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Investigation of nonlinear vibration behavior of the stepped nanobeam

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Abstract. Nonlinearity plays an important role in control systems and the application of design. For this reason, in addition to linear vibrations, nonlinear vibrations of the stepped nanobeam are also discussed in this manuscript. This study investigated the vibrations of stepped nanobeams according to Eringen's nonlocal elasticity theory. Eringen's nonlocal elasticity theory was used to capture the nanoscale effect. The nanoscale stepped Euler Bernoulli beam is considered. The equations of motion representing the motion of the beam are found by Hamilton's principle. The equations were subjected to nondimensionalization to make them independent of the dimensions and physical structure of the material. The equations of motion were found using the multi-time scale method, which is one of the approximate solution methods, perturbation methods. The first section of the series obtained from the perturbation solution represents a linear problem. The linear problem's natural frequencies are found for the simple-simple boundary condition. The second-order part of the perturbation solution is the nonlinear terms and is used as corrections to the linear problem. The system's amplitude and phase modulation equations are found in the results part of the problem. Nonlinear frequency-amplitude, and external frequency-amplitude relationships are discussed. The location of the step, the radius ratios of the steps, and the changes of the small-scale parameter of the theory were investigated and their effects on nonlinear vibrations under simple-simple boundary conditions were observed by making comparisons. The results are presented via tables and graphs. The current beam model can assist in designing and fabricating integrated such as nano-sensors and nano-actuators.

Keywords: analytical modelling; nonlocal elasticity; stepped nanobeams; vibration

1. Introduction

The discovery of nanomaterials in the 21st century has earned a new dimension to the science world. Iijima (1991, 1993) opened a new window of nanotechnology developments, then worldwide investments in nanotechnology applications are increasing day by day. Today, nanotechnology is preferred in a lot of fields such as the electronics industry (Ekinci *et al*. 2005, Bhushan 2007), aerospace industry (Haynes *et al*. 2013), medical applications (Saji *et al*. 2010), pharmaceuticals (Arpagus *et al*. 2018), cosmetics, and environmental processes. Nanostructures are used as important elements such as switches (Taghavi and Nahvi 2013), nanoshell (Arefi 2018), nano actuators (Siegmar 2002), nanosensors (Anker 2010), nano-resonators (Li and Chou 2004), and fluid transport devices (Sheikholeslami 2019). To improve the nano-systems, it is necessary to be able to predict behavior for various conditions.

Undoubtedly, one of the main areas of nano-scale studies is nanomechanics. The study area of nanomechanics covers force and displacement relationships, vibration and frequency analysis, and functional and strain characteristics of nanoscale structures. With the emergence of smart materials technology, the carbon nanotubes (CNT) (Iijima 1991, Eltaher 2020, 2021, Civelek 2022) discovery in the scientific world, the addition of microelectromechanical systems (MEMS) (Jack 2001, Xie *et al*. 2003, Alkharabsheh 2013, Fu *et al*. 2013), and nanoelectromechanical systems (NEMS) (Ebrahimy 2016a, b, 2019, Bornassi 2017, Sharma 2012) to research topics have greatly increased the interest in nanomechanics.

Recently, the scientific world has been trying to produce alternative methods to study the mechanics of nano-structures. Thanks to these efforts, billions of meters of models such as nanoparticles, nano-scale structures, namely rods, beams, plates, have taken their place in many analyses. Study of the mechanics of the models (Eric *et al*. 1997, Aydogdu 2009, Şimşek 2016, Farajpour *et al*. 2011, Gul *et al*. 2017, Eltaher *et al*. 2018, Yapanmis 2023, Nalbant *et al*. 2023a), Analysis of functionally graded models (Eltaher *et al*. 2013, Natarajan *et al*. 2012, Barretta *et al*. 2015), and NEMS models (Gusso *et al*. 2019, Cha *et al*. 2018, Kim and Ahn 2008).

Researchers faced the challenge of adequate theory to analyze nanoelements. Because nano-scale structures have dislocations, atomic grain boundaries, steps same as in Fig. 1, or small-scale discontinuities such as cracks. This situation increases the surface-to-volume ratio at the nanoscale. Also, atomic attraction changes the strength of the elements. For these reasons, it has been observed that the classical continuum theory is not suitable for analysis.

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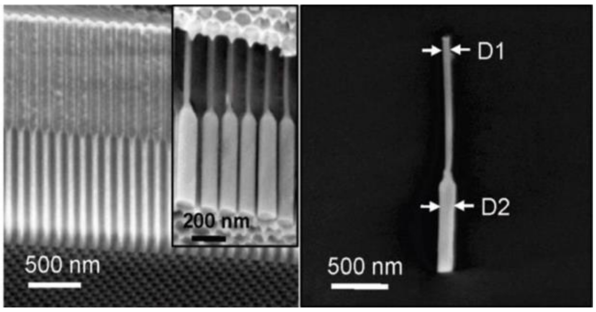

Fig. 1 On the left, cross-sectional SEM images of a blank AAM with dual-diameter pores and the Ge DNPLs (inset) after the growth. On the right, SEM of a single Ge DNPL after harvesting and drop-casting on a siliconsubstrate (Fan *et al*. 2010)

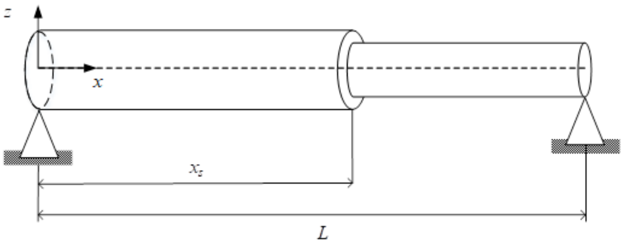

Fig. 2 Schematic representation of stepped nanobeam

Modeling the continuum mechanics of nanoscale structures must be replaced by theories that involve measurements for non-classical phenomena. (Shaat and Abdelkefi 2010). Experimental studies of nano-sized structures are both uneconomical and physically very difficult. For this reason, various numerical methods have been produced to work on these structures. (Li and Chou 2006). Numerical methods are divided into two main branches. The first is the atomistic approach (Liew 2006), and the second is density functional theory (Sanchez-Portal 1999). Although these methods are not advantageous in terms of time, they cannot respond to small-scale effects.

