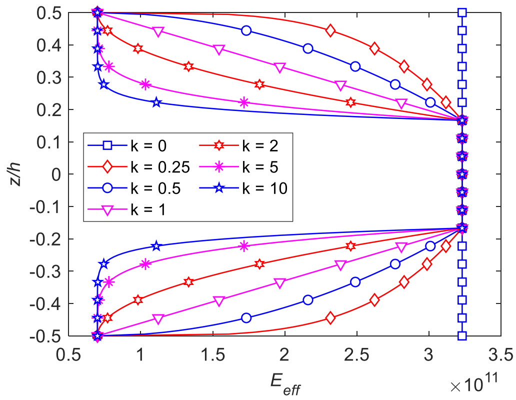


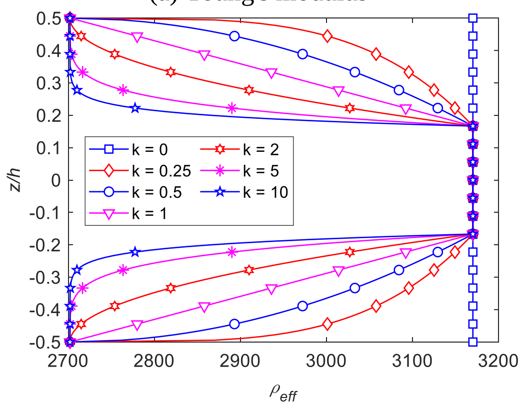

Fig. 1 The model of the FGS nanobeams

Table 1 The material properties of individual materials

| Materials | Young's modulus (GPa) | Mass density (kg/m$^3$) | Poison's ratio |
|---|---|---|---|
| Al | 70 | 2702 | 0.3 |
| $Al_2O_3$ | 280 | 3960 | 0.3 |
| Ti-6Al-4V | 66.2 | 4420 | 1/3 |
| $Si_3N_4$ | 323 | 3170 | 0.3 |
| SUS304 | 207 | 8166 | 0.3 |




(a) Young's modulus




(b) Mass density

Fig. 2 The variation of effective Young's modulus and mass density through the thickness of a $(Al_2O_3/Al)$ FGS nanobeams with (1-1-1) scheme

where $E_c$, $E_m$, $\rho_c$, $\rho_m$, $\nu_c$, $\nu_m$ are, respectively, Young's moduli, the mass densities, and Posson's ratio of the ceramic and metal components of the beams, Table 1 gives the material properties of some individual materials which are used in this study.

The effective Young's modulus $(E_{eff})$ and mass density $(\rho_{eff})$ of the material through the thickness of the $(Al/Si_3N_4)$ FGS nanobeams with the skin-core-skin thicknesses of (1-2-1) are presented in Fig. 2.

### *2.2 The quasi-3D deformation theory*

The displacement field of the FGS nanobeams can be written as

$$\begin{aligned} u(x,z,t) &= u(x,t) - z\frac{\partial w_b(x,t)}{\partial x} - f(z)\frac{\partial w_s(x,t)}{\partial x} \\ w(x,z,t) &= w_b(x,t) + w_s(x,t) + g(z)\phi(x,t) \end{aligned} \tag{3}$$

The function $f(z)$ defines the distribution of the transverse shear stress through the thickness of the beams, and the function $g(z) = 1 - f'(z)$ satisfies the free-conditions of the transverse shear stress on the bottom and top surfaces of the FGS beams. In this study, the function $f(z)$ is chosen as follows

$$f(z) = \frac{5}{3}\frac{z^3}{h^2} - \frac{1}{4}z \tag{4}$$

Therefore, the formulation of $g(z)$ is obtained as follows

$$g(z) = 5\left(\frac{1}{4} - \frac{z^2}{h^2}\right) \tag{5}$$

The strains fields of the beam are written as

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f\frac{\partial^2 w_s}{\partial x^2} \\ \varepsilon_z &= g'\phi \\ \gamma_{xz} &= g\frac{\partial w_s}{\partial x} + g\frac{\partial \phi}{\partial x} \end{aligned} \tag{6}$$

The relations between the stresses and strains of the FGS nanobeams are

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{7}$$

where

$$\begin{aligned} C_{11} &= C_{33} = \frac{E(z)}{1-\nu(z)^2},\ C_{13} = \nu C_{11}, \\ C_{55} &= \frac{E(z)}{2(1+\nu(z))} \end{aligned} \tag{8}$$

To establish the equations of motion of the FGS nanobeams, Hamilton's principle is employed as follows

$$0 = \int_0^T (\delta\Pi + \delta K - \delta T)dt \tag{9}$$

where $\delta\Pi$, $\delta K$ and $\delta T$ are the variations of the strain energy, the potential energy stored in elastic foundations and the kinematic energy. The variation of the strain energy is obtained as the following expression

$$\delta\Pi = \int_0^L \int_A (\sigma_x\delta\varepsilon_x + \sigma_z\delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})dAdx \tag{10}$$

Integrating through the thickness of the FGS nanobeams, one gets

$$\delta\Pi = \int_0^L \left( \begin{array}{c} N\dfrac{\partial \delta u}{\partial x} - M\dfrac{\partial^2 \delta w_b}{\partial x^2} - P\dfrac{\partial^2 \delta w_s}{\partial x^2} \\ +Q\dfrac{\partial \delta w_s}{\partial x} + S\dfrac{\partial \delta \phi}{\partial x} + R\delta\phi \end{array} \right) dx \tag{11}$$

where $N$, $M$, $P$, $Q$, $S$ and $R$ are the stress resultants which are calculated by

$$\begin{aligned} (N, M, P) &= \int_A (1, z, f)\sigma_x dA \\ (Q, S) &= \int_A (g, g)\tau_{xz} dA \\ R &= \int_A g' \sigma_z dA \end{aligned} \tag{12}$$

