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Thermoelastic deformation of a double-porous half-space medium with three-phase-lag model taking fractional derivatives

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Abstract. A problem of deformation of a double porous thermoelastic half space medium with fractional order heat transfer having Three-Phase-Lag (TPL) has been considered and discussed, due to the application of a thermo-mechanical force. A transformed procedure is taken to obtain the transformed form of result of the formulated problem. Inverse transformation of the solution is performed through a computer program for a specific model. The numerical solutions are drawn graphically for different cases. The effect of Three-Phase-Lag on Dual-Phase-Lag and the effect of fractional order and depth parameters on deformation is observed.

Keywords: double-porosity; fractional order; heat transfer; inversion; thermoelastic; Three-Phase-Lag; transforms

1. Introduction

Modern engineering structures frequently comprise multiphase porous media, where the presence of pores and fractures originating from erosion, corrosion, fatigue, or unintended damage has a significant impact on their dynamic responses. These structural complexities have driven the development of the double porosity model, an advanced extension of the classical porous medium theory. Unlike conventional models that consider a single porosity framework, the double porosity approach simultaneously accounts for pores within the matrix and fractures, enabling a more accurate depiction of naturally fractured reservoirs and geomaterial. Traditional single-phase porous media theories inadequately capture the mechanical behaviour of such materials, especially in scenarios where both pore systems are saturated with fluid. To overcome this shortcoming, the

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double porosity theory emerged as a natural progression, offering a more comprehensive characterization.

The foundational contribution by Barenblatt *et al.* (1960) initiated extensive exploration into double porosity systems. Building upon this, Wilson and Aifantis (1982) incorporated the concept of dual permeability into consolidation models. Subsequent developments include a model by Li *et al.* (2018) addressing liquid transport unsaturated double porosity solids, and a comprehensive continuum-based framework by Zhang *et al.* (2021) for anisotropic, deformable porous media with extremely low matrix permeability.

Further computational advancements were made by Khoei and Taghvaei (2024), who simulated the coupled hydro-mechanical behaviour of fractured porous materials using double-porosity theory.

Heat is the energy exchanged between particles-atoms, molecules, or electrons-when a temperature difference exists, and in solids or stationary fluids it is carried mainly through conduction. This mode of transfer moves thermal energy from regions of higher temperature to those of lower temperature without involving bulk motion of the material. Although heat flux cannot be measured directly, it is fundamentally connected to the temperature field, which can be observed and quantified. Once the spatial and temporal temperature distribution is known, the associated heat flow can be determined through the governing relations that link flux to temperature gradients. Thus, a primary objective in heat-conduction analysis is to compute the temperature field within a solid medium.

Thermoelasticity extends the theory of heat conduction by investigating the interaction between thermal fields and mechanical responses. It studies how temperature variations influence stress and strain, and conversely, how mechanical deformation affects heat transfer. The earliest hypothesis on thermal-mechanical coupling was given by Duhamel (1837), followed by Biot (1956), who formalized coupled thermoelastic equations and the thermal conductivity relation. A major limitation of classical coupled thermoelastic theories is their prediction of instantaneous thermal signal propagation, which contradicts physical observations. Generalized thermoelastic theories address this issue. Lord and Shulman (1967) introduced a model incorporating finite thermal wave speeds. Green and Naghdi (1993) contributed further by treating the thermal displacement gradient as a separate constitutive variable, leading to a formulation devoid of energy dissipation, setting it apart from prior models. Marin (1995) established existence and uniqueness results in the thermoelasticity of micropolar bodies. Othman *et al.* (2009) analyzed transient wave propagation generated by a line heat source moving at a constant velocity within an isotropic, homogeneous, perfectly conducting thermoelastic half-space subjected to a uniform magnetic field. In a later study, Lotfy (2014) investigated a two-dimensional electro-magneto-thermoelastic interaction in a homogeneous, isotropic half-space that conducts both heat and electricity, where the surface undergoes a thermal shock under a two-temperature heat-transfer model. Marin *et al.* (2014) analyzed thermoelastic initial boundary value problems with intrinsic rotations and microstructural expansion-contraction. Furthermore, Lotfy *et al.* (2020) applied photothermal theory to determine the primary field variables within a rotating medium based on generalized thermoelasticity.

A notable evolution of this theory is the Dual-Phase-Lag (DPL) model developed by Tzou (1995), which accounts for microstructural interactions in high-rate thermal processes. It introduces two independent phase lags in which one associated with heat flux and the other with the temperature gradient. Zenkour and Abouelregal (2013) investigated wave reflection in thermoelastic DPL media, including effects of variability in material properties. Kumar et.al (2017) Extended DPL theory to micropolar porous thermoelastic plates using eigenvalue analysis.

Roy Choudhury (2007) proposed the Three-Phase-Lag (TPL) model, incorporating the thermal displacement gradient into the formulation. Bharti and Sharma (2018) offered a comprehensive review of the TPL model, especially in the context of surface wave propagation and energy dissipation in thermoelastic systems. For deeper insight into TPL theories, refer to Abouelregal (2019).

Building on these frameworks, Abo-Dahab and Mahmoud (2018) analysed wave interaction at media interfaces under initial stress, gravitational, and magnetic fields within the Coupled Theory (CT), Green-Lindsay (GL) model, and Dual-Phase-Lag (DPL) frameworks. Hobiny *et al.* (2020) presented analytical solutions for thermal and mechanical responses in porous media under TPL, using Laplace-Fourier transforms. Jha and ovelade (2022) examined natural convection in a vertical channel, highlighting how DPL parameters influence early-stage temperature and velocity profiles. Singh *et al.* (2023) studied wave reflection in rotating, non-local semi-conductors using an extended TPL formulation. Chen *et al.* (2024) presented an analytical DPL solution for 3D solids subject to laser-induced heating, illustrating phase-lag effects through superposition and Green's functions. Katouzian *et al.* (2024) evaluated effective elastic moduli of two-phase composites with rectangular fiber arrays, highlighting microstructural geometry effects.