Therefore, the theory of continuum mechanics is much preferred and can capture effects considered important for the nanoscale. (Yapanmis *et al*. 2022). Examples of these numerical methods are modified the couple stress theory (Liu and Reddy 2011, Li *et al*. 2021), micropolar elasticity theory (Faraji-Oskouie 2019), the strain gradient theory (Nix and Gao 1998, Thai *et al*. 2021, Monaco *et al*. 2021), surface elasticity theory (Lu *et al*. 2018, Abdelrahman *et al*. 2021), doublet mechanics (Eltaher *et al*. 2020), and Eringen’s nonlocal elasticity theory (Eringen 1983).

Many researchers have adopted Eringen’s theory of nonlocal elasticity and applied it to study the mechanical behavior of various types of nanostructures. (Eringen 2002, 2006, Azandariani and Nikzad 2022). Eltaher *et al*. (2016) preferred nonlocal elastic models to study topics such as bending of nanoscale beams, vibrations, and wave propagation, Bagdatli (2015) used that theory to study the nonlinear vibrations of nanobeams. Pradhan and Phadikar (2015) took up this theory on plate theory. Shaat and Abdelkefi (2010) developed new insights to increase the applicability of this theory and applied it to different types of materials. Reddy *et al*. (2009) established a high-order nonlocal strain gradient theory in thermodynamics. Abdelrahman *et al*. (2021) studied the dynamics of hollow high order nanobeams subjected to live load using nonlocal strain gradient theory. Alazwari *et al*. (2022) carried out the dynamic behavior of temperature-dependent Reddy functionally graded (RFG) nanobeam subjected to thermo-magnetic effects under the action of moving point load.

The steps observed in nano-scale structures, which are given in Fig. 1, are neglected in theoretical nanobeam studies. This situation is likely to cause problems at the point of transforming the design into practice. (Assaidi *et al*. 2021) In their study, he emphasized that advanced micro/nano systems should be developed and their mechanical behavior should be predicted correctly, and it is not correct to say that nanobeams are continuous as in classical beams.

In the literature, there are studies on stepped beams (Nalbant *et al*. 2023b, Nalbant and Tekin 2023). However, when the studies on the nanoscale are examined, there are very few studies on stepped nanobeams. The first three works by Jaan and Lenbaum (2018, 2019) and Masih (2018) stand out in free vibration analyses of stepped nanobeams. Taima *et al*. (2021) conducted vibration analysis on multistep nanobeams, and Assadi and Nazemzadeh (2021) examined the nonlinear vibrations of stepped nanobeams for the first time. They used the theory of surface elasticity as a methodology and worked on specific materials. For the analytical solution, the variable separation method was preferred.

This study will be the second study in which nonlinear vibrations of stepped nanobeams are investigated. In this study, vibration analysis of Euler-Bernoulli stepped nanobeam was carried out. The effects of non-local parameters of Eringen’s theory were considered in the analysis. The fundamental frequencies were defined for different step locations, step ratios, and non-local parameters. All values were affected by local parameters, step rate, and step position. Unlike the other nonlinear work, an infrastructure was created for the design of much more materials by working without dimensions. The perturbation method was preferred for the first time as an analytical solution. In this way, the possibility of nonlinear analysis is presented. The multi-scale method, which is one of these methods, was preferred and was used for the approximate solution of the motion equations of the stepped nanobeam. The Two terms were obtained from the perturbation expansion. The first creates the linear problem. From this part, natural frequencies and mode shapes were obtained for simple-simple boundary conditions. Data are given in tables.

From the second term of the expansion, nonlinear effects are obtained. Here the amplitude and phase modulation equations are found. Using the equations found, the nonlinearity of the system was examined. The effects of non-local parameters, beam step ratio, and step location on nonlinearity were investigated.

This study consists of five parts. In the second part, the constitutive equations of the nanobeam are obtained by using Eringen’s non-local elasticity theory, in the third part, the equations of motion and boundary conditions are obtained by mathematical modeling. In section 3, linear and nonlinear vibration behaviors are examined and the effects of step ratio and location on vibration and natural frequency of the beam are investigated in nonlocal elasticity theory. The data obtained in section 4 are interpreted by showing them with tables and graphs. The conclusions and main points are summarized in section 5.

2. Nonlocal elasticity theory

According to the nonlocal elasticity theory, the stress at a reference point in the body of the body depends not only on the stresses at that point, but also on the stresses at all other points of the body. This observation is in line with the atomic theory of lattice dynamics and experimental observations on phonon dispersion. The classical (local) theory of elasticity is obtained when the effects of strains at points other than the reference point are neglected in the limit (Eringen 2002).

Nanobeam's material is defaulted to be a non-linear elastic material that conforms to a nonlocal elasticity theory. According to Eringen's (1983) concepts, the constitutive equations for materials conforming to nonlocal elasticity be expressed as:

$$\sigma_{ij}^{n}(x^*) = \iiint_{(V)} a(|x^{*\prime} - x^*|, \tau)\sigma_{ij}^{c}(x^{*\prime})dV \tag{1}$$

Eq. (1), σ_{ij}^{n} denotes the tension tensor at non-local elasticity, σ_{ij}^{c} the classical (hooke) tension tensor, and $V$the volume. Here, a is the kernel function, which is assumed to express the effect of the stress state in $x^{*\prime} \in V$and the stress-strain state in$x^* \in V$, and τ is the physical constant (Eringen 2002).

Different forms of kernel function $a(x^*)$ in eq. (1) describe different approximate models of nonlocal elasticity. Suppose $a(x^*)$ is a linear differential operator L function. In this case,

$$La([x^{*\prime} - x^*]) = \delta([x^{*\prime} - x^*]) \tag{2}$$

Here δ is Dirac's δ - function. Eringen (2006), Lu *et al.* (2006) have shown that the function can be obtained by taking a simple two-dimensional kernel function (Jaan and Lenbaum 2018).

$$L(a) = (1 - (e_0a)^2\nabla^2)a(x^*) \tag{3}$$

Here ∇ is laplace operator. Eq. (3), e_0 is a physical constant. a is the repetitive interatomic distance parameter (lattice size) in the lattice structure of nanomaterials. Eringen named the e_0a expression as a small-scale parameter and suggested that its value should be taken in $e_0a < 2$ nanometer scales (Eringen 1983).