Inserting Eq. (7) into Eq. (12), one gets

$$\begin{Bmatrix} N \\ M \\ P \\ R \end{Bmatrix} = \begin{bmatrix} A & B & E & X \\ B & D & F & Y \\ E & F & H & V \\ X & Y & V & W \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial x^2} \\ \phi \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} Q \\ S \end{Bmatrix} = \begin{bmatrix} As & As \\ As & As \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_s}{\partial x} \\ \dfrac{\partial \phi}{\partial x} \end{Bmatrix} \tag{14}$$

where

$$(A, B, D, E, F, H) = \int_A C_{11}(1, z, z^2, f, zf, f^2) dA \tag{15}$$

$$(X, Y, V) = \int_A C_{13}(1, z, f) g' dA \tag{16}$$

$$(W) = \int_A C_{33}(g')^2 dA \tag{17}$$

$$(As) = \int_A C_{55}(g)^2 dA \tag{18}$$

The variation of the energy stored in the elastic foundations is calculated as

$$\delta K = \int_0^L R_f(\delta w_b + \delta w_s + g(-h/2)\delta\phi) dx \tag{19}$$

where $R_f$ is the force of the elastic foundation that acts on the bottom surface of the beam, and can be computed as follows

$$\begin{aligned} R_f &= k_w\left(w_b + w_s + g\left(-\frac{h}{2}\right)\phi\right) \\ &-k_s\nabla^2(w_b + w_s + g(-h/2)\phi) \end{aligned} \tag{20}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}$.

It is noticed that $g(-h/2) = 0$, therefore, the Eqs. (19) and (20) become

$$\delta K = \int_0^L R_f \delta(\delta w_b + \delta w_s) dx \tag{21}$$

$$R_f = k_w(w_b + w_s) - k_s\nabla^2(w_b + w_s) \tag{22}$$

The variation of the kinematic energy of the FGS beams is calculated as

$$\delta T = \int_0^L \int_A (\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})\rho dA dx \tag{23}$$

$$\begin{aligned} \delta T = \int_0^L \int_A \Bigg[ & \left(\dot{u} - z\frac{\partial \dot{w}_b}{\partial x} - f\frac{\partial \dot{w}_s}{\partial x}\right) \\ & \rho(z)\left(\delta\dot{u} - z\frac{\partial \delta\dot{w}_b}{\partial x} - f\frac{\partial \delta\dot{w}_s}{\partial x}\right) \\ & +(\dot{w}_b + \dot{w}_s + g\dot{\phi})\rho(z)(\delta\dot{w}_b + \delta\dot{w}_s + g\delta\dot{\phi})\Bigg] dA dx \end{aligned} \tag{24}$$

After integrating through the thickness of the FGS beam, one gets

$$\begin{aligned} \delta T = \int_0^L & \Big[I_0(\dot{u}\delta\dot{u} + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)) \\ & +I_1\left(\dot{u}\frac{\partial \delta\dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial x}\delta\dot{u}\right) + I_2\left(\dot{u}\frac{\partial \delta\dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x}\delta\dot{u}\right) + \\ & +I_3\left(\frac{\partial \dot{w}_b}{\partial x}\frac{\partial \delta\dot{w}_b}{\partial x}\right) + I_4\left(\frac{\partial \dot{w}_b}{\partial x}\frac{\partial \delta\dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x}\frac{\partial \delta\dot{w}_b}{\partial x}\right) \\ & +I_5\left(\frac{\partial \dot{w}_s}{\partial x}\frac{\partial \delta\dot{w}_s}{\partial x}\right) + J_1(\dot{w}_b + \dot{w}_s)\delta\dot{\phi} + \\ & +J_1\dot{\phi}(\delta\dot{w}_b + \delta\dot{w}_s) + J_2\dot{\phi}\delta\dot{\phi}\Big] dx \end{aligned} \tag{25}$$

where

$$\begin{aligned} & (I_0, I_1, I_2, I_3, I_4, I_5, J_1, J_2) \\ & = \int_A \rho(z)(1, -z, -f, z^2, fz, f^2, g, g^2) dA \end{aligned} \tag{26}$$

Substituting Eqs. (11), (21) and (25) into Eq. (9) and integrating by parts, the equilibrium equations of the beams are obtained as following

$$\begin{aligned} & \delta u: -\frac{\partial N}{\partial x} = -I_0\ddot{u} - I_1\frac{\partial \ddot{w}_b}{\partial x} - I_2\frac{\partial \ddot{w}_s}{\partial x}; \\ & \delta w_b: -\frac{\partial^2 M}{\partial x^2} = \left( \begin{array}{c} -k_w(w_b + w_s) + k_s\nabla^2(w_b + w_s) \\ -I_0(\ddot{w}_b + \ddot{w}_s) + I_1\dfrac{\partial \ddot{u}}{\partial x} \\ +I_3\dfrac{\partial^2 \ddot{w}_b}{\partial x^2} + I_4\dfrac{\partial^2 \ddot{w}_s}{\partial x^2} - J_1\ddot{\phi} \end{array} \right); \\ & \delta w_s: -\frac{\partial^2 P}{\partial x^2} - \frac{\partial Q}{\partial x} \\ & = \left( \begin{array}{c} -k_w(w_b + w_s) + k_s\nabla^2(w_b + w_s) \\ -I_0(\ddot{w}_b + \ddot{w}_s) + I_2\dfrac{\partial \ddot{u}}{\partial x} \\ +I_4\dfrac{\partial^2 \ddot{w}_b}{\partial x^2} + I_5\dfrac{\partial^2 \ddot{w}_s}{\partial x^2} - J_1\ddot{\phi} \end{array} \right); \\ & \delta\phi: -\frac{\partial S}{\partial x} + R = -J_1(\ddot{w}_b + \ddot{w}_s) - J_2\ddot{\phi} \end{aligned} \tag{27}$$