Thermoelastic theories tailored for double porosity systems were initially proposed to model the mechanical responses of fractured reservoirs. A seminal contribution in this field was made by Iesan and Quintanilla (2014), who provided the theoretical basis for studying thermoelastic media with a double-porosity structure. The relevance of such models has since expanded, given their applicability to geomechanics, seismology, and non-destructive evaluation, as well as to materials science, geophysics, biomechanics, petroleum engineering, and chemical processing. Marin and Flora (2014) investigated temporal behaviour and stability of solutions in porous micropolar thermoelastic bodies. Kumar *et al.* (2016) investigated reflection of plane waves in thermoelastic double-porosity media under Lord-Shulman theory. Abdou *et al.* (2019) derived exact solutions for isotropic, homogeneous, generalized thermoelastic half-spaces using the Lord-Shulman theory. Singh *et al.* (2020) addressed time-harmonic wave motion in infinite thermoelastic double porosity media. Liu *et al.* (2023) integrated Biot's poroelasticity with the DPL framework to study the behaviour of fast and slow P-waves, as well as shear and thermal waves in porous rocks. Kalkal *et al.* (2023) analysed a Three-Phase-Lag thermoelastic model involving functional grading, gravity, and double porosity effects. Mahato and Biswas (2024) proposed a diffusion-based thermoelastic model incorporating Eringen's nonlocal elasticity, TPL heat conduction, and double porosity theory.

Fractional-order differential models have emerged as robust tools for representing systems with memory effects, widely applicable in fields such as fluid dynamics, viscoelasticity, biological systems, and materials engineering. Unlike integer-order derivatives, which are inherently local, fractional derivatives introduce a non-local framework, making system behaviour dependent on historical states. Seminal works by Caputo (1967), Mainardi (1997), Podlubny (1999) laid the foundation for this field. Building upon this, Ezzat (2010-11) proposed fractional heat conduction models based on the time-fractional Taylor expansion introduced by Jumarie (2010). Awad (2012) conducted a further investigation into thermal lagging behavior by re-examining the Danilovskaya problem within the context of fractional Dual-Phase-Lag (DPL) thermoelasticity, employing the modified Tzou model. Ezzat *et al.* (2016) applied fractional order thermoelasticity to a 3D thermal shock problem in a half-space. Mahdy *et al.* (2020) examined how an existing hydrostatic stress field affects the behaviour of the time-fractional heat equation. Marin *et al.* (2020) studied Moore-Gibson-Thompson thermoelasticity in dipolar bodies and derived qualitative solution properties.

Awad *et al.* (2021) investigated fractional Jeffrey's equations and their links to stochastic processes like continuous-time random walks. More recently, Roy and Lahiri (2023) analysed thermoelastic media with voids using fractional TPL models. Othman *et al.* (2023) investigated memory-dependent thermoelasticity in rotating systems with voids under TPL conduction. Pathania *et al.* (2024) studied TPL wave propagation in transversely isotropic, double-porous thermoelastic half-spaces adjacent to inviscid fluids via normal mode analysis. Miglani *et al.* (2024) formulated a mathematical model for an axisymmetric problem in double-porous thermoelastic media incorporating a time-fractional heat conduction equation.

Given the significance of the Three-Phase-Lag (TPL) thermoelasticity model and the growing relevance of fractional-order derivatives, both of which have been extensively studied by researchers. We propose to formulate mathematical models for two-dimensional problems involving a double-porous thermoelastic medium within the framework of the fractional-order Three-Phase-Lag model.

2. Basic equations

Following Iesan and Quintanilla (2014), Roy Choudhuri (2007), Ezzat *et al.* (2012), the basic equations for double porous thermoelastic medium having Three-Phase-Lag and fractional order derivatives, in absence of body forces are as under

Constitutive Relations

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\delta_{ij}\phi + d\delta_{ij}\psi - \beta\delta_{ij}T, \tag{1}$$

$$\sigma_i = \alpha\phi_{,i} + b_1\psi_{,i}, \tag{2}$$

$$\tau_i = b_1\phi_{,i} + \gamma_0\psi_{,i}, \tag{3}$$

Equation of motion

$$\mu\Delta\vec{u} + (\lambda+\mu)\nabla\nabla.\vec{u} + b\nabla\phi + d\nabla\psi - \beta\nabla T = \rho\ddot{\vec{u}}, \tag{4}$$

Equilibrated stress equations of motion

$$\alpha\Delta\phi + b_1\Delta\psi - b\nabla.\vec{u} - \alpha_1\phi - \alpha_3\psi + \gamma_1 T = \chi_1\ddot{\phi}, \tag{5}$$

$$b_1\Delta\phi + \gamma_0\Delta\psi - d\nabla.\vec{u} - \alpha_3\phi - \alpha_2\psi + \gamma_2 T = \chi_2\ddot{\psi}, \tag{6}$$

