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Discretization of laser model with bifurcation analysis and chaos control

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Abstract. This paper investigates the dynamics and stability of steady states in a continuous and discrete-time single-mode laser system. By using an explicit criteria we explored the Neimark-Sacker bifurcation of the single mode continuous and discrete-time laser model at its positive equilibrium points. Moreover, we discussed the parametric conditions for the existence of period-doubling bifurcations at their positive steady states for the discrete time system. Both types of bifurcations are verified by the Lyapunov exponents, while the maximum Lyapunov ensures chaotic and complex behaviour. Furthermore, in a three-dimensional discrete-time laser model, we used a hybrid control method to control period-doubling and Neimark-Sacker bifurcation. To validate our theoretical discussion, we provide some numerical simulations.
Keywords: chaos control; neimark-sacker bifurcation; period-doubling bifurcation; single-mode laser model; stability

1. Introduction

Recent discoveries have revealed that chaos occurs in physical, chemical, and biological systems (Gleick and Berry 1987, Copeland and Moon 1992, Hilborn 2000, Hikihara *et al.* 2012). The majority of engineers and scientists have started to acknowledge this unavoidable truth. The development of computers has brought about this awakening, not only in meteorology but also in astronomy and biology. Dynamical systems theory, a novel area of mathematics, has progressed quickly and is now the standard mathematical terminology for describing chaotic systems in science and industry. The field of nonlinear dynamics involves theoretical and experimental studies of chaos and instability.

Hermann Haken made the connection between chaotic dynamics and laser instabilities when he discovered that the semi-classical equations which describe the functioning of a single-mode laser is equivalent to the equations associated with moving fluid studied by Lorenz some twelve years earlier (Harris *et al*. 2010, Haken 1975). When the variables for the laser's electric field, polarization, and population inversion were changed, the three equations of motion had the same form as the Lorenz equations. These equations, however, mirrored the Lorenz equations, and even without any noise sources projected that a laser would experience irregular deterministic fluctuations in some parameter regimes.

Due to this, several groups from all around began searching for the perfect laser technology to demonstrate Lorenz's chaos and novel modified semi-classical system that able to produce hyper chaos, (Barakat *et al*. 2021, Bonatto 2018, Narducci and Abraham 1988, Milonni *et al*. 1987, Weiss *et al*. 1995). An electric field-based single-mode laser system that decays at the same rate as polarization, population inversion and polarization decay is required for a meaningful assessment of Haken-Lorenz equations. As polarization decays faster than the inversion and field decay in laser systems (He-Ne, CO2, semi-conductors, Nd: YAG, etc.), the three-variable system of equations is reduced to two. A two-dimensional system cannot have an explicit chaotic behaviour; a mathematical theory states that it can only have periodic or stable behaviour.

To witness this sort of chaotic behaviour, we must locate a single-mode laser with comparable decay rates for all three decay rates. Finally, a somewhat laser system, the far-infrared ammonia laser, was chosen because it has the necessary characteristics to produce Lorenz-like chaos. Laser light is an excellent example of bistability, oscillation, and chaos in optics (Gang 2007). It is presently one of the most important fields of laser research.

Numerous theoretical works published over the last several decades have demonstrated the existence of self-pulsing instabilities and chaos in laser models (Miles 1964, Haken 1975, Abraham *et al*. 1985). This is primarily due to chaotic synchronization's application in a variety of areas, including biomedical research, biology, optics, chemistry, social sciences, information processing, and pattern generation (Grzybowski *et al*. 2009, Chang 2009, Illing 2009, Li and Zhang 2011, Li and Xu 2004, Yu *et al*. 2007, Wang and Wang 2011, Xie *et al*. 2002, Li 2011). Many methods of chaotic synchronization have developed in recent years, including total synchronization Pecora and Carroll (1990) and Grzybowski *et al*. (2009) anti-synchronization (Li and Xu 2011), phase synchronization, generalized synchronization (Li and Zhang 2011, Li and Xu 2004) and so on. Ling *et al*. (1998) first contributed to the chaotic behaviour of a single-mode laser model in 1998, which is in the form:

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$$\begin{cases} \dfrac{dx(t)}{dt} = -ax(t) + y(t), \\ \dfrac{dy(t)}{dt} = -by(t) + x(t)z(t), \\ \dfrac{dz(t)}{dt} = c - z(t) - x(t)y(t). \end{cases} \tag{1}$$

where x refers to atoms being driven by an electromagnetic field, y refers to the dipole moment of atoms and z refers to the population inversion. For the detailed explanation of the parameters in the model (1) one can see (Ling *et al*. 1998).

Inspired by the previously cited work, we have investigated the system (1), connected to period-doubling bifurcation, Neimark-Sacker bifurcation, and chaos control for the discrete counterpart of (1). Because the discrete dynamical system is beneficial in exploring the rich and complex behaviour of the system as it has not happened before to the system (1). Over the past few decades, discrete-time models have been thoroughly studied. It is standard practice to construct discrete-time models from continuous-time models using the forward Euler method or Mickens' non-conventional discretization, (Franke and Yakubu 2008, Castillo-Chavez and Yakubu 2001, Willox *et al*. 2003). Moreover, discrete-time models approached by the forward Euler scheme had flip bifurcations, Hopf bifurcations, and chaos dynamics that explored rich and complex dynamics which differed from their continuous-time models (Ghaziani *et al*. 2012, Hu *et al*. 2011, He and Lai 2011). Moreover, many authors explored rich dynamics for the discrete dynamical systems in three-dimensional systems such as (Ishaque *et al*. 2019, Taj *et al*. 2021, Din and Ishaque 2020, Öztürk 2017, Zhang *et al*. 2015, 2022, Atabakhshian and Shooshtaria 2020, Boutaleb *et al*. 2019, Liao and Bose 2022, Liao *et al*. 2020, Din and Zulfiqar 2022), and the references are there in. To do this, we use forward Euler's approach on the system (1) as follows:

$$\begin{cases} x_{n+1} = x_n + h(y_n - a\,x_n), \\ y_{n+1} = y_n + h(-by_n + x_n z_n) \\ z_{n+1} = z_n + h(c - z_n - x_n y_n), \end{cases} \tag{2}$$

where h is the step size for Euler's method.

This article's primary contributions are summarized as follows:

• First, we discussed some qualitative behaviour of the system (1). Due to the complexity seen in the system (1) to analyze its qualitative behaviour at its positive equilibrium; we proposed a discrete-time single-mode laser system (2) by using forward Euler's approach.

• Secondly, we investigated Neimark-Sacker bifurcation by using an explicit criterion of bifurcation for a system (1) and (2) at its positive steady state, and demonstrate that it is dynamically consistent.