### *2.3 Nonlocal theory*

To take into account for the small-scale effects on the behavior of the nanobeams, the Eringen's nonlocal theory (Eringen 1967, 1972, 1983, Eringen and Edelen 1972) is adopted herein. In the Eringen's nonlocal theory, the stress at a point depends on the strains at all neighbor points of the body, hence the nonlocal stress tensor $\sigma_{ij}$ at a point $x$ is obtained via the local stress tensor $t_{ij}$ as the following formula

$$\sigma_{ij} - \mu \nabla^2 \sigma_{ij} = t_{ij} \tag{28}$$

where $\mu = (e_0 a)^2$ is the nonlocal parameter which incorporates the small-scale effect, $e_0$ is a constant appropriate to each material, $a$ is the internal characteristic length. Thus, the nonlocal constitutive relation can be written as

$$(1-\mu\nabla^2)\begin{Bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix}C_{11} & C_{13} & 0\\ C_{13} & C_{33} & 0\\ 0 & 0 & C_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{Bmatrix} \tag{29}$$

As a consequence, the stress resultants are calculated as the following formula

$$\begin{Bmatrix}N\\ M\\ P\\ R\end{Bmatrix} - \mu\nabla^2\begin{Bmatrix}N\\ M\\ P\\ R\end{Bmatrix} = \begin{bmatrix}A & B & E & X\\ B & D & F & Y\\ E & F & H & V\\ X & Y & V & W\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ -\dfrac{\partial^2 w_b}{\partial x^2}\\ -\dfrac{\partial^2 w_s}{\partial x^2}\\ \phi\end{Bmatrix} \tag{30}$$

$$\begin{Bmatrix}Q\\ S\end{Bmatrix} - \mu\nabla^2\begin{Bmatrix}Q\\ S\end{Bmatrix} = \begin{bmatrix}A_s & A_s\\ A_s & A_s\end{bmatrix}\begin{Bmatrix}\dfrac{\partial w_s}{\partial x}\\ \dfrac{\partial \phi}{\partial x}\end{Bmatrix} \tag{31}$$

### 2.4 Equations of motion

By substituting Eq. (30) into Eq. (27), the following equations of motion of the nanobeams are achieved as the following formulae

$$\begin{aligned}
&\delta u:\ -\frac{\partial N}{\partial x} = -(1-\mu\nabla^2)\left(I_0\ddot{u} + I_1\frac{\partial \ddot{w}_b}{\partial x} + I_2\frac{\partial \ddot{w}_s}{\partial x}\right);\\
&\delta w_b:-\frac{\partial^2 M}{\partial x^2}\\
&= -(1-\mu\nabla^2)\begin{pmatrix}k_w(w_b+w_s) - k_s\nabla^2(w_b+w_s)\\ +I_0(\ddot{w}_b+\ddot{w}_s) - I_1\dfrac{\partial \ddot{u}}{\partial x}\\ -I_3\dfrac{\partial^2\ddot{w}_b}{\partial x^2} - I_4\dfrac{\partial^2\ddot{w}_s}{\partial x^2} + J_1\ddot{\phi}\end{pmatrix};\\
&\delta w_s:-\frac{\partial^2 P}{\partial x^2} - \frac{\partial Q}{\partial x}\\
&= -(1-\mu\nabla^2)\begin{pmatrix}k_w(w_b+w_s) - k_s\nabla^2(w_b+w_s)\\ +I_0(\ddot{w}_b+\ddot{w}_s) - I_2\dfrac{\partial \ddot{u}}{\partial x}\\ -I_4\dfrac{\partial^2\ddot{w}_b}{\partial x^2} - I_5\dfrac{\partial^2\ddot{w}_s}{\partial x^2} + J_1\ddot{\phi}\end{pmatrix};\\
&\delta\phi:-\frac{\partial S}{\partial x} + R = -(1-\mu\nabla^2)\big(J_1(\ddot{w}_b+\ddot{w}_s) + J_2\ddot{\phi}\big)
\end{aligned} \tag{32}$$

### 2.5 Solution

In this study, a simply-simply supported FGS nanobeam is considered. The Navier's solution technique is employed to solve the equations of motion, the unknown displacement functions of the beams are assumed as the following formulae

$$\begin{aligned}
u(x,t) &= \sum_{m=1}^{\infty} U_m e^{i\omega t}\cos\alpha_m x;\\
\phi(x,t) &= \sum_{m=1}^{\infty} \Phi_m e^{i\omega t}\sin\alpha_m x;\\
w_b(x,t) &= \sum_{m=1}^{\infty} Wb_m e^{i\omega t}\sin\alpha_m x;\\
w_s(x,t) &= \sum_{m=1}^{\infty} Ws_m e^{i\omega t}\sin\alpha_m x
\end{aligned} \tag{33}$$

where $\alpha_m = m\pi/L$, $i^2 = -1$, $\omega$ is the natural frequency of the FGS nanobeams, $U_m$, $Wb_m$, $Ws_m$, $\Phi_m$ are the unknown coefficients.

Substituting Eq. (33) into Eq. (32), one gets

$$\left(\begin{bmatrix}k_{11} & k_{12} & k_{13} & k_{14}\\ & k_{22} & k_{23} & k_{24}\\ & & k_{33} & k_{34}\\ sys & & & k_{44}\end{bmatrix} - \omega^2\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ & m_{22} & m_{23} & m_{24}\\ & & m_{33} & m_{34}\\ sys & & & m_{44}\end{bmatrix}\right)\begin{Bmatrix}U_m\\ Wb_m\\ Ws_m\\ \Phi_m\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix} \tag{34}$$

The formulations of $k_{ij}$ and $m_{ij}$ are provided in Appendix.

By solving the Eq. (34) using the common manner, the vibration behaviors of the beams are achieved.