Equation of heat conduction

$$k^*\left(1 + \frac{\tau_v{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta T + k_1{}^*\left(1 + \frac{\tau_T{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta\dot{T} = \left(1 + \frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}} + \frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\left(\beta T_0\nabla.\ddot{\vec{u}} + \gamma_1 T_0\ddot{\phi} + \gamma_2 T_0\ddot{\psi} + \rho C^*\ddot{T}\right), \tag{7}$$

where t_{ij} denotes the stress tensor; e_{ij} represents the strain tensor and e_{rr} indicating the cubical dilatation; the Lame's constants λ and μ characterize the elastic behaviour of the medium; ϕ and ψ describe the volume fraction fields corresponding to pores ν_1 and fissures ν_2, respectively; the Kronecker's delta is expressed as δ_{ij}; σ_i is interpreted as the equilibrated stress corresponding to pores; τ_i denotes the equilibrated stress corresponding to fissures; b, d, α, b_1, γ_0, α_1, α_2, α_3 γ_1, γ_2

are the constitutive coefficients. $\beta = (3\lambda + 2\mu)\alpha_t$; α_t is the linear thermal expansion; $C^*$is the specific heat under constant strain conditions; K^* is the material characteristic, $K^* = c^*\left(\frac{\lambda+2\mu}{4}\right)$. The thermal conductivity is expressed by ${K_1}^*$. T is the temperature change measured from the absolute temperature T_0 $(T_0 \neq 0)$. α^* is fractional order. τ_ν , τ_T , τ_q are the phase lags corresponding to thermal displacement, temperature gradient and heat flux respectively. $\vec{u}$ is the displacement vector; ρ is the mass density; χ_1 and χ_2 are the coefficients of equilibrated inertia. And T is the temperature change measured from the absolute temperature T_0 $(T_0 \neq 0)$. α^* refers to fractional order parameter. Additionally, the model incorporates phase-lag parameters: τ_ν, τ_T, τ_q corresponding to thermal displacement, temperature gradient and heat flux respectively. The displacement vector is denoted by $\vec{u}$.

3. Formulation of the problem

An infinite homogenous isotropic double porous thermoelastic medium with fractional order heat transfer having Three-Phase-Lag (TPL) is considered, and is acted upon by a thermo-mechanical force, which is a kind of normal line load moving along the boundary and point heat step source at the origin. Such sources exist typically in the problems related to the field of engineering, mechanics and thermal analysis. Commonly used to model situations where a force travels along a structure or continuum like moving wheel load on roads, runways, rail tracks or bridges along with the model for instantaneous heat release at a point, e.g., microseismic events, magma injections.

Taking cartesian system of axis (x_1, x_2, x_3), x_3- axis is considered perpendicularly downward in half space and the plane free surface is taken as $x_3 = 0$, so that the half apace is represented as $x_3 \geq 0$. As such, two-dimensional plain strain problem in x_1x_3-plane is considered,
The initial temperature is taken as a constant temperature T_0.

4. Solution of the problem

For the considered two-dimensional plane strain problem in x_1x_3-plane, all the variables depend upon x_1,x_3 and *t*, and the displacement vector $\vec{u}$ is of the form $\vec{u} = (u_1, 0, u_3)$. As such, the equation of motion, equilibrated stress equations of motion, equation of heat conduction and constitutive relations given by Eqs. (1)-(7) for the two-dimensional plane strain problem are reduced to the following component form of equations

$$\mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) + b\left(\frac{\partial \phi}{\partial x_1}\right) + d\left(\frac{\partial \psi}{\partial x_1}\right) - \beta\left(\frac{\partial T}{\partial x_1}\right) = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{8}$$

$$\mu\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + b\left(\frac{\partial \phi}{\partial x_3}\right) + d\left(\frac{\partial \psi}{\partial x_3}\right) - \beta\left(\frac{\partial T}{\partial x_3}\right) = \rho\frac{\partial^2 u_3}{\partial t^2}, \tag{9}$$

$$\alpha\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_3^2}\right) + b_1\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2}\right) - b\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) - \alpha_1\phi - \alpha_3\psi + \gamma_1 T = \chi_1\frac{\partial^2 \phi}{\partial t^2}, \tag{10}$$

$$b_1\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_3^2}\right) + \gamma_\circ\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2}\right) - d\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) - \alpha_3\phi - \alpha_2\psi + \gamma_2 T = \chi_2\frac{\partial^2 \psi}{\partial t^2}, \tag{11}$$

$$k^*\left(1+\frac{\tau_v{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}\right)T+k_1{}^*\left(1+\frac{\tau_T{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}\right)\dot{T}$$
$$=\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\left(\beta T_0\frac{\partial^2 e}{\partial t^2}+\gamma_1 T_0\frac{\partial^2\phi}{\partial t^2}+\gamma_2 T_0\frac{\partial^2\psi}{\partial t^2}+\rho C^*\frac{\partial^2 T}{\partial t^2}\right), \tag{12}$$

$$t_{11}=\lambda e_{rr}+2\mu e_{11}+b\phi+d\psi-\beta T, \tag{13}$$

$$t_{33}=\lambda e_{rr}+2\mu e_{33}+b\phi+d\psi-\beta T, \tag{14}$$

$$t_{31}=2\mu e_{31}, \tag{15}$$

$$\sigma_1=\alpha\frac{\partial\phi}{\partial x_1}+b_1\frac{\partial\psi}{\partial x_1}, \tag{16}$$

$$\sigma_3=\alpha\frac{\partial\phi}{\partial x_3}+b_1\frac{\partial\psi}{\partial x_3}, \tag{17}$$

$$\tau_1=b_1\frac{\partial\phi}{\partial x_1}+\gamma_0\frac{\partial\psi}{\partial x_1}, \tag{18}$$

$$\tau_3=b_1\frac{\partial\phi}{\partial x_3}+\gamma_0\frac{\partial\psi}{\partial x_3}, \tag{19}$$

where

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\text{ and }e_{rr}=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}=e=\text{div}\,\vec{u}.$$