• We investigated flip bifurcation for system (2) at its positive steady state.

• As a result of the emergence of Neimark-Sacker and flip bifurcations, system (2) is chaotic. In this case, a hybrid control method has been used to control the bifurcation.

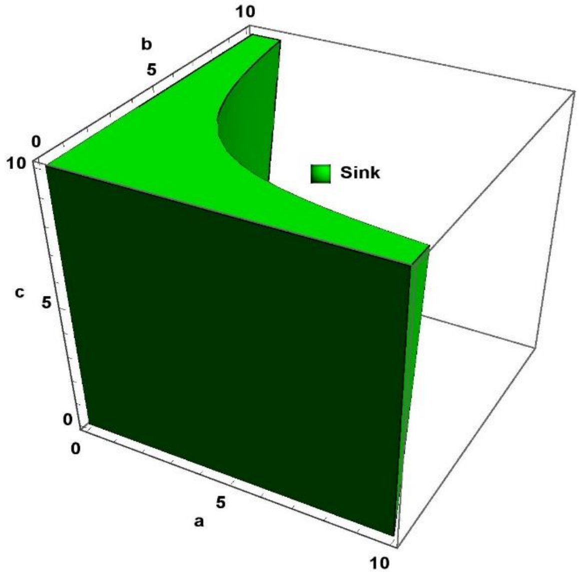


Fig. 1 Sink region for positive equilibrium of system (1)

• Lastly, we provide appropriate numerical examples showing the effectiveness of both the bifurcation theory and control method.

2. Dynamics of a contnious system

In this section, our main purpose is to investigate some qualitative behaviour of continuous system (1). For this, it is quite simple to see that $\Xi_0 = (0,0,c)$ and $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}},\ \sqrt{a}\sqrt{c-ab},\ ab\right)$ are two equilibrium points for this system (1). Furthermore, assume that $c > ab$, then $\Xi_1(x^*, y^*, z^*)$ is the unique positive equilibrium point. On the other hand, Jacobian matrices about these equilibria are given as follows:

Moreover, in order to study the Hopf bifurcation about interior equilibrium of the system (1), an explicit criterion is used for Hopf bifurcation. For this, the following Theorem is presented (cf. (Liu 1994)

Theorem 1. Taking into account an n-dimensional system of the following form:

$$\frac{d\,Z}{d\,t} = F(Z, \theta) \tag{4}$$

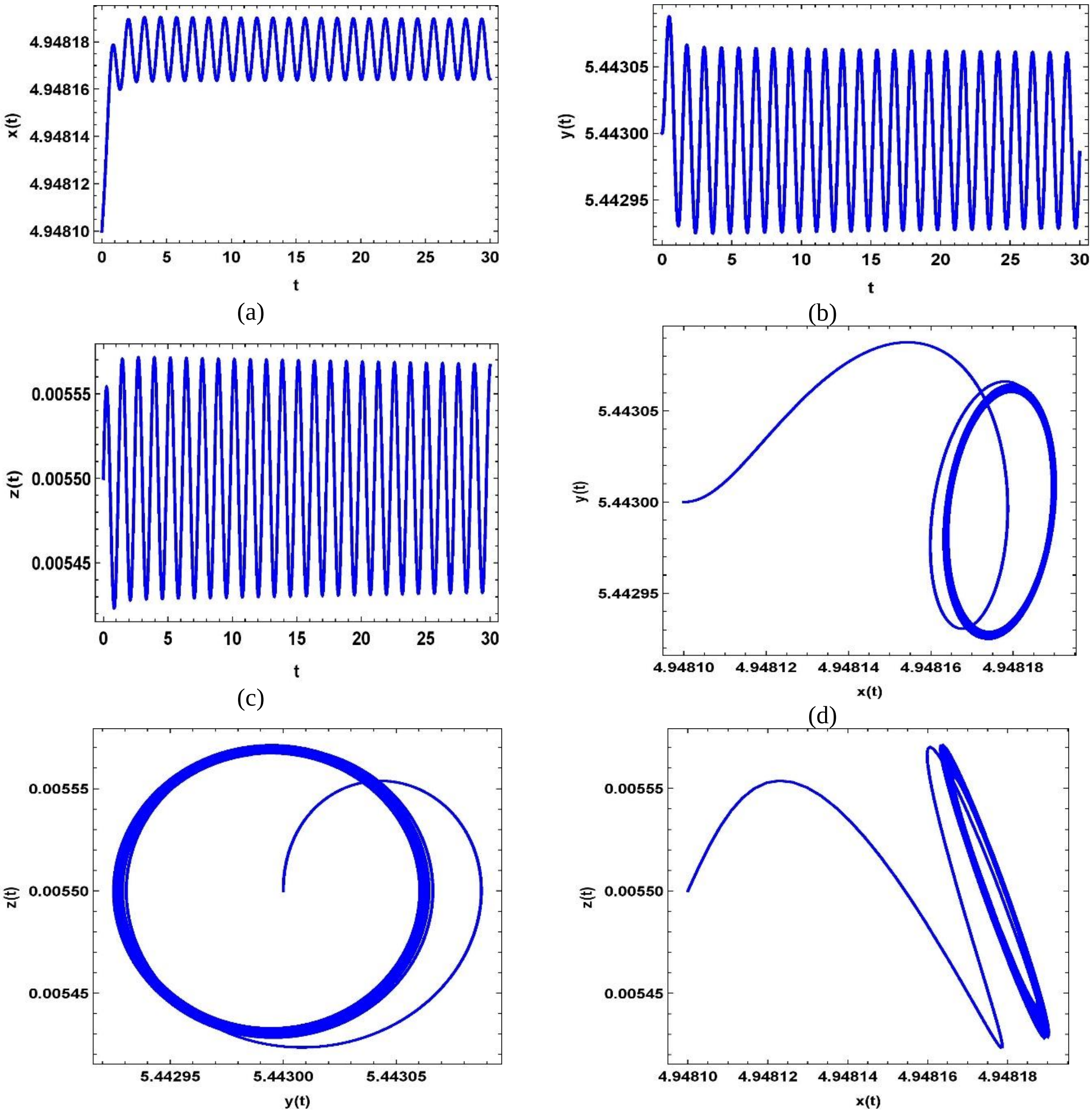
where $Z \in \mathbb{R}^n$, $F \in C^\infty$, and $\theta \in \mathbb{R}$. On the other hand, suppose that (Z^*, θ^*) be a steady state for the system (4). Next, we assume that $P(\lambda, \theta)$ denotes the characteristic polynomial for the Jacobian matrix of the system (4) given by:

$$P(\lambda, \theta) = a_n(\theta)\,\lambda^n + a_{n-1}(\theta)\,\lambda^{n-1} + \cdots + a_1(\theta)\,\lambda + a_0(\theta) \tag{5}$$

Furthermore, assume that the following hold true:

(i) $a_0(\theta^*) > 0$, $\Delta_1(\theta^*) > 0$, $\cdots$, $\Delta_{n-2}(\theta^*) > 0$, $\Delta_{n-1}(\theta^*) = 0$,

(ii) $\frac{d\,\Delta_{n-1}(\theta^*)}{d\,\theta} \neq 0$,



(a) (b) (c) (d) (e) (f)

Fig. 2 Plots of system (1) at $a = 1.1$, $b = 0.005$ and $c = 26.9384$

where $\Delta_n(\theta) := det \begin{pmatrix} a_1(\theta) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{2n-1}(\theta) & \cdots & a_n(\theta) \end{pmatrix}$.