## 3. Numerical results

### 3.1 Validations

Firstly, a functionally graded sandwich beam resting on elastic foundations is examined in this section to confirm that the present algorithm is accurate for analysis of beams on elastic foundations. The considered sandwich beam is made from $Al_2O_3$ as ceramic and Al as metal components with ceramic core and two FG layers. The length of the beam is $L$, the height is $h$, and the beams are simply supported at both end sides. The material properties of these materials are given in Table 1. For this example, the nonlocal parameter equals zero, $\mu = 0$.

The dimensionless quantities are calculated via following formulae

$$K_w = k_w\frac{L^2}{A_{110}};\ K_s = k_s\frac{1}{A_{110}};\ \bar{\omega} = \omega L\sqrt{\frac{I_{00}}{A_{110}}} \tag{35}$$

where:

$$I_{00} = \rho_m h;\ A_{110} = E_m h; \tag{36}$$

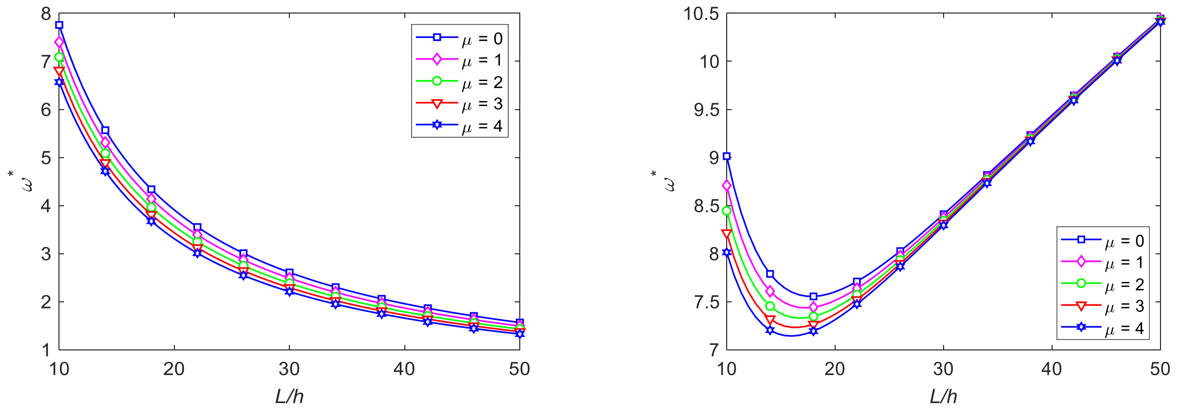
The present numerical results and the solutions of Tossapanon and Wattanasakulpong (2016) are presented in Table 2. According to this table, the present numerical results are in good agreement with the solution of Tossapanon and Wattanasakulpong (2016). Therefore, it

Table 2 Comparison frequencies of the FGS beams resting on elastic foundations

| $K_w$ | $K_s$ | Sources | 1-0-1 | 1-1-1 | 2-1-2 | 1-5-1 |
|---|---|---|---|---|---|---|
| 0 | 0 | Tossapanon and Wattanasakulpong (2016) | 0.4271 | 0.4486 | 0.4384 | 0.4920 |
| | | Present | 0.4277 | 0.4491 | 0.4390 | 0.4924 |
| 0.2 | 0 | Tossapanon and Wattanasakulpong (2016) | 0.5779 | 0.5891 | 0.5833 | 0.6178 |
| | | Present | 0.5776 | 0.5889 | 0.5831 | 0.6176 |
| 0.2 | 0.2 | Tossapanon and Wattanasakulpong (2016) | 1.3525 | 1.3363 | 1.3420 | 1.3267 |
| | | Present | 1.3493 | 1.3332 | 1.3389 | 1.3238 |

Table 3 Comparison the frequencies of FGS nanobeams

| (1-0-1) FGS nanobeams | | | | (2-2-1) FGS nanobeams | | | |
|---|---|---|---|---|---|---|---|
| $k$ | $\mu(\text{nm}^2)$ | Luat *et al.* (2021) | Present | $k$ | $\mu(\text{nm}^2)$ | Luat *et al.* (2021) | Present |
| 0 | 0 | 12.0030 | 12.0069 | 0.5 | 0 | 9.5949 | 9.6099 |
| | 2 | 10.9691 | 10.9727 | | 2 | 8.7684 | 8.7822 |
| | 4 | 10.1633 | 10.1667 | | 4 | 8.1243 | 8.1370 |
| 1 | 0 | 7.2770 | 7.2980 | 1 | 0 | 8.5053 | 8.5316 |
| | 2 | 6.6502 | 6.6694 | | 2 | 7.7727 | 7.7967 |
| | 4 | 6.1617 | 6.1795 | | 4 | 7.2017 | 7.2240 |
| 10 | 0 | 5.1965 | 5.2493 | 10 | 0 | 6.3882 | 6.4494 |
| | 2 | 4.7489 | 4.7971 | | 2 | 5.8379 | 5.8939 |
| | 4 | 4.4000 | 4.4447 | | 4 | 5.4091 | 5.4610 |




(a) $K_w = 0,\ K_s = 0$ (b) $K_w = 2,\ K_s = 1$

Fig. 3 The model of the FGS nanobeams

can be concluded that the present algorithm is accurate for predicting the frequencies of the sandwich beams resting on elastic foundations.

Secondly, the present algorithm is applied to analyze the free vibration of the functionally graded sandwich nanobeams. The beam with a ceramic core is made from $Si_3N_4$ as ceramic phase; the bottom surface is an FG layer made from Ti-6Al-4V as meal phase and $Si_3N_4$ as ceramic phase; and the top surface is an FG layer made from SUS304 as metal phase and $Si_3N_4$ as ceramic phase. Their material properties are given in Table 1. The length of the beam is $L$, the height is $h$ and the beams are simply supported at both end sides.