According to Eqs. (1) - (3) the constitutive equation of nonlocal elasticity can be determined as follows,

$$(1 - e_0a\nabla^2)\sigma_{ij}^{n} = \sigma_{ij}^{c} \tag{4}$$

for homogeneous isotropic Euler Bernoulli beam,

$$\sigma(x^*) - (e_0a)^2\frac{\partial^2\sigma(x^*)}{\partial x^2} = E\varepsilon(x^*) \tag{5}$$

3. Equation of motion

The considered simple simply supported stepped nanobeam is shown in Fig. 2. The size of the stepped nanobeam is L, and the expression representing the location of the step is x_s.

In order to obtain the equation of motion of the stepped nanobeam, Hamilton's principle, which is one of the energy methods, was preferred. The Lagrangian expression for the system is written as:

$$\begin{aligned}
£ =& \frac{1}{2}\int_0^{x_s} \mathrm{\rho A_1}\left(\frac{\partial {w_1}^*}{\partial t^*}\right)^2 \mathrm{dx}^* + \frac{1}{2}\int_L^{x_s} \mathrm{\rho A_2}\left(\frac{\partial {w_2}^*}{\partial t^*}\right)^2 \mathrm{dx}^* \\
&-\frac{1}{2}\int_0^{x_s}\left(\begin{matrix} EI_1\dfrac{\partial^2 {w_1}^*}{\partial x^{*2}} + (e_0a)^2N\dfrac{\partial^2 {w_1}^*}{\partial x^{*2}} \\ -(e_0a)^2\rho A_1\dfrac{\partial^2 {w_1}^*}{\partial t^{*2}}\end{matrix}\right)\frac{\partial^2 {w_1}^*}{\partial x^{*2}}\,dx^* \\
&-\frac{1}{2}\int_0^{x_s} N\left(\frac{\partial {w_1}^*}{\partial x^*}\right)^2 \\
&-\frac{1}{2}\int_{x_s}^{L}\left(\begin{matrix} EI_2\dfrac{\partial^2 {w_2}^*}{\partial x^{*2}} + (e_0a)^2N\dfrac{\partial^2 {w_2}^*}{\partial x^{*2}} \\ -(e_0a)^2\rho A_2\dfrac{\partial^2 {w_2}^*}{\partial t^{*2}}\end{matrix}\right)\frac{\partial^2 {w_2}^*}{\partial x^{*2}}\,dx^* \\
&-\frac{1}{2}\int_{x_s}^{L} N\left(\frac{\partial {w_2}^*}{\partial x^*}\right)^2
\end{aligned} \tag{6}$$

Here, $()^*$ represents dimensional parameters. w^* denotesthe transverse displacement of the beam. t^* is the time. ρ represents the density of the stepped nanobeam and represents the cross-sectional areas of the stepped nanobeam. A_1 and A_2 represent the cross-sectional areas of the stepped nanobeam. E is the modulus of elasticity of the stepped nanobeam, and I_1 and I_2 are the cross-section moments of inertia. N is defined as the axial force. L is the length scale parameter of the stepped nanobeam.

Also, in Eq. (6), the first integral relates to the kinetic energy of the beam, the second integral to the potential energy due to the bending moment, and the last integrals to the axial stress for before and after the step respectively. The equations of motion of the system are found by Hamilton's principle. Invoking Hamilton's principle,

$$\delta\int_{t_1}^{t_2} Ldt^* = 0 \tag{7}$$

The equations of motion and boundary conditions (10) of the stepped nanobeam before (8) and after (9) the step have been found as follows,

$$\begin{aligned}
&EI_1\frac{\partial^4 {w_1}^*}{\partial x^{*4}} + \rho A_1\left(\frac{\partial^2 {w_1}^*}{\partial t^{*2}} - (e_0a)^2\frac{\partial^4 {w_1}^*}{\partial t^{*2}\partial x^{*2}}\right) \\
&= \frac{EA_1}{2\left[x_s + \frac{(L-x_s)}{\alpha^2}\right]}\left[\begin{matrix}\displaystyle\int_0^{x_s}\left(\frac{\partial {w_1}^*}{\partial x^*}\right)^2 dx^* \\ \displaystyle+\int_{x_s}^{L}\left(\frac{\partial {w_2}^*}{\partial x^*}\right)^2 dx^*\end{matrix}\right] \\
&\left(\frac{\partial^2 {w_1}^*}{\partial x^{*2}} - (e_0a)^2\frac{\partial^4 {w_1}^*}{\partial x^{*4}}\right)
\end{aligned} \tag{8}$$

$$EI_2\frac{\partial^4 {w_2}^*}{\partial x^{*4}} + \rho A_2\left(\frac{\partial^2 {w_2}^*}{\partial t^{*2}} - (e_0a)^2\frac{\partial^2 {w_2}^*}{\partial t^{*2}\partial x^{*2}}\right) \tag{9}$$

$$= \frac{EA_1}{2\left[x_s + \frac{(L-x_s)}{\alpha^2}\right]} \left[\begin{array}{l} \int_0^{x_s} \left(\frac{\partial {w_1}^*}{\partial x^*}\right)^2 dx^* \\ + \int_{x_s}^{L} \left(\frac{\partial {w_2}^*}{\partial x^*}\right)^2 dx^* \end{array} \right] \left(\frac{\partial^2 {w_2}^*}{\partial {x^*}^2} - (e_0 a)^2 \frac{\partial^4 {w_2}^*}{\partial {x^*}^4}\right)$$