Then, system (4) undergoes Hopf bifurcation about (Z^*, θ^*).

Lemma 1. Taking $n = 3$, then conditions (i) and (ii) reduce to the following:

$a_0(\theta^*) > 0$, $\Delta_1(\theta^*) = a_1(\theta^*) > 0$, $\Delta_2(\theta^*) = a_1(\theta^*)\, a_2(\theta^*) - a_0(\theta^*) = 0$, and $\frac{d\,\Delta_2(\theta^*)}{d\,\theta} \neq 0$.

Considering the system (1) and its characteristic polynomial (3). Moreover, taking c as bifurcation parameter, then we have the following result.

Lemma 2. Let $c > a\,b$, and $a > 1 + b$, then system (1) undergoes Hopf bifurcation about its positive equilibrium $\left(\frac{\sqrt{c-ab}}{\sqrt{a}},\ \sqrt{a}\sqrt{c-ab},\ ab\right)$as c varies in a small neighborhood of $c^* = \frac{a^2(1+a+3b)}{a-1-b}$, if the following hold true:

$(1 + a + b)\left(a + \frac{c}{a}\right) = 2(c - ab)$ and $\frac{1-a+b}{a} \neq 0$.

For numerical verification, we take $a = 1.1$, $b = 0.005$ and $c \in [25,35]$, then system (1) undergoes Hopf bifurcation about its interior equilibrium $(4.948, 5.443, 0.0055)$ at $c^* = 26.93842105263156$. Furthermore, considering $a = 1.1, b = 0.005,$ $c = 26.93842105263156,$ and initial conditions $\big(x(0), y(0), z(0)\big) = (4.9481, 5.443, 0.0055)$ various plots for the system (1) are depicted in Fig. 2.

3. Stability analysis of discrete system

The purpose of this section is to examine the local dynamical behaviour of the system (2). We begin by looking at steady-states for this system. According to (1),

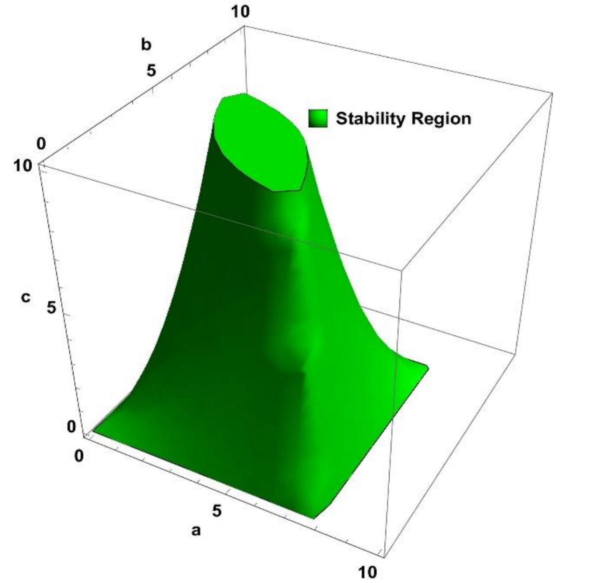


Fig. 3 Stability region of system (2) about Ξ_0 at $h = 0.25$

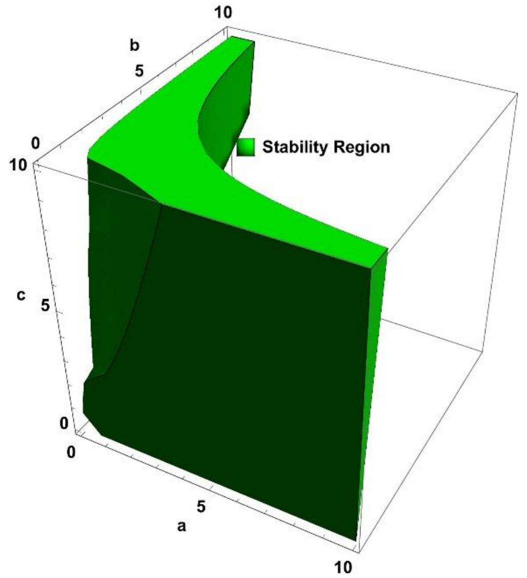


Fig. 4 Stability region of system (2) about Ξ_1 at $h = 0.1$

the steady-states solve the following 3-dimensional algebraic system:

$$\begin{cases} -a\,x + y = 0, \\ -b\,y + x\,z = 0, \\ c - z - x\,y = 0. \end{cases} \quad (6)$$

Therefore, we must assume that $c > ab$ so that system (6) will have the following boundary and a positive equilibrium:

$\Xi_0 = (0,0,c)$, $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}},\ \sqrt{a}\sqrt{c-ab},\ ab\right)$, where $x^* = \frac{\sqrt{c-ab}}{\sqrt{a}}$, $y^* = \sqrt{a}\sqrt{c-ab}$ and $z^* = ab$

The Jacobian matrix evaluated at $\Xi_0 = (0,0,c)$ of system (2) is given as follows:

$$J_{\Xi_0} = \begin{pmatrix} 1-ah & h & 0 \\ ch & 1-bh & 0 \\ 0 & 0 & 1-h \end{pmatrix}.$$

The cubic characteristic polynomial of the above-mentioned Jacobian matrix J_{Ξ_0} is computed as:

$$P(\mu) = (\mu - 1 + h)(\mu^2 - (2 - (a+b)h)\mu + 1 + ah(bh-1) - h(b + ch)).$$

Simple computation yields that the eigenvalues for J_{Ξ_0} can easily be calculated as follows:

$\mu_1 = 1 - h,\ \mu_{2,3} = \frac{1}{2}\left(2 - ah - bh \pm \sqrt{a^2 - 2ab + b^2 + 4c}h\right).$