Now, we define the dimensionless parameters as

$$\begin{gathered}x_1'=\frac{\omega_1}{c_1}x_1,\ x_3'=\frac{\omega_1}{c_1}x_3,\ u_1'=\frac{\omega_1}{c_1}u_1,\\ u_3'=\frac{\omega_1}{c_1}u_3,\ \phi'=\frac{\chi_1\omega_1^2}{\alpha_1}\phi,\ \psi'=\frac{\chi_1\omega_1^2}{\alpha_1}\psi,\\ t_{ij}'=\frac{t_{ij}}{\beta T_0},\ \sigma_1'=\left(\frac{c_1}{\alpha\omega_1}\right)\sigma_1,\ \sigma_3'=\left(\frac{c_1}{\alpha\omega_1}\right)\sigma_3,\\ \tau_1'=\left(\frac{c_1}{\alpha\omega_1}\right)\tau_1,\ \tau_3'=\left(\frac{c_1}{\alpha\omega_1}\right)\tau_3,\ t'=\omega_1 t,\\ \tau_v'=(\omega_1)\tau_v,\ \tau_T'=(\omega_1)\tau_T,\ \tau_q'=(\omega_1)\tau_q,\\ T'=\frac{T}{T_0},\end{gathered} \tag{20}$$

where

$$c_1^2=\frac{\lambda+2\mu}{\rho},\ \omega_1=\frac{\rho C^* c_1^2}{K_1{}^*}\text{ and }K^*=c^*\left(\frac{\lambda+2\mu}{4}\right). \tag{21}$$

are the constants having the dimensions of velocity and frequency in the medium, respectively. Using dimensionless parameters given by (20) in Eqs. (8)-(19), and after suppressing the primes, we obtain the dimension-less form of equations as

$$\left(\frac{\lambda+\mu}{\rho c_1^2}\right)\frac{\partial e}{\partial x_1}+\frac{\mu}{\rho c_1^2}\nabla^2 u_1+\delta_1\nabla\phi+\delta_2\nabla\psi-\delta_3\nabla T=\frac{\partial^2 u_1}{\partial t^2}, \tag{22}$$

$$\left(\frac{\lambda+\mu}{\rho c_1^2}\right)\frac{\partial e}{\partial x_3}+\frac{\mu}{\rho c_1^2}\nabla^2 u_3+\delta_1\nabla\phi+\delta_2\nabla\psi-\delta_3\nabla T=\frac{\partial^2 u_3}{\partial t^2}, \tag{23}$$

$$\delta_4\nabla^2\phi+\delta_5\nabla^2\psi-\delta_6 e-\delta_7\phi-\delta_8\psi+\delta_9 T=\frac{\partial^2\phi}{\partial t^2}, \tag{24}$$

$$\delta_{10}\nabla^2\phi+\delta_{11}\nabla^2\psi-\delta_{12} e-\delta_{13}\phi-\delta_{14}\psi+\delta_{15} T=\frac{\partial^2\psi}{\partial t^2}, \tag{25}$$

$$\begin{gathered}\delta_{16}\left(1+\frac{\tau_\nu{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta T+\delta_{17}\left(1+\frac{\tau_T{}^{\alpha^*}}{\alpha^*!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta\dot{T}-\delta_{18}\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\frac{\partial^2}{\partial t^2}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\frac{\partial^2 e}{\partial t^2}\\ -\delta_{19}\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\frac{\partial^2\phi}{\partial t^2}-\delta_{20}(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\frac{\partial^2\psi}{\partial t^2}\\ =\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\frac{\partial^2 T}{\partial t^2},\end{gathered} \tag{27}$$

$$t_{11}=P_1\frac{\partial u_1}{\partial x_1}+P_2\frac{\partial u_3}{\partial x_3}+P_3\phi+P_4\psi-T, \tag{27}$$

$$t_{33}=P_1\frac{\partial u_3}{\partial x_3}+P_2\frac{\partial u_1}{\partial x_1}+P_3\phi+P_4\psi-T, \tag{28}$$

$$t_{31}=P_8\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right), \tag{29}$$

$$\sigma_1=P_5\frac{\partial\phi}{\partial x_1}+P_6\frac{\partial\psi}{\partial x_1}, \tag{30}$$

$$\sigma_3=P_5\frac{\partial\phi}{\partial x_3}+P_6\frac{\partial\psi}{\partial x_3}, \tag{31}$$

$$\tau_1=P_6\frac{\partial\phi}{\partial x_1}+P_7\frac{\partial\psi}{\partial x_1}, \tag{32}$$

$$\tau_3=P_6\frac{\partial\phi}{\partial x_3}+P_7\frac{\partial\psi}{\partial x_3}, \tag{33}$$

where

$$\begin{gathered}\delta_1=\frac{b\alpha_1}{\rho c_1^2\chi_1\omega_1^2},\ \delta_2=\frac{d\alpha_1}{\rho c_1^2\chi_1\omega_1^2},\ \delta_3=\frac{\beta T_0}{\rho c_1^2},\\ \delta_4=\frac{\alpha}{c_1^2\chi_1},\ \delta_5=\frac{b_1}{c_1^2\chi_1},\ \delta_6=\frac{b}{\alpha_1},\\ \delta_7=\frac{\alpha_1}{\chi_1{\omega_1}^2},\ \delta_8=\frac{\alpha_3}{\chi_1\omega_1^2},\ \delta_9=\frac{\gamma_1 T_0}{\alpha_1},\\ \delta_{10}=\frac{b_1}{c_1^2\chi_2},\ \delta_{11}=\frac{\gamma_\circ}{c_1^2\chi_2},\ \delta_{12}=\frac{d\chi_1}{\alpha_1\chi_2},\\ \delta_{13}=\frac{\alpha_3}{\chi_2{\omega_1}^2},\ \delta_{14}=\frac{\alpha_2}{\chi_2{\omega_1}^2},\ \delta_{15}=\frac{\gamma_2 T_0\chi_1}{\alpha_1\chi_2},\\ \delta_{16}=\frac{K^*}{\rho C^* c_1^2},\ \delta_{17}=\frac{K_1^*\omega_1}{\rho C^* c_1^2}\ \delta_{18}=\frac{\beta}{\rho C^*},\\ \delta_{19}=\frac{\gamma_1\alpha_1}{\rho C^*\chi_1{\omega_1}^2},\ \delta_{20}=\frac{\gamma_2\alpha_1}{\rho C^*\chi_1{\omega_1}^2},\ P_1=\frac{\lambda+2\mu}{\beta T_0},\\ P_2=\frac{\lambda}{\beta T_0},\ P_3=\frac{b\alpha_1}{\beta T_0\chi_1\omega_1^2},\ P_4=\frac{d\alpha_1}{\beta T_0\chi_1\omega_1^2},\\ P_5=\frac{\alpha_1}{\chi_1\omega_1^2},\ P_6=\frac{b_1\alpha_1}{\alpha\chi_1\omega_1^2},\ P_7=\frac{\gamma_0\alpha_1}{\alpha\chi_1\omega_1^2},\\ P_8=\frac{\mu}{\beta T_0}.\end{gathered} \tag{34}$$