For Simple-Simple Support,

$$\begin{gathered} \frac{\partial^2 {w_1}^*(0)}{\partial {x^*}^2} = 0, \delta \;\; {w_1}^*(0) = 0, \frac{\partial^2 {w_2}^*(L)}{\partial {x^*}^2} = 0, \delta \\ {w_2}^*(L) = 0 \;\; \delta {w_1}^*(x_s) = \delta {w_2}^*(x_s), \\ \frac{\partial\left(\delta {w_1}^*(x_s)\right)}{\partial x^*} = \frac{\partial\left(\delta {w_2}^*(x_s)\right)}{\partial x^*}, \\ -EI_1 \frac{\partial^2 {w_1}^*(x_s)}{\partial {x^*}^2} + EI_2 \frac{\partial^2 {w_2}^*(x_s)}{\partial {x^*}^2} = 0, E \\ I_1 \frac{\partial^3 {w_1}^*(x_s)}{\partial {x^*}^3} - EI_2 \frac{\partial^3 {w_2}^*(x_s)}{\partial {x^*}^3} = 0 \end{gathered} \tag{10}$$

Dimensionless parameters are associated with dimensional values marked with an asterisk and equations are nondimensionalized

$$\begin{gathered} x = \frac{x^*}{L}, w_{1,2} = \frac{{w_{1,2}}^*}{R_{1,2}}, t = \beta t^*, \gamma = \frac{e_0 a}{L}, \\ \alpha = \frac{r_2}{r_1}, \eta = \frac{x_s}{L}, \beta = \frac{1}{L^2}\sqrt{\frac{EI_1}{\rho A_1}} \end{gathered} \tag{11}$$

α is a dimensionless parameter that indicates the ratio of the radii of the steps at Eq. (11). γ is a dimensionless non-local parameter. η is a dimensionless parameter expressing the step location. R is the parameter expressing the radius of inertia of the circular cross section stepped beam.

The dimensionless states of Eqs. (8) and (9) are as follows:

$$\begin{gathered} \frac{\partial^4 w_1}{\partial x^4} + \left(\frac{\partial^2 w_1}{\partial t^2} - \gamma^2 \frac{\partial^4 w_1}{\partial x^2 \partial t^2}\right) \\ = \frac{1}{2\left(\eta + \frac{1}{\alpha^2}(1-\eta)\right)} \left[\begin{array}{l} \int_0^{\eta} \left(\frac{\partial w_1}{\partial x}\right)^2 dx \\ + \int_{\eta}^{1} \alpha^2 \left(\frac{\partial w_2}{\partial x}\right)^2 dx \end{array} \right] \left(\frac{\partial^2 w_1}{\partial x^2} - \gamma^2 \frac{\partial^4 w_1}{\partial x^4}\right) \end{gathered} \tag{12}$$

$$\begin{gathered} \frac{\partial^4 w_2}{\partial x^4} + \frac{1}{\alpha^2}\left(\frac{\partial^2 w_2}{\partial t^2} - \gamma^2 \frac{\partial^4 w_2}{\partial x^2 \partial t^2}\right) \\ = \frac{1}{2\left(\eta + \frac{1}{\alpha^2}(1-\eta)\right)} \frac{1}{\alpha^4} \left[\begin{array}{l} \int_0^{\eta} \left(\frac{\partial w_1}{\partial x}\right)^2 dx \\ + \int_{\eta}^{1} \alpha^2 \left(\frac{\partial w_2}{\partial x}\right)^2 dx \end{array} \right] \left(\frac{\partial^2 w_2}{\partial x^2} - \gamma^2 \frac{\partial^4 w_2}{\partial x^4}\right) \end{gathered} \tag{13}$$

And the dimensionless boundary conditionss are:

$$\begin{gathered} \frac{\partial^2 w_1(0)}{\partial x^2} = 0, w_1(0) = 0, \\ \frac{\partial^2 w_2(1)}{\partial x^2} = 0, w_2(1) = 0 \; w_1(\eta) = \alpha w_2(\eta), \\ \frac{\partial w_1(\eta)}{\partial x} = \alpha \frac{\partial w_2(\eta)}{\partial x}, \quad \frac{\partial^2 w_1(\eta)}{\partial x^2} = \alpha^5 \frac{\partial^2 w_2(\eta)}{\partial x^2}, \\ \frac{\partial^3 w_1(\eta)}{\partial x^3} = \alpha^5 \frac{\partial^3 w_2(\eta)}{\partial x^3} \end{gathered} \tag{14}$$

4. Analytical solution

The equations of motions (12-13) and boundary conditions (14) were found and dimensionless in the previous section. In this section, the approximate solution is obtained by the perturbation method. The multi-scale method, which is one of the perturbation methods, is applied for the solution. (Nayfeh 1979, 1981).

If damping and forcing terms are added to the equations of motion, the following equations are obtained,

$$\begin{gathered} w_1^{iv} + \ddot{w}_1 - \gamma^2 \ddot{w}_1'' \\ = \frac{1}{2\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)} \left[\int_0^{\eta} {w_1'}^2 dx + \alpha^2 \int_{\eta}^{1} {w_2'}^2 dx\right] \\ \left[w_1'' - \gamma^2 w_1^{iv}\right] + \bar{F} \cos \Omega t - 2\bar{\mu}\dot{w}_1 \end{gathered} \tag{15}$$

$$\begin{gathered} w_2^{iv} + \frac{1}{\alpha^2}(\ddot{w}_2 - \gamma^2 \ddot{w}_2'') \\ = \frac{1}{2\alpha^4\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)} \left[\int_0^{\eta} {w_1'}^2 dx + \alpha^2 \int_{\eta}^{1} {w_2'}^2 dx\right] \\ \left[w_2'' - \gamma^2 w_2^{iv}\right] + \bar{F} \cos \Omega t - 2\bar{\mu}\dot{w}_2 \end{gathered} \tag{16}$$

To add forcing and damping expressions to the system within the scope of weakly nonlinear effects, ε, deflection w_1 and w_2 are transformed $w_1 = \sqrt{\varepsilon} y_1$ and $w_2 = \sqrt{\varepsilon} y_2$. The following expansion can be suggested for the displacement functions.

$$y_1(x, t{:}\, \varepsilon) = \varepsilon^0 y_{10}(x, T_0, T_1) + \varepsilon y_{11}(x, T_0, T_1) \tag{17}$$

$$y_2(x, t{:}\, \varepsilon) = \varepsilon^0 y_{20}(x, T_0, T_1) + \varepsilon y_{21}(x, T_0, T_1) \tag{18}$$