Then, obviously $|\mu_1| < 1$ if $0 < h < 2$ and $|\mu_{2,3}| < 1$ if the following conditions are satisfied:

$$a\,b > c,\ \sqrt{\frac{(a-b)^2 + 4c}{(-ab+c)^2}} + h < \frac{a+b}{ab-c}.$$

Secondly, in order to discuss linearized stability for the aforementioned equilibrium $\Xi_1(x^*, y^*, z^*)$, we need to consider the following Lemma regarding necessary and sufficient conditions for local asymptotic stability of discrete dynamical systems in three dimensions:

Lemma 3.1 (Camouzis and Ladas 2007) Suppose that ϕ_2, ϕ_1 and ϕ_0 are real numbers are defined above. Then, all roots of the equation $\chi^3 + \phi_2\chi^2 + \phi_1\chi + \phi_0 = 0$, must satisfy the necessary and sufficient conditions to lie inside the open unit disk iff $|\phi_2 + \phi_0| < 1 + \phi_1$, $|\phi_2 - 3\phi_0| < 3 - \phi_1$, and $\phi_0^2 + \phi_1 - \phi_0\phi_2 < 1$.

Next, the Jacobian matrices evaluated at $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}},\ \sqrt{a}\sqrt{c-ab},\ ab\right)$ are given as follows:

$$J_{\Xi_1(x^*,y^*,z^*)} = \begin{pmatrix} 1-ah & h & 0 \\ abh & 1-bh & \frac{\sqrt{-ab+c}\,h}{\sqrt{a}} \\ -\sqrt{a}\sqrt{-ab+c}h & -\frac{\sqrt{-ab+c}\,h}{\sqrt{a}} & 1-h \end{pmatrix}.$$

The cubic polynomial of the above mentioned Jacobian matrix is computed as:

$$\varsigma^3 + \omega_2\varsigma^2 + \omega_1\varsigma + \omega_0 = 0 \quad (7)$$

where $\omega_2 = 3 - \mathrm{a}h - (1+b)h)$, $\omega_1 = -3 + 2\mathrm{a}h + 2(1+b)h - \mathrm{a}h^2 - cah^2$ and

$$\omega_0 = 1 - \mathrm{a}h - (1+b)h + \mathrm{a}h^2 + cah^2 + 2\mathrm{a}bh^3 - 2ch^3 \quad (8)$$

The following Lemma establishes the parametric conditions for local stability of positive steady states for system (2).

Lemma 3.2 (Camouzis and Ladas 2007) Suppose that ω_2, ω_1 and ω_0 are real numbers are defined above. Then, all roots of the equation $\varsigma^3 + \omega_2\varsigma^2 + \omega_1\varsigma + \omega_0 = 0$, must satisfies the necessary and sufficient conditions to lie inside the open unit disk iff $|\omega_2 + \omega_0| < 1 + \omega_1$, $|\omega_2 - 3\omega_0| < 3 - \omega_1$, and $\omega_0^2 + \omega_1 - \omega_0\omega_2 < 1$.

Taking into account the above computation, the stability regions for the system (2) about Ξ_0 and $\Xi_1(x^*, y^*, z^*)$ are depiced in Figs. 3 and 4, respectively.

4. Neimark-Sacker bifurcation

The purpose is to examine the parametric conditions under which equilibrium points for a discrete model (2) undergo a Neimark-Sacker bifurcation. As a result of this investigation, the Neimark-Sacker bifurcation criterion was applied without specifying the eigenvalues of the Jacobian matrix (see also Din *et al*. (2017), Matouk *et al*. (2015)). As a result, the following Lemma presents clear criteria for Neimark-Sacker bifurcation.

Lemma 4.1 (Wen 2005) Suppose we have an n-dimensional discrete system as follows: $M_{k+1} = P_\zeta(M_k)$, in which ζ represents the bifurcation parameter. Let us now consider an evaluation of the Jacobian matrix $J(M^*) = (\Upsilon_{ij})_{n\times n}$ at a fixed point $M^* \in \mathbb{R}^n$ of P_ζ with the following characteristic polynomial:

$$K_\zeta(\alpha) = \alpha^n + \Gamma_1\alpha^{n-1} + \cdots + \Gamma_{n-1}\alpha + \Gamma_n \quad (9)$$

It is assumed that $\chi_i = \chi_i(\chi, v)$, where $i = 1,2,\cdots n$ and v represents a control parameter, or some other parameter to be determined. Take into account a sequence $(\mathbb{D}_i^\pm(\chi, v))_{i=0}^3$ of determinants with $\mathbb{D}_0^\pm(\chi, v) = 1$, as follows:

$$\mathbb{E}_i^\pm(\chi, v)\det(E_1 \pm E_2) \quad (10)$$

where

$$E_1 = \begin{bmatrix} 1 & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{i-1} \\ 0 & 1 & \Gamma_1 & \cdots & \Gamma_{i-2} \\ 0 & 0 & 1 & \cdots & \Gamma_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} \Gamma_{n-i+1} & \Gamma_{n-i+2} & \cdots & \Gamma_{n-1} & \Gamma_n \\ \Gamma_{n-i+2} & \Gamma_{n-i+3} & \cdots & \Gamma_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma_{n-1} & \Gamma_n & \cdots & 0 & 0 \\ \Gamma_n & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (11)$$

Furthermore, if the following conditions are satisfied:

(I) Eigenvalue criterion:

$\mathbb{E}_{m-1}^-(\chi_0, v) = 0$, $\mathbb{E}_{m-1}^+(\chi_0, v) > 0$, $K_{\chi_0}(1) > 0$, $(-1)^n K_{\chi_0}(-1) > 0$, $\mathbb{E}_i^\pm(\chi, v) > 0$, for $i = m-3,\ m-5,\ \cdots$, 2 (or 1), when m is even or odd respectively.

(II) $\left(\frac{d}{d\chi}(\mathbb{E}_{n-1}^-(\chi, v))\right)_{\chi=\chi_0} \neq 0$, known as transversality condition.

(III) In addition assume that $\cos\left(\frac{2\pi}{k}\right) \neq \phi$, or $\cos\left(\frac{2\pi}{k}\right) = \phi$, where $k = 3,4,5,\cdots$, and $\phi = 1 - 0.5K_{\chi_0}(1)\mathbb{E}_{m-3}^-(\chi_0, v)/\ \mathbb{E}_{m-2}^+(\chi_0, v)$ defined as non-resonance or resonance conditions respectively. It follows that Hopf bifurcation occurs at the critical value χ_0. Based on the following Lemma, we can state how system (2) undergoes Neimark-Sacker bifurcation when v is taken as a parameter.