Following Helmholtz's decomposition technique, the non-dimensional displacement components u_1 and u_3 are related to the scalar potential functions ϕ_1 and ψ_1 as

$$u_1 = \frac{\partial \phi_1}{\partial x_1} - \frac{\partial \psi_1}{\partial x_3},\ u_3 = \frac{\partial \phi_1}{\partial x_3} + \frac{\partial \psi_1}{\partial x_1}. \tag{35}$$

Substituting the values of u_1 and u_3 from Eq. (35) in Eqs. (22)-(26), we obtain

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right)\phi_1 + \delta_1\phi + \delta_2\psi - \delta_3 T = 0, \tag{36}$$

$$\left(\frac{\mu}{\rho c_1^2}\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\psi_1 = 0, \tag{37}$$

$$\delta_4\Delta\phi + \delta_5\Delta\psi - \delta_6(\Delta\phi_1) - \delta_7\phi - \delta_8\psi + \delta_9 T = \frac{\partial^2\phi}{\partial t^2}, \tag{38}$$

$$\delta_{10}\Delta\phi + \delta_{11}\Delta\psi - \delta_{12}(\Delta\phi_1) - \delta_{13}\phi - \delta_{14}\psi + \delta_{15} T = \frac{\partial^2\psi}{\partial t^2}, \tag{39}$$

$$\delta_{16}\left(1 + \frac{{\tau_\nu}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta T + \delta_{17}\left(1 + \frac{{\tau_T}^{\alpha^*}}{\alpha^*!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta\dot{T} - \left(1 + \frac{{\tau_q}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}} + \frac{{\tau_q}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\left(\delta_{18}\frac{\partial^2(\Delta\phi_1)}{\partial t^2} + \delta_{19}\frac{\partial^2(\phi)}{\partial t^2} + \delta_{20}\frac{\partial^2(\psi)}{\partial t^2} + \frac{\partial^2 T}{\partial t^2}\right) = 0. \tag{40}$$

By applying Laplace transform as defined by Debnath (1995), on Eqs. (36)-(40), we obtain

$$(\Delta - s^2)\bar{\phi}_1 + \delta_1\bar{\phi} + \delta_2\bar{\psi} - \delta_3\bar{T} = 0, \tag{41}$$

$$\left(\frac{\mu}{\rho c_1^2}\nabla^2 - s^2\right)\bar{\psi}_1 = 0, \tag{42}$$

$$\delta_4\Delta\bar{\phi} + \delta_5\Delta\bar{\psi} - \delta_6(\Delta\bar{\phi}_1) - \delta_7\bar{\phi} - \delta_8\bar{\psi} + \delta_9\bar{T} = s^2\bar{\phi}, \tag{43}$$

$$\delta_{10}\Delta\bar{\phi} + \delta_{11}\Delta\bar{\psi} - \delta_{12}(\Delta\bar{\phi}_1) - \delta_{13}\bar{\phi} - \delta_{14}\bar{\psi} + \delta_{15}\bar{T} = s^2\bar{\psi}, \tag{44}$$

$$\delta_{16}\left(1 + \frac{{\tau_\nu}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}\right)\Delta\bar{T} + \delta_{17}\left(1 + \frac{{\tau_T}^{\alpha^*}}{\alpha^*!}s^{\alpha^*}\right)\Delta s\bar{T} - \left(1 + \frac{{\tau_q}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{{\tau_q}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)(\delta_{18}s^2(\Delta\bar{\phi}_1) + \delta_{19}s^2\bar{\phi} + \delta_{20}s^2\bar{\psi} + s^2\bar{T}) = 0. \tag{45}$$

Further, applying Fourier transform as given by Sneddon (1979), on Eqs. (41)-(45), we obtain

$$\left(\frac{d^2}{dx_3^2} - \xi^2 - s^2\right)\tilde{\phi}_1 + \delta_1\tilde{\phi} + \delta_2\tilde{\psi} - \delta_3\tilde{T} = 0, \tag{46}$$

$$\delta_6\left(\frac{d^2}{dx_3^2} - \xi^2\right)\tilde{\phi}_1 - \left(\delta_4\left(\frac{d^2}{dx_3^2} - \xi^2\right) - \delta_7 - s^2\right)\tilde{\phi} - \left(\delta_5\left(\frac{d^2}{dx_3^2} - \xi^2\right) - \delta_8\right)\tilde{\psi} - \delta_9\tilde{T} = 0, \tag{47}$$

$$\delta_{12}\left(\frac{d^2}{dx_3^2} - \xi^2\right)\tilde{\phi}_1 - \left(\delta_{10}\left(\frac{d^2}{dx_3^2} - \xi^2\right) - \delta_{13}\right)\tilde{\phi} - \left(\delta_{11}\left(\frac{d^2}{dx_3^2} - \xi^2\right) - \delta_{14} - s^2\right)\tilde{\psi} - \delta_{15}\tilde{T} = 0, \tag{48}$$