ε is a small parameter used in calculations. $T_0 = \varepsilon^0 t$ is a fast time scale, $T_1 = \varepsilon t$ is slow time scale. Forcing and damping expressions to the system are expressed as (Bagdatli 2015):

$$\overline{F} = \varepsilon\sqrt{\varepsilon} F \;\; \overline{\mu} = \varepsilon \mu \tag{19}$$

According to the time derivative expressions are written in terms of new time variables,

$$\begin{gathered} \partial/\partial t = D_0 + \varepsilon D_1 \\ \partial^2/\partial t^2 = {D_0}^2 + 2\varepsilon D_0 D_1 \qquad \text{where,} \qquad D_n = \partial/\partial T \end{gathered} \tag{20}$$

After expansion, the first and second terms of the expansion are separated as follows:

$$\begin{gathered} \text{Order} \\ (\varepsilon^0) \;\; y_{10}^{iv} + D_0^2 y_{10} - \gamma^2 D_0^2 y_{10}'' = 0 \end{gathered} \tag{21}$$

$$y_{20}^{iv} + \frac{1}{\alpha^2} D_0^2 y_{20} - \frac{\gamma^2}{\alpha^2} D_0^2 y_{20}'' = 0 \tag{22}$$

Order (ε)

$$y_{11}^{\mathrm{iv}} + {D_0}^2 y_{11} + 2 D_0 D_1 y_{10} - 2\gamma^2 D_0 D_1 y_{10}'' - \gamma^2 {D_0}^2 y_{11}'' \tag{23}$$

$$
\begin{aligned}
&= \Gamma_1 \left[\int_0^{\eta} \left({y'}^2_{10}\right) dx + \alpha^2 \int_{\eta}^{1} \left({y'}^2_{20}\right) dx \right] y''_{10} \\
&-\Gamma_1 \gamma^2 \left[\int_0^{\eta} \left({y'}^2_{10}\right) dx + \alpha^2 \int_{\eta}^{1} \left({y'}^2_{20}\right) dx \right] y^{\mathrm{iv}}_{10} \\
&+\mathrm{F} \cos \Omega \mathrm{t} - 2\mu D_0 y_{10}
\end{aligned}
$$

$$
\begin{aligned}
&y^{\mathrm{iv}}_{21} + \frac{1}{\alpha^2} {D_0}^2 y_{21} + \frac{2}{\alpha^2} D_0 D_1 y_{20} - 2\frac{\gamma^2}{\alpha^2} D_0 D_1 y''_{20} \\
&-\frac{\gamma^2}{\alpha^2} {D_0}^2 y''_{21} = \Gamma_2 \left[\int_0^{\eta} \begin{matrix} \left({y'}^2_{10}\right) dx \\ +\alpha^2 \int_{\eta}^{1} \left({y'}^2_{20}\right) dx \end{matrix} \right] y''_{20} \\
&-\Gamma_2 \gamma^2 \left[\int_0^{\eta} \left({y'}^2_{10}\right) dx + \alpha^2 \int_{\eta}^{1} \left({y'}^2_{20}\right) dx \right] y^{\mathrm{iv}}_{20} \\
&+\mathrm{F} \cos \Omega \mathrm{t} - 2\mu D_0 y_{20}
\end{aligned} \tag{24}
$$

where, $\Gamma_1 = \frac{1}{2\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)}$ and $\Gamma_2 = \frac{1}{2\alpha^4\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)}$

The equations in the ε^0 Order give the linear equation of motion and the linear frequency equation of the system. The equations in ε Order show the effects coming from the nonlinear part. The boundary conditions can be represented as

$$
\begin{aligned}
&y_{10}(0) = 0, \qquad y_{20}(1) = 0 \\
&y''_{11}(0) = 0, y''_{21}(1) = 0 \\
&y_{11}(\eta) = \alpha y_{21}(\eta), y'_{11}(\eta) = \alpha y'_{21}(\eta) \\
&y''_{11}(\eta) = \alpha^5 y''_{21}(\eta), y'''_{11}(\eta) = \alpha^5 y'''_{21}(\eta)
\end{aligned} \tag{25}
$$

The first perturbation order is linear given in Eqs. (21) and (22); The solution can be represented as

$$
\begin{aligned}
&y_{10}(x, T_0, T_1) \\
&= A_1(T_1) e^{i\omega T_0} Y_1(x) + \bar{A}_1(T_1) e^{-i\omega T_0} \bar{Y}_1(x)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&y_{20}(x, T_0, T_1) \\
&= A_2(T_1) e^{i\omega T_0} Y_2(x) + \bar{A}_2(T_1) e^{-i\omega T_0} \bar{Y}_2(x)
\end{aligned} \tag{27}
$$

If Eqs. (26) and (27) are applied to Eqs. (21) and (22),

$$
Y_1^{iv}(x) - \omega^2 Y_1(x) + \gamma^2 \omega^2 {Y_1}''(x) = 0 \tag{28}
$$

$$
Y_2^{iv}(x) - \frac{1}{\alpha^2} \omega^2 Y_2(x) + \frac{1}{\alpha^2} \gamma^2 \omega^2 {Y_2}''(x) = 0 \tag{29}
$$

Eqs. (30) and (31) can be used to solve Eqs. (28) and (29).