The dimensionless frequency is calculated via the following formula

$$\bar{\omega} = \omega L^2 \sqrt{\frac{12\rho_0}{\hbar_0^2 E_0}},\ E_0 = 200\ \text{GPa}; \quad \rho_0 = 3000\ \text{kg/m}^3;\ \hbar_0 = L/10 \tag{37}$$

Table 3 gives a comparison between the present numerical results and the solutions of Luat *et al.* (2021) with different values of nonlocal parameters $\mu$. It can be seen that the present results are close to the published data of Luat *et al.* (2021).

Table 4 The dimensionless fundamental frequency of the FGS nanobeams with $K_w = 10,\ K_s = 5$

| $L/h$ | $k$ | $\mu(\mathrm{nm}^2)$ | 1-0-1 | 1-1-1 | 2-1-2 | 2-1-1 | 2-2-1 | 1-8-1 |
|---|---|---|---|---|---|---|---|---|
| 10 | 0 | 0 | 15.5754 | 15.5754 | 15.5754 | 15.5754 | 15.5754 | 15.5754 |
| | | 1 | 15.1539 | 15.1539 | 15.1539 | 15.1539 | 15.1539 | 15.1539 |
| | | 2 | 14.7928 | 14.7928 | 14.7928 | 14.7928 | 14.7928 | 14.7928 |
| | | 3 | 14.4795 | 14.4795 | 14.4795 | 14.4795 | 14.4795 | 14.4795 |
| | | 4 | 14.2051 | 14.2051 | 14.2051 | 14.2051 | 14.2051 | 14.2051 |
| | 1 | 0 | 13.0829 | 13.4592 | 13.2668 | 13.3994 | 13.6313 | 14.7200 |
| | | 1 | 12.8581 | 13.1976 | 13.0227 | 13.1444 | 13.3556 | 14.3592 |
| | | 2 | 12.6673 | 12.9750 | 12.8154 | 12.9275 | 13.1209 | 14.0509 |
| | | 3 | 12.5033 | 12.7833 | 12.6369 | 12.7408 | 12.9186 | 13.7840 |
| | | 4 | 12.3608 | 12.6164 | 12.4817 | 12.5783 | 12.7423 | 13.5508 |
| | 2 | 0 | 12.5273 | 12.8650 | 12.6682 | 12.8192 | 13.0669 | 14.4402 |
| | | 1 | 12.3547 | 12.6531 | 12.4764 | 12.6140 | 12.8375 | 14.0999 |
| | | 2 | 12.2087 | 12.4734 | 12.3140 | 12.4401 | 12.6427 | 13.8093 |
| | | 3 | 12.0836 | 12.3190 | 12.1746 | 12.2908 | 12.4752 | 13.5580 |
| | | 4 | 11.9751 | 12.1849 | 12.0537 | 12.1611 | 12.3296 | 13.3385 |
| | 10 | 0 | 12.2691 | 12.2933 | 12.2022 | 12.3457 | 12.4931 | 14.0565 |
| | | 1 | 12.1333 | 12.1349 | 12.0590 | 12.1882 | 12.3155 | 13.7449 |
| | | 2 | 12.0186 | 12.0011 | 11.9381 | 12.0551 | 12.1652 | 13.4791 |
| | | 3 | 11.9206 | 11.8864 | 11.8346 | 11.9411 | 12.0363 | 13.2496 |
| | | 4 | 11.8357 | 11.7871 | 11.7450 | 11.8424 | 11.9245 | 13.0494 |
| 20 | 0 | 0 | 15.3499 | 15.3499 | 15.3499 | 15.3499 | 15.3499 | 15.3499 |
| | | 1 | 15.2415 | 15.2415 | 15.2415 | 15.2415 | 15.2415 | 15.2415 |
| | | 2 | 15.1504 | 15.1504 | 15.1504 | 15.1504 | 15.1504 | 15.1504 |
| | | 3 | 15.0727 | 15.0727 | 15.0727 | 15.0727 | 15.0727 | 15.0727 |
| | | 4 | 15.0057 | 15.0057 | 15.0057 | 15.0057 | 15.0057 | 15.0057 |
| | 1 | 0 | 15.1997 | 15.1237 | 15.1433 | 15.1496 | 15.1323 | 15.2151 |
| | | 1 | 15.1509 | 15.0649 | 15.0894 | 15.0927 | 15.0696 | 15.1268 |
| | | 2 | 15.1100 | 15.0157 | 15.0442 | 15.0450 | 15.0170 | 15.0527 |
| | | 3 | 15.0752 | 14.9738 | 15.0058 | 15.0044 | 14.9723 | 14.9895 |
| | | 4 | 15.0453 | 14.9378 | 14.9727 | 14.9695 | 14.9338 | 14.9351 |
| | 2 | 0 | 15.2484 | 15.0971 | 15.1424 | 15.1420 | 15.0987 | 15.1748 |
| | | 1 | 15.2126 | 15.0516 | 15.1019 | 15.0982 | 15.0487 | 15.0929 |
| | | 2 | 15.1826 | 15.0135 | 15.0681 | 15.0615 | 15.0067 | 15.0242 |
| | | 3 | 15.1572 | 14.9811 | 15.0393 | 15.0303 | 14.9711 | 14.9657 |
| | | 4 | 15.1353 | 14.9532 | 15.0145 | 15.0035 | 14.9404 | 14.9153 |
| | 10 | 0 | 15.4555 | 15.1368 | 15.2442 | 15.2264 | 15.1175 | 15.1246 |
| | | 1 | 15.4281 | 15.1044 | 15.2153 | 15.1941 | 15.0805 | 15.0515 |
| | | 2 | 15.4052 | 15.0773 | 15.1911 | 15.1671 | 15.0495 | 14.9901 |
| | | 3 | 15.3858 | 15.0542 | 15.1705 | 15.1442 | 15.0232 | 14.9379 |
| | | 4 | 15.3690 | 15.0344 | 15.1528 | 15.1245 | 15.0005 | 14.8929 |