$$s^2\delta_{18}\left(1 + \frac{{\tau_q}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{{\tau_q}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)\left(\frac{d^2}{dx_3^2} - \xi^2\right)\tilde{\phi}_1 + s^2\delta_{19}\left(1 + \frac{{\tau_q}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{{\tau_q}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)\tilde{\phi}$$

$$+s^2\delta_{20}\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)\tilde{\psi}-\left(\left(\delta_{16}\left(1+\frac{\tau_\nu{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}\right)+\delta_{17}s\left(1+\frac{\tau_T{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}\right)\right)\left(\frac{d^2}{dx_3^2}-\xi^2\right)-s^2\left(1+\frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}+\frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)\right)\tilde{T}=0, \tag{49}$$

$$\left(\frac{\mu}{\rho c_1^2}\left(\frac{d^2}{dx_3^2}-\xi^2\right)-s^2\right)\tilde{\psi}_1=0. \tag{50}$$

Also, applying the Laplace and Fourier transforms on the stress components given by (27)-(33), we obtain

$$\tilde{t}_{11}=-i\xi P_1\tilde{u}_1+P_2\frac{\partial\tilde{u}_3}{\partial x_3}+P_3\tilde{\phi}+P_4\tilde{\psi}-\tilde{T}, \tag{51}$$

$$\tilde{t}_{33}=P_1\frac{\partial\tilde{u}_3}{\partial x_3}-i\xi P_2\tilde{u}_1+P_3\tilde{\phi}+P_4\tilde{\psi}-\tilde{T}, \tag{52}$$

$$\tilde{t}_{31}=P_8\left(-i\xi\tilde{u}_3+\frac{\partial\tilde{u}_1}{\partial x_3}\right), \tag{53}$$

$$\tilde{\sigma}_1=-i\xi P_5\tilde{\phi}-i\xi P_6\tilde{\psi}, \tag{54}$$

$$\tilde{\sigma}_3=P_5\frac{\partial\tilde{\phi}}{\partial x_3}+P_6\frac{\partial\tilde{\psi}}{\partial x_3}, \tag{55}$$

$$\tilde{\tau}_1=-i\xi P_6\tilde{\phi}-i\xi P_7\tilde{\psi}, \tag{56}$$

$$\tilde{\tau}_3=P_6\frac{\partial\tilde{\phi}}{\partial x_3}+P_7\frac{\partial\tilde{\psi}}{\partial x_3}. \tag{57}$$

The Eqs. (46)-(49) are four homogeneous equations in four unknown variables $\tilde{\phi}_1,\ \tilde{\phi},\ \tilde{\psi}$ and $\tilde{T}$. This system of equations has a non-trivial solution, if the determinant of the coefficient matrix of $(\ \tilde{\phi}_1,\ \tilde{\phi},\ \tilde{\psi},\ \tilde{T}\ )$ vanishes, which on solving gives the polynomial characteristic equations as

$$\left(E_1\frac{d^8}{dx_3^8}+E_2\frac{d^6}{dx_3^6}+E_3\frac{d^4}{dx_3^4}+E_4\frac{d^2}{dx_3^2}+E_5\right)\left(\ \tilde{\phi}_1,\tilde{\phi},\tilde{\psi},\tilde{T}\ \right)=0, \tag{58}$$

where the coefficients E_1, E_2, E_3, E_4 and E_5 of Eq. (58) are given in appendix.

Making use of the transformed form of radiation conditions, that is

$$\tilde{\phi}_1,\ \tilde{\phi},\ \tilde{\psi},\ \tilde{T},\ \tilde{\psi}_1\to 0 \text{ as } x_3\to\infty, \tag{59}$$

the solutions for Eq. (58) are taken as

$$(\ \tilde{\phi}_1,\ \tilde{\phi},\ \tilde{\psi},\ \tilde{T}\ )=\textstyle\sum_{i=1}^{4}(\ 1,\ a_i,\ b_i,\ d_i\ )\,A_i\,e^{-m_i x_3}, \tag{60}$$

Similarly, solution for Eq. (50) are taken as

$$\tilde{\psi}_1=A_5\,e^{-m_5x_3}, \tag{61}$$

In (60) and (61), A_i (i=1, 2, 3, 4, 5) are the arbitrary constants and m_i (i=1, 2, 3, 4) are the roots of the equation

$$E_1\frac{d^8}{dx_3^8}+E_2\frac{d^6}{dx_3^6}+E_3\frac{d^4}{dx_3^4}+E_4\frac{d^2}{dx_3^2}+E_5=0, \tag{62}$$

and

$$m_5=\sqrt{\xi^2+\frac{s^2}{a_5^2}}. \tag{63}$$

The coupling constants in Eq. (60) are obtained as

$$a_i=-\frac{D_{1i}}{D_{0i}},\ b_i=-\frac{D_{2i}}{D_{0i}},\ \ d_i=-\frac{D_{3i}}{D_{0i}},\ \text{for}\ \ i=1,2,3,4, \tag{41}$$

and

$$a_5^2=\frac{\mu}{\rho c_1^2}, \tag{65}$$

where

$$D_{0i}=\begin{vmatrix}\delta_4 m_i^2+q_3 & \delta_5 m_i^2+q_4 & \delta_9\\ \delta_{10}m_i^2+q_6 & \delta_{11}m_i^2+q_7 & \delta_{15}\\ \delta_{22} & \delta_{23} & \delta_{24}m_i^2+q_9\end{vmatrix},$$