Lemma 4.2 A Neimark-Sacker bifurcation occurs at $v = v_0$ for the positive equilibrium point $\Xi_1(x^*, y^*, z^*)$ of (2) if the following conditions are satisfied:

$$1 - \omega_1 + \omega_0(\omega_2 - \omega_0) = 0,$$
$$1 + \omega_1 - \omega_0(\omega_2 + \omega_0) > 0,$$
$$1 + \omega_2 + \omega_1 + \omega_0 > 0,$$
$$1 - \omega_2 + \omega_1 - \omega_0 > 0,$$
$$\frac{d}{dv} = \left(1 - \omega_1 + \omega_0(\omega_2 - \omega_0)\right)_{v=v_0} \neq 0$$
$$\text{and } \cos\left(\frac{2\pi}{k}\right) \neq 1 - \frac{1+\omega_2+\omega_1+\omega_0}{2(1+\omega_0)},\ k = 3,4,5,\ldots$$

where ω_2, ω_1, and ω_0 are defined in (8), and v_0 denotes possible real root for $1 - \omega_1(v) + \omega_0(v)(\omega_2(v) - \omega_0(v)) = 0$.

Proof: Based on Lemma 4.1 and equation (7), in the case of $n = 3$ and v as the bifurcation parameter we have,

$$\mathbb{E}_2^-(v) = 1 - \omega_1 + \omega_0(\omega_2 - \omega_0) = 0,$$
$$\mathbb{E}_2^+(v) = 1 + \omega_1 - \omega_0(\omega_2 + \omega_0) = 0,$$
$$K_v(1) = 1 + \omega_2 + \omega_1 + \omega_0 > 0,$$
$$(-1)^3 K_v(-1) = 1 - \omega_2 + \omega_1 - \omega_0 > 0,$$
$$\left(\left(\frac{d}{dv}(\mathbb{E}_2^-(v)\right)\right)_{v=v_0} = \frac{d}{dv}\ (1 - \omega_1 + \omega_0(\omega_2 - \omega_0))_{v=v_0} \neq 0$$
$$\text{and } \frac{1-0.5K_v(1)\mathbb{E}_2^+(v)}{\mathbb{E}_1^-(v)} = 1 - \frac{1+\omega_2+\omega_1+\omega_0}{2(1-\omega_0)}.$$

5. Period-doubling bifurcation

This section discusses the parametric conditions that lead to period-doubling bifurcations in the discrete-time model (2). The explicit criterion of period-doubling bifurcation is used without the need to determine the eigenvalues of the system (2). We explicitly provided the following period-doubling bifurcation criterion to demonstrate this.

Lemma 5.1 (Wen, Chen *et al*. 2008) Consider a discrete-time n-dimensional dynamical system $G_{k+1} = F_\rho(G_k)$ hence, this fulfills the same conditions (9) (10) and (11) given in Lemma 4.1. In addition, if the following conditions are satisfied, it follows that:

(I) $P_{\rho_0}(-1) = 0$, $\mathbb{E}_{n-1}^-(\rho_0, \beta) > 0$, $\mathbb{E}_i^\pm(\rho_0, \beta) > 0$, $i = r-2, r-4, \ldots, 1$ (or 2), where r is even or odd, respectively, and it is known as eigenvalue assignment.

(II) $\frac{\sum_{i=1}^3(-1)^{3-i}\Gamma_i'}{\sum_{i=1}^3(-1)^{3-i}(n-i+1)\Gamma_{i-1}} \neq 0$ Γ_i', denotes derivative of $\Gamma(\rho)$ at $\rho = \rho_0$, and this is known as transversality condition, then flip bifurcation occurs at critical value ρ_0.

It follows that flip bifurcation occurs at the critical value ρ_0. Based on the following Lemma, we can state how system (2) undergoes flip bifurcation when h is taken as a bifurcation parameter.

Lemma 5.2 The unique steady state $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}},\ \sqrt{a}\sqrt{c-ab},\ ab\right)$ for model (2) shows flip bifurcation at critical value $h = h_0$ if the next presented inequalities are satisfied:

$$1 - \omega_1 + \omega_0(\omega_2 - \omega_0) > 0,$$
$$1 + \omega_1 - \omega_0(\omega_2 + \omega_0) > 0,$$
$$1 + \omega_2 + \omega_1 + \omega_0 > 0,$$
$$1 - \omega_2 + \omega_1 - \omega_0 = 0,$$
$$1 \pm \omega_0 > 0,$$
$$\text{and } \frac{\omega_2' - \omega_1' + \omega_0'}{3 - 2\omega_2 + \omega_1} \neq 0.$$

where ω_2, ω_1 and ω_0 is already given in equation (8), ω_i' is derivative of $\omega(t)$at the crucial value $h = h_0$, and h_0 denotes a probable real root for $1 - \omega_2(h) + \omega_1(h) - \omega_0(h) = 0$.

6. Chaos control

Computational chemistry calculations were carried out using ChemBio version 18 (for drawing of chemical structure), hyperchem release 10 (for calculation of binding/ adsorption energy) and Spartan for DFT calculations.

This section discusses the chaos control method applied to a discrete-time model. For sack of simplicity, we will adopt the hybrid control approach initially proposed by (Luo *et al*. 2003). In which a single regulating parameter spans the open unit interval. Furthermore, because it depends a parameter perturbation and a state-feedback control technique, such a hybrid control system is simple to construct. It's worth mentioning that there have been several other techniques to deal with chaos in discrete-time systems (Din 2017a, b, 2018, Din *et al*. 2018, 2017, Ishaque *et al*. 2019, 2021, Taj *et al*. 2021). A hybrid control technique is used to create an equivalent control system of the following type:

$$\begin{aligned} x_{n+1} &= \beta[x_n + h(y_n - a\,x_n)] + (1-\beta)x_n, \\ y_{n+1} &= \beta[y_n + h(-by_n + x_n z_n)] + (1-\beta)y_n \\ z_{n+1} &= \beta[z_n + h(c - z_n - x_n y_n)] + (1-\beta)z_n. \end{aligned} \quad (12)$$

where β is a control parameter.