### *3.2 Parameter study*

In the parametric study, an FGS beam made from Al as metal phase and $\mathrm{Si}_3N_4$ as ceramic phase is considered. The length of the nanobeam is $L = 10$ nm, the depth of the nanobeam is $b = 1$ nm, the height is $h$. The boundary conditions of the nanobeam are simply supported at both end sides. The following dimensionless quantities are used for convenience

Table 5 The first six frequencies of the FGS nanobeams with $L/h = 10,\ K_w = 10,\ K_s = 5$

| Schemes | $k$ | $\mu(\mathrm{nm}^2)$ | $\omega_1^*$ | $\omega_2^*$ | $\omega_3^*$ | $\omega_4^*$ | $\omega_5^*$ | $\omega_6^*$ |
|---|---|---|---|---|---|---|---|---|
| 1-1-1 | 0 | 0 | 15.5754 | 49.4345 | 100.5144 | 163.6304 | 234.7268 | 310.9977 |
| | | 1 | 15.1539 | 42.9689 | 75.3647 | 105.3886 | 130.9851 | 152.2623 |
| | | 2 | 14.7928 | 38.8904 | 63.9263 | 85.6606 | 103.7302 | 118.7371 |
| | | 3 | 14.4795 | 36.0455 | 57.1448 | 75.1177 | 90.0937 | 102.6702 |
| | | 4 | 14.2051 | 33.9315 | 52.5809 | 68.4010 | 81.6785 | 92.9433 |
| | 1 | 0 | 13.4592 | 38.9446 | 78.1899 | 128.1227 | 185.9226 | 249.3702 |
| | | 1 | 13.1976 | 34.4428 | 59.8396 | 84.3205 | 106.0498 | 124.7850 |
| | | 2 | 12.9750 | 31.6492 | 51.6547 | 69.7855 | 85.4895 | 98.9668 |
| | | 3 | 12.7833 | 29.7282 | 46.8804 | 62.1417 | 75.3477 | 86.7216 |
| | | 4 | 12.6164 | 28.3185 | 43.7116 | 57.3347 | 69.1524 | 79.3456 |
| | 2 | 0 | 12.8650 | 35.5963 | 70.6426 | 115.6153 | 168.1478 | 226.2905 |
| | | 1 | 12.6531 | 31.7650 | 54.6875 | 77.0354 | 97.1209 | 114.6370 |
| | | 2 | 12.4734 | 29.4070 | 47.6455 | 64.3764 | 79.0353 | 91.7162 |
| | | 3 | 12.3190 | 27.7966 | 43.5716 | 57.7722 | 70.1716 | 80.8835 |
| | | 4 | 12.1849 | 26.6216 | 40.8853 | 53.6433 | 64.7770 | 74.3579 |
| | 10 | 0 | 12.2933 | 31.9571 | 62.0522 | 100.9329 | 146.7583 | 197.9261 |
| | | 1 | 12.1349 | 28.9010 | 48.9342 | 68.6498 | 86.5708 | 102.3514 |
| | | 2 | 12.0011 | 27.0426 | 43.2387 | 58.2434 | 71.5013 | 82.9970 |
| | | 3 | 11.8864 | 25.7853 | 39.9830 | 52.8760 | 64.1758 | 73.8699 |
| | | 4 | 11.7871 | 24.8751 | 37.8555 | 49.5450 | 59.7300 | 68.3479 |
| 2-1-2 | 0 | 0 | 15.5754 | 49.4345 | 100.5144 | 163.6304 | 234.7268 | 310.9977 |
| | | 1 | 15.1539 | 42.9689 | 75.3647 | 105.3886 | 130.9851 | 152.2623 |
| | | 2 | 14.7928 | 38.8904 | 63.9263 | 85.6606 | 103.7302 | 118.7371 |
| | | 3 | 14.4795 | 36.0455 | 57.1448 | 75.1177 | 90.0937 | 102.6702 |
| | | 4 | 14.2051 | 33.9315 | 52.5809 | 68.4010 | 81.6785 | 92.9433 |
| | 1 | 0 | 13.2668 | 37.7936 | 75.5469 | 123.6614 | 179.4614 | 240.8227 |
| | | 1 | 13.0227 | 33.5260 | 58.0420 | 81.7349 | 102.8291 | 121.0708 |
| | | 2 | 12.8154 | 30.8851 | 50.2623 | 67.8774 | 83.1821 | 96.3477 |
| | | 3 | 12.6369 | 29.0733 | 45.7372 | 60.6108 | 73.5153 | 84.6417 |
| | | 4 | 12.4817 | 27.7464 | 42.7408 | 56.0510 | 67.6193 | 77.5932 |
| | 2 | 0 | 12.6682 | 34.2689 | 67.4678 | 110.1130 | 160.0113 | 215.3367 |
| | | 1 | 12.4764 | 30.7233 | 52.5647 | 73.9018 | 93.1321 | 109.9468 |
| | | 2 | 12.3140 | 28.5503 | 46.0248 | 62.0958 | 76.2112 | 88.4348 |
| | | 3 | 12.1746 | 27.0713 | 42.2577 | 55.9631 | 67.9471 | 78.2849 |
| | | 4 | 12.0537 | 25.9953 | 39.7820 | 52.1403 | 62.9255 | 72.1665 |
| | 10 | 0 | 12.2022 | 30.9921 | 59.4882 | 96.1704 | 139.3059 | 187.3922 |
| | | 1 | 12.0590 | 28.1736 | 47.2913 | 66.0568 | 83.0863 | 98.0576 |
| | | 2 | 11.9381 | 26.4674 | 42.0321 | 56.4281 | 69.1260 | 80.0964 |
| | | 3 | 11.8346 | 25.3171 | 39.0400 | 51.4851 | 62.3616 | 71.6277 |
| | | 4 | 11.7450 | 24.4866 | 37.0914 | 48.4255 | 58.2574 | 66.4860 |

$$K_w = k_w \frac{L^4}{D_0};\ K_s = k_s \frac{L^2}{D_0};\ \omega^* = \omega L \sqrt{\frac{\rho_0}{E_0}},$$
$$D_0 = \frac{E_c h_0^3}{12(1-\nu_c^2)};\ E_0 = 200\ \text{GPa};\ \rho_0 = 3000\ \text{kg/m}^3;\ h_0 = L/10. \tag{38}$$

The dimensionless fundamental frequencies of the FGS nanobeams for various values of the length-to-thickness ratios, power-law indexes, nonlocal parameters, and schemes are given in Table 4, while the first six frequencies of the FGS nanobeams are given in Table 5. The parameters of the elastic foundations are $K_w = 10,\ K_s = 5$.