$$D_{1i}=\begin{vmatrix}-\delta_6 m_i^2+q_2 & \delta_5 m_i^2+q_4 & \delta_9\\ -\delta_{12}m_i^2+q_5 & \delta_{11}m_i^2+q_7 & \delta_{15}\\ \delta_{21}m_i^2+q_8 & \delta_{23} & \delta_{24}m_i^2+q_9\end{vmatrix},$$

$$D_{2i}=\begin{vmatrix}-\delta_6 m_i^2+q_2 & \delta_4 m_i^2+q_3 & \delta_9\\ -\delta_{12}m_i^2+q_5 & \delta_{10}m_i^2+q_6 & \delta_{15}\\ \delta_{21}m_i^2+q_8 & \delta_{22} & \delta_{24}m_i^2+q_9\end{vmatrix}, \tag{66}$$

$$D_{3i}=\begin{vmatrix}-\delta_6 m_i^2+q_2 & \delta_4 m_i^2+q_3 & \delta_5 m_i^2+q_4\\ -\delta_{12}m_i^2+q_5 & \delta_{10}m_i^2+q_6 & \delta_{11}m_i^2+q_7\\ \delta_{21}m_i^2+q_8 & \delta_{22} & \delta_{23}\end{vmatrix}.$$

Applying the Laplace and Fourier transforms on Eq. (35), and then using Eqs. (60) and (61), we get

$$\tilde{u}_1=-i\xi(A_1e^{-m_1x_3}+A_2e^{-m_2x_3}+A_3e^{-m_3x_3}+A_4e^{-m_3x_3})+m_5A_5e^{-m_5x_3}, \tag{67}$$

$$\tilde{u}_3=-(m_1A_1e^{-m_1x_3}+m_2A_2e^{-m_2x_3}+m_3A_3e^{-m_3x_3}+m_4A_4e^{-m_3x_3})-i\xi A_5e^{-m_5x_3}, \tag{68}$$

Further, the relevant stress components and temperature change are obtained from Eqs. (51)-(57) using Eqs. (67)-(68) as

$$\tilde{t}_{33}=Q_1A_1e^{-m_1x_3}+Q_2A_2e^{-m_2x_3}+Q_3A_3e^{-m_3x_3}+Q_4A_4e^{-m_4x_3}+Q_5A_5e^{-m_5x_3}, \tag{69}$$

$$\tilde{t}_{31}=R_6\{R_1A_1e^{-m_1x_3}+R_2A_2e^{-m_2x_3}+R_3A_3e^{-m_3x_3}+R_4A_4e^{-m_4x_3}+R_5A_5e^{-m_5x_3}\}, \tag{70}$$

$$\tilde{\sigma}_3=U_1A_1e^{-m_1x_3}+U_2A_2e^{-m_2x_3}+U_3A_3e^{-m_3x_3}+U_4A_4e^{-m_4x_3}, \tag{71}$$

$$\tilde{\tau}_3=V_1A_1e^{-m_1x_3}+V_2A_2e^{-m_2x_3}+V_3A_3e^{-m_3x_3}+V_4A_4e^{-m_4x_3}, \tag{72}$$

$$\tilde{T}=d_1A_1e^{-m_1x_3}+d_2A_2e^{-m_2x_3}+d_3A_3e^{-m_3x_3}+d_4A_4e^{-m_4x_3}, \tag{73}$$

where

$$Q_1 = P_1 m_1^2 - \xi^2 P_2 + P_3 a_1 + P_4 b_1 - d_1,$$
$$Q_2 = P_1 m_2^2 - \xi^2 P_2 + P_3 a_2 + P_4 b_2 - d_2,$$
$$Q_3 = P_1 m_3^2 - \xi^2 P_2 + P_3 a_3 + P_4 b_3 - d_3,$$
$$Q_4 = P_1 m_4^2 - \xi^2 P_2 + P_3 a_4 + P_4 b_4 - d_4,$$
$$Q_5 = i\xi(P_1 - P_2)m_5,\ R_1 = 2m_1,\ R_2 = 2m_2,$$
$$R_3 = 2m_3,\ R_4 = 2m_4,\ R_5 = i\xi\left(1 + \frac{m_5^2}{\xi^2}\right), \tag{75}$$
$$R_6 = i\xi P_8,\ U_1 = -(P_5 a_1 + P_6 b_1)m_1,$$
$$U_2 = -(P_5 a_2 + P_6 b_2)m_2,\ U_3 = -(P_5 a_3 + P_6 b_3)m_3,$$
$$U_4 = -(P_5 a_4 + P_6 b_4)m_3,\ V_1 = -(P_6 a_1 + P_7 b_1)m_1,$$
$$V_2 = -(P_6 a_2 + P_7 b_2)m_2,\ V_3 = -(P_6 a_3 + P_7 b_3)m_3,$$
$$V_4 = -(P_6 a_4 + P_7 b_4)m_4.$$

5. Boundary conditions

A thermo-mechanical force, a kind of normal line load moving along the boundary with velocity v and a concentrated mechanical source is acted on the boundary surface $x_3 = 0$. So, the boundary conditions, at $x_3 = 0$ are obtained as

$$\text{(i) } t_{33} = F_1 \delta(x_3 - vt), \tag{75}$$

$$\text{(ii) } t_{31} = 0, \tag{76}$$

$$\text{(iii) } \sigma_3 = 0, \tag{77}$$

$$\text{(iv) } \tau_3 = 0, \tag{78}$$

$$\text{(v) } T = F_3 \mathrm{H}(t - v)\ \delta(x_1). \tag{79}$$

Now, by applying Laplace and Fourier transforms on boundary conditions we have

$$\text{(i) } \tilde{t}_{33}(\xi, x_3, s) = \frac{F_1}{v} e^{-\left(\frac{s}{v}\right)x_3}, \tag{80}$$