$$
\begin{aligned}
Y_1(x) &= c_{11} e^{ir_{11}x} + c_{12} e^{ir_{12}x} + c_{13} e^{ir_{13}x} + c_{14} e^{ir_{14}x} \\
&= c_{11} \left(e^{ir_{11}x} + \frac{c_{12}}{c_{11}} e^{ir_{12}x} + \frac{c_{13}}{c_{11}} e^{ir_{13}x} + \frac{c_{14}}{c_{11}} e^{ir_{14}x} \right)
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
Y_2(x) &= c_{21} e^{ikr_{21}x} + c_{22} e^{ikr_{22}x} + c_{23} e^{ikr_{23}x} \\
&\quad + c_{24} e^{ikr_{24}x} \\
&= c_{21} \left(e^{ikr_{21}x} + \frac{c_{22}}{c_{21}} e^{ikr_{22}x} + \frac{c_{23}}{c_{21}} e^{ikr_{23}x} + \frac{c_{24}}{c_{21}} e^{ikr_{24}x} \right)
\end{aligned} \tag{31}
$$

where, $k = \frac{1}{\sqrt{\alpha}}$

To obtain the nonlinear correction terms of order ε, a solution assumption is made as follows (Tekin *et al.* 2009). To determine this condition, we first separate the secular and non-secular terms, assuming a solution like this:

$$
\begin{aligned}
y_{11}(x, T_0, T_1) &= \phi_{11}(x, T_0, T_1) e^{i\omega T_0} + W_{11}(x, T_0, T_1) \\
&\quad + ke
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
y_{21}(x, T_0, T_1) &= \phi_{21}(x, T_0, T_1) e^{i\omega T_0} + W_{21}(x, T_0, T_1) \\
&\quad + ke
\end{aligned} \tag{33}
$$

And separate the secular and non-secular terms. Assuming the system is in a dominant resonance state:

$$
\Omega = \omega + \varepsilon\sigma \tag{34}
$$

where σ is a detuning parameter. After some algebraic operations, the solvability condition is obtained:

$$
\begin{aligned}
&2i\omega(A' + \mu A) - 2\gamma^2 i\omega A' b + \Gamma_1 3A^2 \bar{A} b^2 \\
&-\Gamma_2 \gamma^2 3A^2 \bar{A} bc - f\frac{1}{2} e^{i\varepsilon\sigma T_1} + NST + cc = 0
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
&\int_0^{\eta} Y_1^2(x) dx + \alpha^4 \int_{\eta}^{1} Y_2^2 dx = 1 \\
&\int_0^{\eta} {Y_1}''^2(x)\, dx + \alpha^2 \int_{\eta}^{1} {Y_2}''^2(x) dx = c \\
&\int_0^{\eta} {Y_1}'^2(x)\, dx + \alpha^2 \int_{\eta}^{1} {Y_2}'^2(x) dx = b \\
&\int_0^{\eta} {Y_1}'^2(x) dx + \alpha^2 \int_{\eta}^{1} {Y_2}'^2(x) dx = f
\end{aligned} \tag{36}
$$

The complex amplitude A is written in terms of a real amplitude a and a phase θ

$$
A = \frac{1}{2} a e^{i\theta} \tag{37}
$$

Then the amplitude and phase modulation is obtained as the following equations.

$$
\begin{aligned}
&i\omega a' - a\omega\sigma + a\omega\psi' + \gamma^2 i\omega a' b - \gamma^2 \omega ab\sigma \\
&+\gamma^2 \omega ab\psi' + \Gamma_1 \frac{3}{8} a^3 b^2 + \Gamma_2 \gamma^2 \frac{3}{8} a^3 bc \\
&-\frac{1}{2} f \cos\Psi - i\frac{1}{2} f \sin\Psi = 0
\end{aligned} \tag{38}
$$

where $\theta = \sigma T_1 - \psi$

In this section, the modulation equations for the amplitude and phase stepped nanobeam were determined from the nonlinear analysis.

5. Numerical results

In this part of the study, numerical analysis of the equations found analytically is done.

The following assumptions are made for free vibration

$$
f = 0 \qquad \mu = 0 \qquad \sigma = 0 \tag{39}
$$

If Eq. (39) is placed in Eq. (38),

$$
a' = 0 \qquad \text{and} \qquad a = a_0 \tag{40}
$$

In this case, it is nonlinear from Eq. (37). The frequency is obtained as follows.

Table 1 Comparison of the first three fundamental frequencies according to different small-scale parameters

				$\alpha = 1$		$\eta = 1$			
γ	ω_1			ω_2			ω_3		
	Present	Ref (Bagdatli 2015)	Ref (Taima 2021)	Present	Ref (Bagdatli 2015)	Ref (Taima 2021)	Present	Ref (Bagdatli 2015)	Ref (Taima 2021)
0	9.8696	9.8696	9.8696	39.4784	39.4784	39.4787	88.8264	88.8264	88.8280
0.1	9.4158	9.4159	9.4159	33.4277	33.4277	33.4279	64.6414	64.6414	64.6420
0.2	8.3569	8.3569	8.3569	24.5823	24.5823	24.5824	41.6284	41.6284	41.6287
0.3	7.1824	7.1824	-	18.5015	18.5016	-	29.6180	29.6180	-
0.4	6.1455	6.1456	-	14.5951	14.5951	-	22.7743	22.7743	-
0.5	5.3003	5.3003	-	11.9744	11.9744	-	18.4389	18.4389	-

Table 2 The first three frequencies and correction terms are for different step ratio and step location values

α	γ	η	ω_1	ω_2	ω_3	λ_1
0.5	0.2	0.1	3.190	10.480	19.086	50.6257
		0.2	3.205	10.850	20.373	49.5054
		0.4	3.425	13.520	25.982	34.8064
		0.5	3.736	15.966	27.764	29.0513
		0.6	4.288	18.349	27.770	25.1905
		0.8	4.609	18.712	35.346	38.8245
0.8	0.2	0.1	6.163	18.858	32.706	3.4512
		0.2	6.211	19.231	33.921	3.1673
		0.4	6.560	21.235	35.672	2.2448
		0.5	6.896	21.810	36.133	1.9455
		0.6	7.329	21.855	37.850	1.8428
		0.8	8.153	23.187	38.997	2.2490
2	0.2	0.1	20.238	50.656	78.613	4806.6
		0.2	17.109	42.730	75.050	1421.7
		0.4	11.245	41.521	60.144	114.4
		0.5	9.833	36.594	59.881	41.6
		0.6	9.022	31.485	56.233	18.12
		0.8	8.414	25.645	44.991	6.1
3	0.2	0.1	30.361	67.620	201.381	58621.3
		0.2	19.329	61.745	197.035	5070.5
		0.4	10.739	48.784	80.765	261.5
		0.5	9.319	38.647	70.601	88.6
		0.6	8.623	31.937	59.063	37.2
		0.8	8.313	25.441	45.076	13.2

$$\omega_{n1} = \omega + \lambda {a_0}^2 \tag{41}$$

where

$$\lambda = \frac{3}{8}\frac{(\Gamma_1 b^2 + \Gamma_2 \gamma^2 bc)}{\omega(1+\gamma^2 b)} \tag{42}$$

Eq. (41) shows that there is a parabolic relationship between amplitude and nonlinear frequency. The coefficient of the ${a_0}^2$ term is defined as λ and shows the corrections due to nonlinearity and is listed in Table 2 for the first frequency.