Moreover, variational matrix of controlled model (12) at positive coexistence $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}}, \sqrt{a}\sqrt{c-ab},\ ab\right)$ can be evaluated as follows:

$$V(x^*, y^*, z^*) := \begin{pmatrix} 1 - ah\beta & h\beta & 0 \\ abh\beta & 1 - bh\beta & \frac{\sqrt{-ab+c}h\beta}{\sqrt{a}} \\ -\sqrt{a}\sqrt{-ab+c}h\beta & -\frac{\sqrt{-ab+c}h\beta}{\sqrt{a}} & 1 - h\beta \end{pmatrix}.$$

Characteristic polynomial corresponding to the above Jacobian matrix at $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}}, \sqrt{a}\sqrt{c-ab},\ ab\right)$ is computed as:

$$P(\mu) = \mu^3 + \varrho_2\mu^2 + \varrho_1\mu + \varrho_0 \quad (13)$$

where

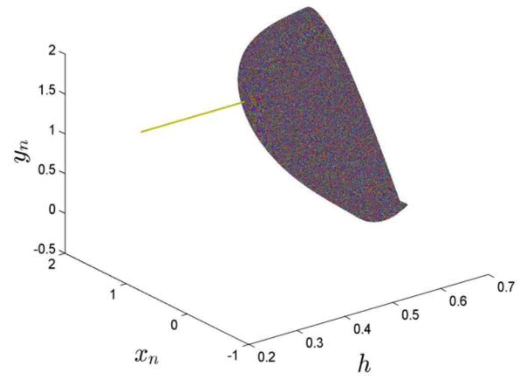
$$\begin{cases} \varrho_2 = -3 + (1+a+b)h\beta, \\ \varrho_1 = 3 + 2(-1-a-b)h\beta + \dfrac{(a^2+c)h^2\beta^2}{a}, \\ \varrho_0 = -1 + (1+a+b)h\beta \\ \quad -\dfrac{(a^2+c)h^2\beta^2}{a} + 2(-ab+c)h^3\beta^3 \end{cases} \quad (14)$$

It is noting that if all of the equation's roots are within the open unit disc, the positive coexistence $\Xi_1(x^*, y^*, z^*)$ for system (12) is a sink. The following Lemma specifies such parametric conditions.

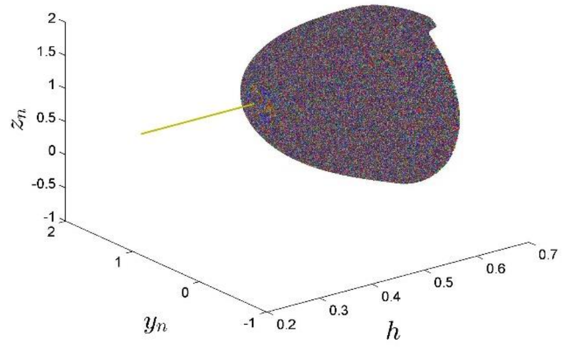
Lemma 6.1 For the model (12) positive coexistence $\Xi_1(x^*, y^*, z^*) = \left(\frac{\sqrt{c-ab}}{\sqrt{a}}, \sqrt{a}\sqrt{c-ab},\ ab\right)$ is sink and locally asymptotically stable if and only if the following conditions are verified:

$$|\varrho_2 + \varrho_0| < 1 + \varrho_1, \quad |\varrho_2 - 3\varrho_0| < 3 - \varrho_1 \quad \text{and } \varrho_0^2 + \varrho_1 - \varrho_0\varrho_2 < 1.$$

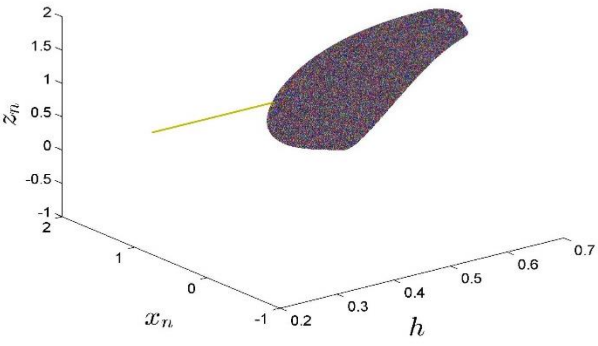
where ϱ_2, ϱ_1 and ϱ_0 are defined above in (14).



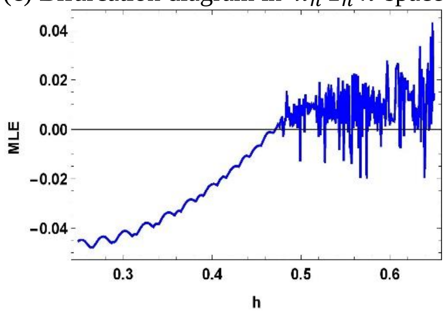
(a) Bifurcation diagram in $x_n\, y_n\, h$-space



(b) Bifurcation diagram in $y_n\, z_n\, h$-space



(c) Bifurcation diagram in $x_n\, z_n\, h$-space

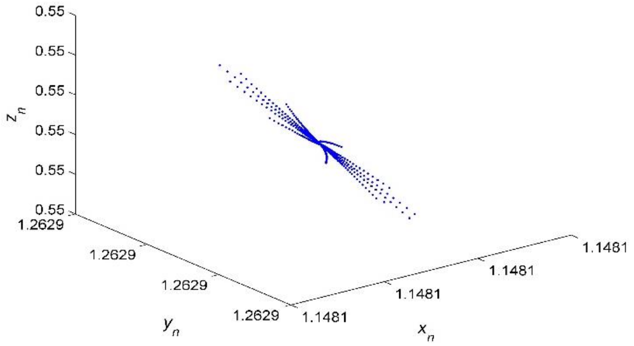


(d) Maximum Lyapunov exponents

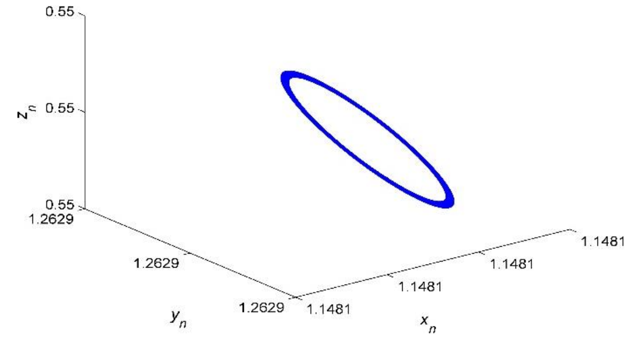
Fig. 5 Bifurcation diagrams and MLE for system (2) with $a = 1.1, b = 0.5, c = 2,$ $h \in [0.25, 0.65]$ and initial conditions $(x_0, y_0, z_0) = (1.14,\ 1.26, 0.55)$