In general, it can be seen that when the power-law index increases, the dimensionless frequencies of the FGS nanobeams decrease. Because when the power-law index increases, the volume fraction of the metal phase increases; therefore, the stiffness of the FGS nanobeams decreases. When the power-law index equals zeros, the FGS nanobeams become homogeneous isotropic ceramic nanobeams, and the stiffness of the beams is the greatest; therefore, the frequency of the beam is maximum in comparison with FGS nanobeams with a power-law index greater than zeros. Besides, it can be seen that the FGS nanobeams are softened when considering the size-effects on the free vibration behaviors.

The first six frequencies of the FGS nanobeams with various values of the power-law indexes and nonlocal parameters with two schemes (1-1-1) and (2-1-2) of FGS nanobeams are presented in Table 5. The ratio of length-to-height of the beam is 10 and two parameters of the elastic foundations are $K_w = 10,\ K_s = 5$. According to this table, it is obvious that the nonlocal parameters have significant effects on the free vibration of the FGS nanobeams. The effects of the nonlocal parameters on the high frequencies are more significant than on the low frequencies. For the first six frequencies of the FGS nanobeams, when the power-law index increases, the frequencies of the FGS nanobeams decrease due to the fact that the volume fraction of the metal phase increases.

For more details on the effects of the length-to-height ratio, Fig. 3 plots the variation of dimensionless frequencies of the FGS nanobeams with respect to the variation of the ratio $L/h$. Fig. 3(a) presents the variation of the frequencies of FGS nanobeams without elastic foundations. It can be seen that when the ratio $L/h$ increases, the frequencies of the beams decrease rapidly. Fig. 3(b) presents the variation of the frequencies of the FGS nanobeams with elastic foundations. When the ratio $L/h$ increases, the frequencies of the FGS nanobeams first decrease and reach the minimum values, then the frequencies of the FGS nanobeams increase quickly. This is a new important finding that should be considered when designing the FGS nanobeams for nano-devices. By comparing Fig. 3(a) and Fig. 3(b), it can be seen that the elastic foundations play a significant role in the free vibration behaviors of the FGS nanobeams.

The effects of the power-law index on the free vibration of the FGS nanobeams are presented in Fig. 4. When the power-law index increases, the dimensionless frequencies of the FGS nanobeams decrease. When the power-law index increases from 0 to 2, the frequencies of the beams decrease rapidly. However, when the power-law index is greater than 2, the frequencies of the FGS nanobeams decrease slowly when the power-law index increases. Additionally, by comparing the frequencies of the FGS nanobeams with four values of the nonlocal parameters $\mu = 0,\ 1,\ 2,\ 3,\ 4$ with variable power-law index, it can be seen that the frequencies of the local FGS nanobeams ($\mu = 0$) are greater than the frequencies of the nonlocal ones ($\mu > 0$).

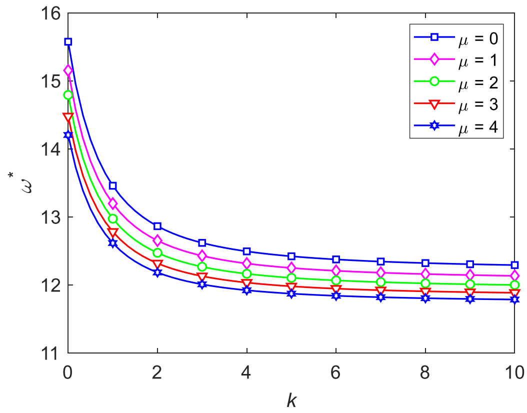



Fig. 4 The effects of $k$ on the fundamental frequencies of the (1-1-1) FGS nanobeams with $L/h = 10,\ K_w = 10,\ K_s = 5$

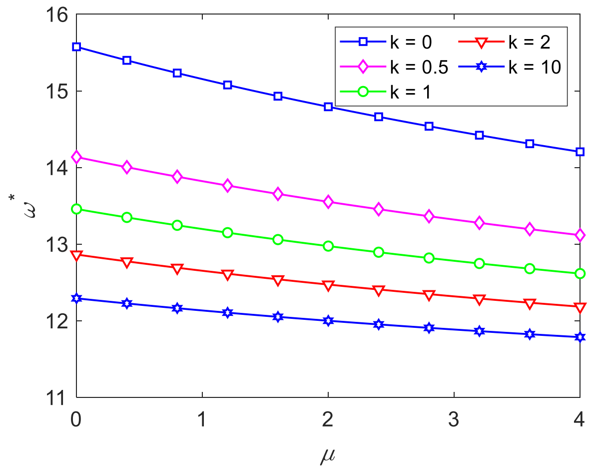



Fig. 5 The effects of $\mu$ on the fundamental frequencies of the (1-1-1) FGS nanobeams with $L/h = 10,\ K_w = 10,\ K_s = 5$