At the steady-state, $a' = 0$ and $\Psi' = 0$ becomes zero. The frequency detuning parameter is as follows

$$\sigma_{1,2} = \lambda a^2 \mp \sqrt{\frac{f^2}{4\omega^2 {a_0}^2 (1+\gamma^2 b)} - \mu^2} \tag{43}$$

To prove the correctness of the solutions of the study's linear problem, the values are compared with the values of (Bagdatli 2015) and (Taima 2021). A stepped beam with a simple-simple boundary condition, $\alpha = 1$ was chosen and the first three fundamental frequency values of the beam are compared with other nanobeam values for different small-scale parameters in Table 1 and show good agreement with the other two studies. From these comparisons, it is also probable to observe that the frequency values increase as

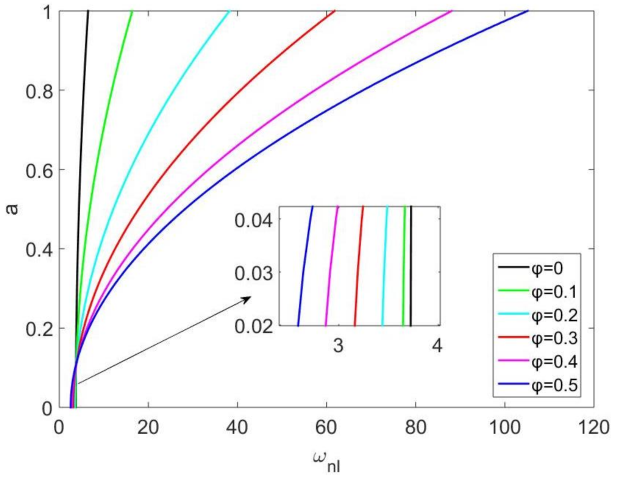

Fig. 3 Nonlinear frequency against amplitude due to different nonlocal parameters, $\alpha = 0.5$ and $\eta = 0.4$

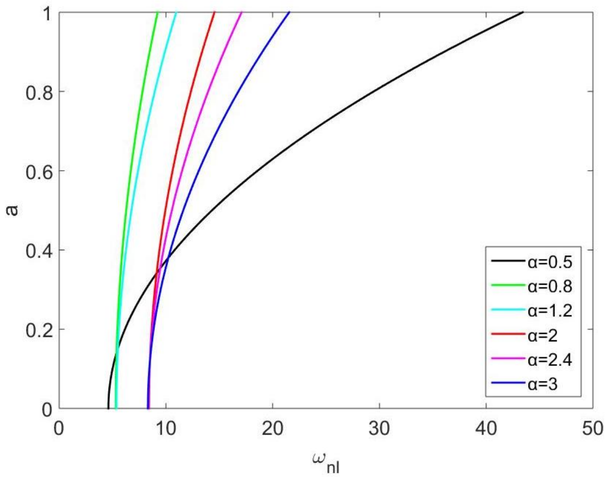

Fig. 5 Nonlinear frequency against amplitude due to different step ratios, $\gamma = 0.2$ and $\eta = 0.8$

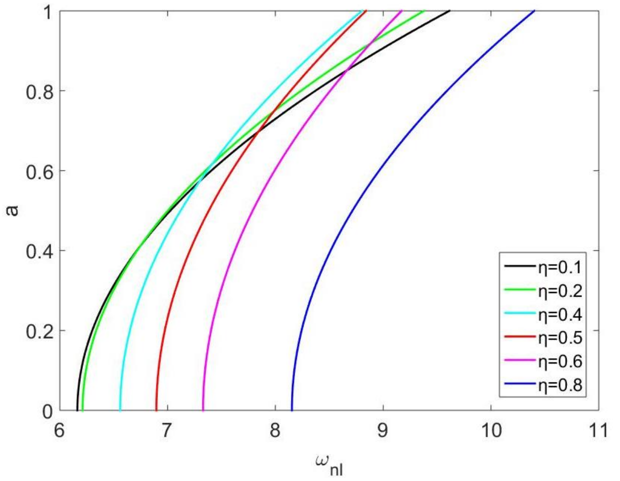

Fig. 4 Nonlinear frequency versus amplitude for different step location values, $\alpha = 0.8$ and $\gamma = 0.2$

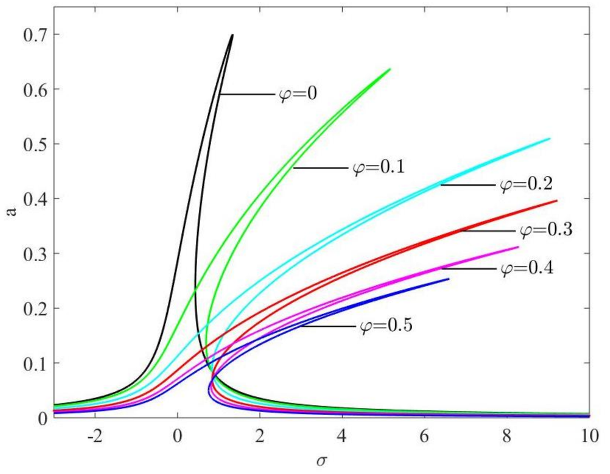

Fig. 6 Frequency-response curves due to different nonlocal small-scale parameters, $\alpha = 0.5$ and $\eta = 0.4$

the small-scale parameter increases. It is seen that this opinion is in parallel with the views of other many studies working at the nanoscale (Aydogdu 2009, Bagdatli 2015, Murmu and Adhikari 2010, 2012).