7. Numerical simulation

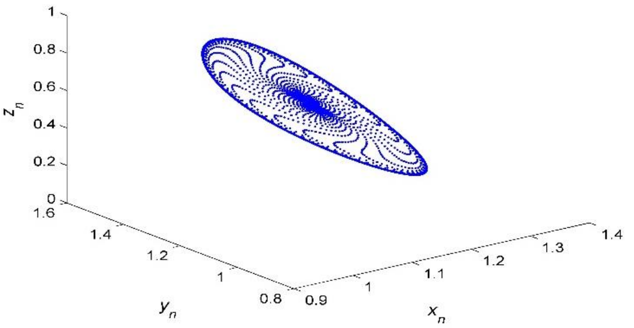
Example 7.1. First, we take $a = 1.1, b = 0.5, c = 2,$ and bifurcation parameter $h \in [0.25, 0.65]$ with initial conditions $(x^*, y^*, z^*) = (1.148, 1.26, 0.55)$ then system (2) undergoes Hopf bifurcation at $a = 1.1, b = 0.5, c = 2$ and $h = 0.471178$, the cubic polynomial of (2) is found as follows:



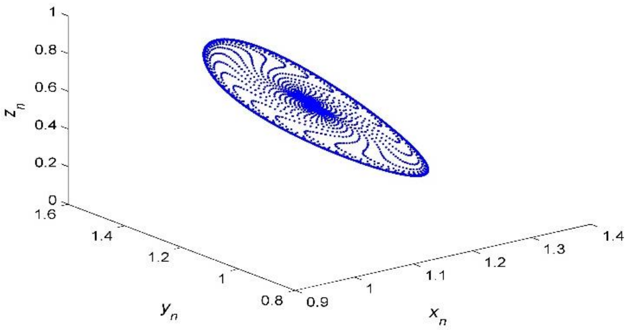
(a) Phase portrait at h=0.45



(b) Phase portrait at $h = 0.471178$



(c) Phase portrait at $h = 0.48$



(d) Phase plot for $h = 0.64$

Fig. 6 Phase portraits of system (2) with $a = 1.1, b = 0.5,\ c = ,\ x_0 = 0.78, y_0 = 1.26, z_0 = 0.55$ and various values of h

$$\mu^3 - 0.6479\mu^2 - 0.9841\mu + 0.6638 = 0 \qquad (15)$$

The solutions of (15) are $\mu_{1,2} = 0.990696 \pm 0.136093i$, and $\mu_3 = -0.135235$ with $|\mu_{1,2}| = 1$. As a result, the Neimark-Sacker bifurcation fulfills the

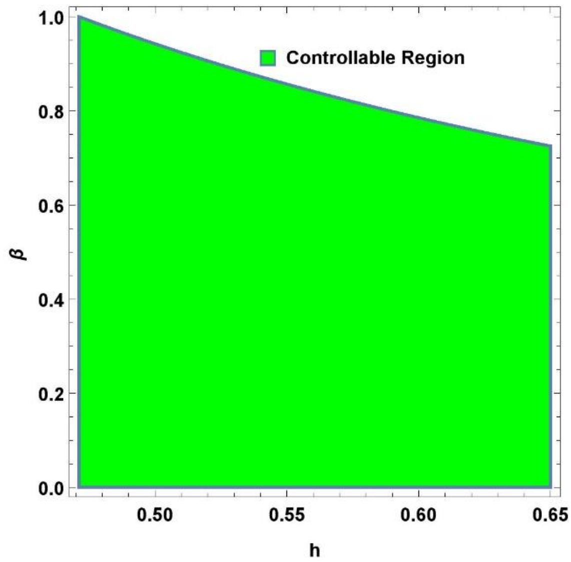


Fig. 7 Controllable region for system (12) for $a = 1.1$, $b = 0.5$, $c = 2$, $h \in [0.471178\,, 0.65]$ and $\beta \in [0,1]$

eigenvalues criteria. Following that, we check the constraints of Lemma 4.2 as follows:

$$\mathbb{D}_2^-(h) = 1 - \omega_1 + \omega_0(\omega_2 - \omega_0) = 0,$$
$$\mathbb{D}_2^+(h) = 1 + \omega_1 - \omega_0(\omega_2 + \omega_0) = 1.97147 > 0,$$
$$P_h(1) = 1 + \omega_2 + \omega_1 + \omega_0 = 0.303358 > 0,$$
$$(-1)^3 P_h(-1) = 1 - \omega_2 + \omega_1 - \omega_0 = 4.09211 > 0,$$
$$\left(\frac{d}{dh}(\mathbb{D}_2^-(h))\right)_{h=h_0} = (1 - \omega_1 + \omega_0(\omega_2 - \omega_0))_{h=h_0} = -0.597014 \neq 0 \text{ and } 1 - \frac{0.5 P_h(1)\mathbb{D}_0^-(h)}{\mathbb{D}_1^+(h)} = 1 - \frac{1+\omega_2+\omega_1+\omega_0}{2(1+\omega_0)} = 0.8277 \neq \pm 10.5472\ , l = 3,4,5, \ldots$$

Solving $\cos\left(\frac{2\pi}{l}\right) = 0.8277$, we have $l = \pm 10.5473$. As a result, the non-resonance criterion is likewise met. Figs. 5(a)-(d) depict the bifurcation diagrams and maximum Lyapunov exponents. Figs. 6(a)-(d) show some phase portraits at specific values of h parameters.

To demonstrate the effectiveness of the control approach (12) and in order to control Neimark-Sacker bifurcation, we use the parameters $a = 1.1$, $b = 0.5$, $c = 2$ and bifurcation parameter h is varied in chaotic region $h \in [0.471178\,, 0.65]$. Then, it is easy to see that $\Xi_1(x^*, y^*, z^*) = (1.148, 1.26, 0.55)$ is interior (positive) fixed point of controlled system (12). On the other hand, the Jacobian matrix for the controlled system (12) is given by:

$$\begin{bmatrix} 1 - 1.1\,h\,\beta & h\,\beta & 0 \\ 0.55\,h\,\beta & 1 - 0.5\,h\,\beta & 1.148121\,h\,\beta \\ -1.2629331\,h\,\beta & -1.148121\,h\,\beta & 1 - h\,\beta \end{bmatrix}.$$