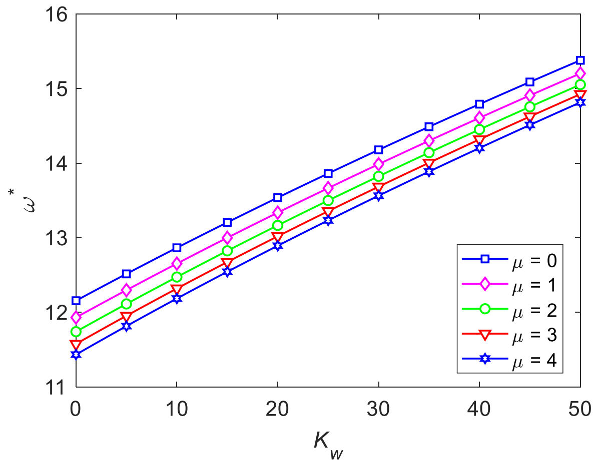



Fig. 6 The effects of $K_w$ on the fundamental frequencies of the (1-1-1) FGS nanobeams with $L/h = 10,\ k = 2,\ K_s = 5$

Continuously, the size-effects of the small-scale on the free vibration behaviors of the FGS nanobeams are exhibited in Fig. 5. It can be seen that when the nonlocal

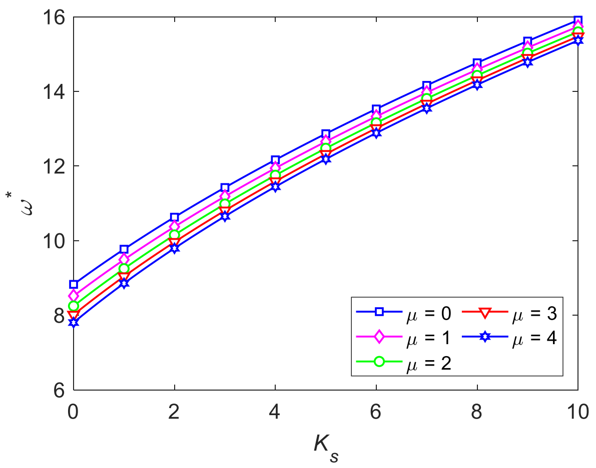



Fig. 7 The effects of $K_s$ on the fundamental frequencies of the (1-1-1) FGS nanobeams with $L/h = 10$, $k = 2$, $K_w = 10$

parameter $\mu$ increases, the dimensionless frequencies of the FGS nanobeams decrease. However, the speed of the variation is different and depends on the power-law index of the FGS nanobeams. The effect of the nonlocal parameters on the full ceramic beam ($k = 0$) is more noteworthy than on the FGS beams ($k > 0$).

The influences of two parameters of the elastic foundations on the free vibration behaviors of the FGS nanobeams are illustrated in Figs. 6 and 7. In general, the frequencies of the FGS nanobeams increase when increasing two parameters $K_w$ and $K_s$. According to Fig. 6, the frequencies of the FGS nanobeams with $K_w = 50$ are about 25% greater than the frequencies of the FGS nanobeams with $K_w = 0$. According to Fig. 7, the frequencies of the FGS nanobeams with $K_s = 10$ are about 85% greater than the frequencies of the FGS nanobeams with $K_s = 0$. Therefore, it can be concluded that the effect of the parameter $K_s$ is more significant than the parameter $K_w$.

## 4. Conclusions

In this study, a nonlocal quasi-3D theory has been developed for size-dependent free vibration analysis of the FGS nanobeams. A combination of Hamilton's principle and nonlocal elasticity theory has been used to establish the governing equations of the motion of the FGS nanobeams. These equations were solved by Navier's technique to achieve the free vibration behaviors of the FGS nanobeams. According to the outcomes of this study, some important conclusions can be drawn as follows:

• Nonlocal parameters: The results shown that considering nonlocal parameters decreases FGS nanobeam frequencies, underlining the importance of accounting for small-scale effects in their analysis.

• Elastic foundations: Elastic foundations significantly stiffen FGS nanobeams, raising their frequencies. This insight has practical implications for nanoscale engineering applications.

• Power-law index: FGS nanobeam frequencies decrease as the power-law index increases, offering greater control over their performance.

• Length-to-height ratio: The influence of the length-to-height ratio on frequencies varies with elastic foundations, highlighting a critical point for frequency optimization.

These findings advance our understanding of FGS nanobeams and have broad engineering and industry applications. They contribute to nanoscale structural design, materials science, electronics, and nanomechanical systems. Our work empowers researchers and engineers to make informed decisions in design and analysis, cementing its importance in nanotechnology.

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## Appendix

The formulations of $k_{ij}$ and $m_{ij}$ in Eq (34) are given as follows

$$k_{11} = \alpha^2 A;\ k_{12} = -\alpha^3 B;\ k_{13} = -\alpha^3 E;\ k_{14} = -X\alpha;$$
$$k_{22} = (k_s\mu + D)\alpha^4 + (k_w\mu + ks)\alpha^2 + k_w;$$
$$k_{23} = (k_s\mu + F)\alpha^4 + (k_w\mu + ks)\alpha^2 + k_w;$$
$$k_{24} = Y\alpha^2;\ k_{33} = (k_s\mu + H)\alpha^4 + (k_w\mu + As + k_s)\alpha^2 + k_w;$$
$$k_{34} = (V + A_s)\alpha^2;\ k_{44} = A_s\alpha^2 + W;$$
$$m_{11} = (\mu\alpha^2 + 1)I_0;\ m_{12} = (\mu\alpha^2 + 1)I_1\alpha;$$
$$m_{13} = (\mu\alpha^2 + 1)I_2\alpha;\ m_{14} = 0;$$
$$m_{22} = (\mu\alpha^2 + 1)(I_3\alpha^2 + I_0);\ m_{23} = (\mu\alpha^2 + 1)(I_4\alpha^2 + I_0);$$
$$m_{24} = (\mu\alpha^2 + 1)J_1;\ m_{33} = (\mu\alpha^2 + 1)(I_5\alpha^2 + I_0);$$
$$m_{34} = (\mu\alpha^2 + 1)J_1;\ m_{44} = (\mu\alpha^2 + 1)J_2;$$