Table 2 gives the λ values for the first three modes corresponding to different values for the cascade nanobeam with simple boundary conditions. When the natural frequency values are examined, it is seen that the step ratio has a significant effect and if the ratio is less than one the natural frequency increases as the location of the step move away from the starting point, but if the step ratio is greater than one, it can see has the opposite effect.

Nonlinear frequency against amplitude curves is charted in Figs. 3-5 for different nonlocal parameters, step location η and step ratio α values respectively. In fig. 3, the graph giving the variation of the nonlinear frequency of the first mode of the system is drawn. The following values of the non-local parameter are taken into account $\gamma = 0$, $\gamma = 0.1$, $\gamma = 0.2$, $\gamma = 0.3$, $\gamma = 0.4$ and $\gamma = 0.5$. When the graph is examined, it is observed that the increase in the non-local parameter (γ) decreases the natural frequency value of the 1st mode of the system. On the other hand, as the amplitude value increases, it is seen that the non-linear frequency values increase.

In Fig. 4, The nonlinear frequency graphs of the first mode of the system are drawn according to the different step locations from $\eta = 0.1$ to $\eta = 0.8$. When this graph is examined, it is seen that the value of nonlinear frequencies increases as the step position (η) moves from the starting point to the beam end. At the same time, the nonlinear frequency increased with increasing amplitude.

Especially the amount of increase is high as it gets closer to the beam ends ($\eta = 0.1$, $\eta = 0.2$ and $\eta = 0.8$). Similar to Figs. 3-4 in Fig. 5, The nonlinear frequency graphs of the first mode of the system are drawn according to the different step ratios, $\alpha = 0.8$, $\alpha = 2$, $\alpha = 2.4$ and $\alpha = 3$. From these figs., as the step ratio (α) increases, the nonlinear frequencies decrease. But, With the increase of the amplitude values, the nonlinear frequency values ω_{nl} increase and even more especially at the beam ends ($\alpha = 0.5$ and $\alpha = 3$).

The detuning parameter (σ) expresses the closeness of the forcing frequency to the natural frequency.

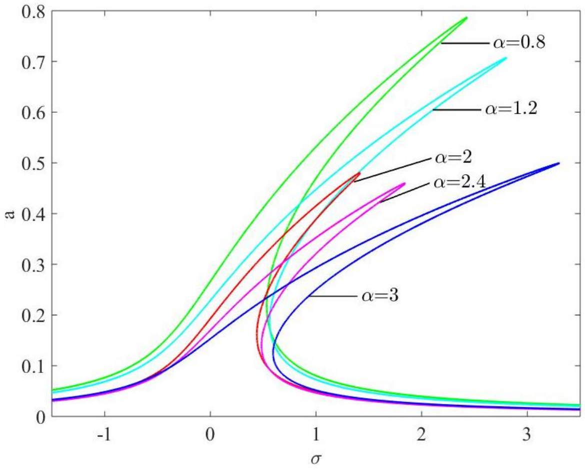

Fig. 7 Frequency-response curves due to different step ratios, $\gamma = 0.2$ and $\eta = 0.8$

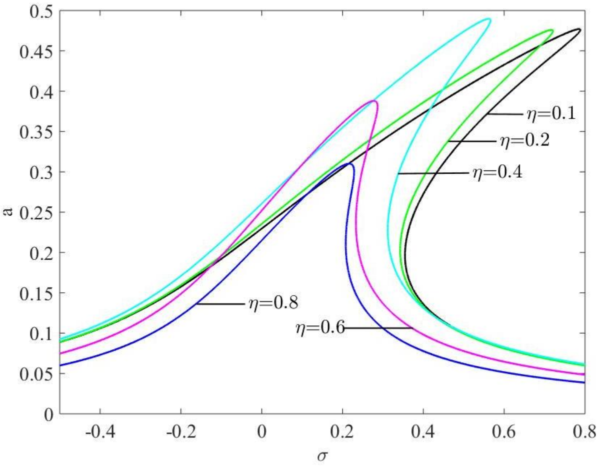

Fig. 8 Frequency-response curves due to different step locations, $\alpha = 0.8$ and $\eta = 0.2$

Nonlinear frequency against amplitude curves is charted in Figs. 6-8, for the different nonlocal parameters, step ratio α, and step location η values respectively. Fig. 6 shows the effect of non-local parameters on frequency response curves for simple-simple boundary conditions. It is seen that the hardening effect increases with the increase of the non-local parameter. A similar result is obtained as the step ratios increase in Fig. 7.

However, when Fig. 8 is examined, contrary to other results, it is seen that the step location moves away from the starting point and reduces the hardening effect.

6. Conclusions

In this article, the vibrational motions of the stepped nanobeam are investigated with the nonlocal elasticity theory. A Euler Bernoulli beam is considered in the case of the Simple-Simple boundary condition. Nonlinear equations of motion are generated, which include the stretching of the neutral axis of the stepped nanobeam. The perturbation method was used to obtain approximate solutions. Dimensionalization was done to make the system independent of its geometry and material properties. First, free vibration analysis of the stage nanobeam was performed. To confirm the results, they were compared with some studies in the literature and were found to agree with other studies. When the step ratio is equal to one, the stepped beam becomes stepless. It was determined that the data obtained from the study were in agreement with the results of a stepless nanobeam study found in the literature.

Then, forcing and damping effects were added to the system and the nonlinear vibrations of the system were examined in detail. The effects of the nonlocal parameter, step ratio, and step location were determined, and frequency response curves were drawn to represent their effects on the vibrations of the nanobeam. The following main results were obtained from the study.

- As the value of the non-local parameter increases, the natural frequency value of the object decreases. That is, the non-local parameter value reduces the stiffness of the stepped nanobeam.
- In cases where the step radius ratio is less than 1, it is seen that the natural frequency values increase as the step location moves away from the starting point, and in the case where the step ratio is greater than 1, the results are reversed, that is, the natural frequency values decrease. In this case, the fact that the step ratio moves away from 1 result in a decrease in the stiffness of the beam.
- In the results related to the step location, it is seen that the sensitivity of the system increases if the fine part is more.
- Step rate away from 1 increases the nonlinearity of the system.
- Appropriate modeling, selection of step rate and location play important roles in determining the vibration behavior of the non-uniform nanobeam.

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