The characteristic polynomial is further computed as follows:

$$P(\rho) = \rho^3 + A_2\,\rho^2 + A_1\,\rho + A_0,$$

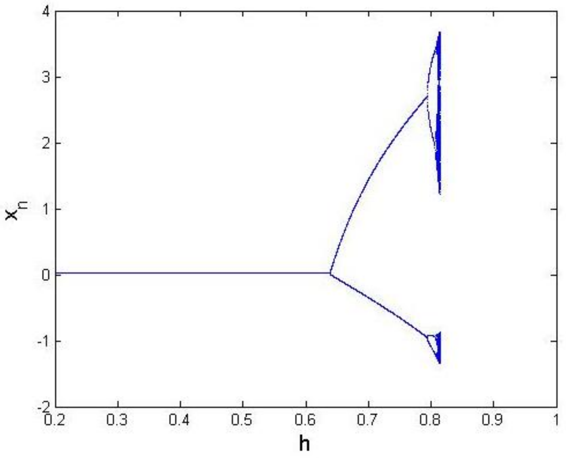
where

$$A_2 = 2.6\,h\,\beta - 3,$$
$$A_1 = 3 - 5.2\,h\,\beta + 2.9181818\,h^2\,\beta^2,$$

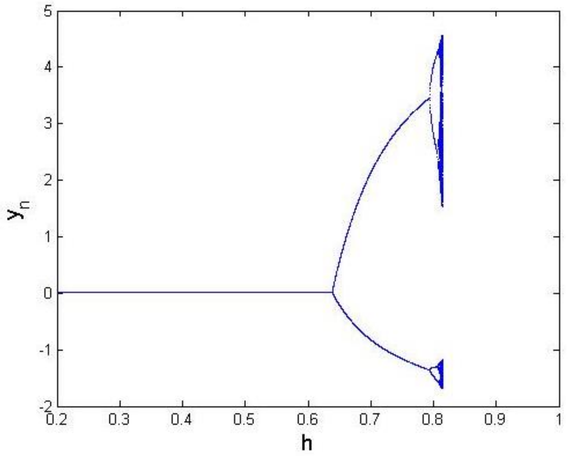
and

$$A_0 = 2.6\,h\,\beta - 2.918181818181818\,h^2\,\beta^2 + 2.9\,h^3\,\beta^3 - 1.$$

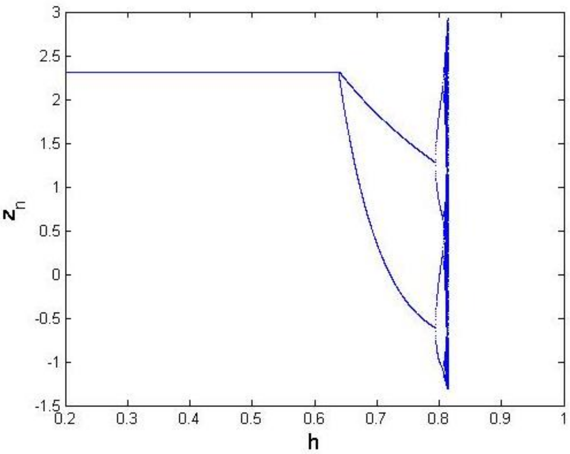
Then, system (12) is controllable if the following inequalities hold true:



(a) Bifurcation diagram for x_n



(b) Bifurcation diagram for y_n



(c) Bifurcation diagram for z_n

Fig. 8 Bifurcation diagrams for system (2) with $a = 1.2$, $b = 1.924$, $c = 2.35$, $h \in [0.2, 1]$ and initial values $(0.185, 0.222, 2.30)$

$$\begin{cases} |A_2 + A_0| < 1 + A_1 \\ |A_2 - 3A_0| < 3 - A_1 \\ A_0^2 + A_1 - A_0 A_2 < 1 \end{cases} \qquad (16)$$

Then, feasible region of (16) will be the controllability region for the system (12), and this controllability region is depicted in Fig. 7 in $h\,\beta$ -plane.

From Fig. 7, it is easy to see that the area of controllability region decreases with increasing the bifurcation parameter in chaotic region.

Example 7.2 We take $a = 1.2$, $b = 1.924$, $c = 2.35$, $(x_0, y_0, z_0) = (0.185, 0.222, 2.30)$ and $h \in [0.2, 1]$ in system (2). At these values, system (2) undergoes period-doubling bifurcation at $h = 0.64097$ as it is depicted in Fig. 8. For $a = 1.2$, $b = 1.924$, $c = 2.35$, and $h = 0.64097$ the cubic polynomial of (2) takes the form:
$\lambda^3 - 0.356635596\,\lambda^2 - 0.9891147067\lambda + 0.36748179 = 0.$

The solutions of the above polynomial are $\lambda_1 = -1$, $\lambda_2 = 0.373963$, $\lambda_3 = 0.98266$, as a result, the eigenvalues requirement for period-doubling bifurcation is met. Furthermore, the following criteria of Lemma 5.2 are justified:

$$\mathbb{D}_2^-(1.55416) = 1 - \omega_1 + \omega_0(\omega_2 - \omega_0)$$
$$= 1.113396514049319 > 0,$$
$$1 + \omega_1 - \omega_0(\omega_2 + \omega_0) = 0.005325450724677161 > 0$$
$$1 + \omega_2 + \omega_1 + \omega_0 = 0.03168086299273655 > 0,$$
$$1 - \omega_2 + \omega_1 - \omega_0 = 0,$$
$$1 + \omega_0 = 1.663806460960574 > 0,$$
$$\text{and } \frac{\omega_2' - \omega_1' + \omega_0'}{3 - 2\omega_2 + \omega_1} = -2.5936945019 \neq 0.$$

Figs. 8(a)-(c) depict the bifurcation diagrams for the system (2).

8. Conclusions

The laser phenomenon exhibits irregular and strange behaviour in particular parameter regimes, according to research by (Haken 1975). The chaotic behaviour and laser instabilities of the single-mode laser model were initially studied by (Ling, Chengren *et al*. 1998). Taking into account a single-mode laser system, both continuous and its discrete counterpart are studied. Our analysis reveals that continuous laser system undergoes Hopf bifuraction, and parametric conditions are derived for which system experiences Hopf bifurcation. On the other hand, discrete couterpart, which is obtained with the implementation of Euler approximation, undergoes period-doubling bifurcation and Neimark-Sacker bifurcation whenever step size of discretization is taken as bifuraction parameter. Explicit criteria are used to discuss both types of bifurcations. Consequently, rich dynamical behaviour is obtained in case of discrete-time laser model. The chaotic behaviour of the discrete-time model is successfully controlled with the utilization of hybrid control method. Our analysis shows that unstable or fluctuating trajectories can be stabilized under certain parametric conditions. Controlling chaotic behaviour and chaotic dynamics have become a remarkable stimulating situation in nonlinear laser models. On the other hand, the advances of chaos control in laser systems are considered mainly based on the ring cavity laser systems. Performance and improvement of laser power, secure communication, synchronized chaos, and information processing are considered as main prospects of chaos control for applications in laser systems.

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