$$\text{(ii) } \tilde{t}_{31}(\xi, x_3, s) = 0, \tag{81}$$

$$\text{(iii) } \tilde{\sigma}_3(\xi, x_3, s) = 0, \tag{82}$$

$$\text{iv) } \tilde{\tau}_3(\xi, x_3, s) = 0, \tag{83}$$

$$\text{(v) } \tilde{T}(\xi, x_3, s) = \frac{F_3}{s} e^{-s v_0}. \tag{84}$$

Now substituting the values of $\tilde{t}_{33}, \tilde{t}_{31,} \tilde{\sigma}_3, \tilde{\tau}_3, \tilde{T}$ from Eqs. (69) to (73) in Eqs. (80) to (84), we get

$$Q_1 A_1 e^{-m_1 x_3} + Q_2 A_2 e^{-m_2 x_3} + Q_3 A_3 e^{-m_3 x_3} + Q_4 A_4 e^{-m_4 x_3} + Q_5 A_5 e^{-m_5 x_3} = \frac{F_1}{v} e^{-\left(\frac{s}{v}\right)x_3}, \tag{85}$$

$$R_6\{R_1A_1e^{-m_1x_3}+R_2A_2e^{-m_2x_3}+R_3A_3e^{-m_3x_3}+R_4A_4e^{-m_4x_3}+R_5A_5e^{-m_5x_3}\}=0, \quad (86)$$

$$U_1A_1e^{-m_1x_3}+U_2A_2e^{-m_2x_3}+U_3A_3e^{-m_3x_3}+U_4A_4e^{-m_4x_3}=0, \quad (87)$$

$$V_1A_1e^{-m_1x_3}+V_2A_2e^{-m_2x_3}+V_3A_3e^{-m_3x_3}+V_4A_4e^{-m_4x_3}=0, \quad (88)$$

$$d_1A_1e^{-m_1x_3}+d_2A_2e^{-m_2x_3}+d_3A_3e^{-m_3x_3}+d_4A_4e^{-m_4x_3}=\frac{F_3}{s}e^{-sv_0}. \quad (89)$$

Taking $x_3=0$, we get

$$Q_1A_1+Q_2A_2+Q_3A_3+Q_4A_4+Q_5A_5=\frac{F_1}{v}, \quad (90)$$

$$R_6\{R_1A_1+R_2A_2+R_3A_3+R_4A_4+R_5A_5\}=0, \quad (91)$$

$$U_1A_1+U_2A_2+U_3A_3+U_4A_4=0, \quad (92)$$

$$V_1A_1+V_2A_2+V_3A_3+V_4A_4=0, \quad (93)$$

$$d_1A_1+d_2A_2+d_3A_3+d_4A_4=\frac{F_3}{s}e^{-sv_0}. \quad (94)$$

Solving the Eqs. (90)-(94) for A_1, A_2, A_3, A_4, A_5 we get

$$A_i=\frac{\Delta_i}{\Delta}, \text{ for } i=1,2,3,4,5. \quad (95)$$

$$\Delta=\begin{vmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\ R_1 & R_2 & R_3 & R_4 & R_5 \\ U_1 & U_2 & U_3 & U_4 & 0 \\ V_1 & V_2 & V_3 & V_4 & 0 \\ d_1 & d_2 & d_3 & d_4 & 0 \end{vmatrix}, \quad (96)$$

and $\Delta_i (i=1,2,3,4,5)$ are obtained by replacing i_{th} column of Eq. (96) with $\begin{bmatrix}\frac{F_1}{v} & 0 & 0 & 0 & \frac{F_3}{s}e^{-sv_0}\end{bmatrix}^T$ where $[-]^T$ stands for the transpose of the matrix.

Substituting these values of $A_1,\ A_2,\ A_3,\ A_4$ and A_5 from Eq. (95) in Eqs. (90)-(94), we obtain

$$\tilde{t}_{33}=\frac{1}{\Delta}(Q_1\Delta_1e^{-m_1x_3}+Q_2\Delta_2e^{-m_2x_3}+Q_3\Delta_3e^{-m_3x_3}+Q_4\Delta_4e^{-m_4x_3}+Q_5\Delta_5e^{-m_5x_3}), \quad (97)$$

$$\tilde{t}_{31}=\frac{1}{\Delta}R_6(R_1\Delta_1e^{-m_1x_3}+R_2\Delta_2e^{-m_2x_3}+R_3\Delta_3e^{-m_3x_3}+R_4\Delta_4e^{-m_4x_3}+R_5\Delta_5e^{-m_5x_3}), \quad (98)$$

$$\tilde{\sigma}_3=\frac{1}{\Delta}(U_1\Delta_1e^{-m_1x_3}+U_2\Delta_2e^{-m_2x_3}+U_3\Delta_3e^{-m_3x_3}+U_4\Delta_4e^{-m_4x_3}), \quad (99)$$

$$\tilde{\tau}_3=\frac{1}{\Delta}(V_1\Delta_1e^{-m_1x_3}+V_2\Delta_2e^{-m_2x_3}+V_3\Delta_3e^{-m_3x_3}+V_4\Delta_4e^{-m_4x_3}, \quad (100)$$

$$\tilde{T}=\frac{1}{\Delta}(d_1\Delta_1e^{-m_1x_3}+d_2\Delta_2e^{-m_2x_3}+d_3\Delta_3e^{-m_3x_3}+d_4\Delta_4e^{-m_4x_3}. \quad (101)$$

6. Special case

If we neglect the double porous effect, i.e., if we take

$$d = b_1 = \gamma_0 = \alpha_3 = \alpha_2 = \gamma_2 \to 0, \tag{102}$$

In the problem considered. Accordingly, the governing equations of the problem will reduce to the following equations:

Constitutive Relations

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\delta_{ij}\phi - \beta\delta_{ij}T, \tag{103}$$

$$\sigma_i = \alpha\phi_{,i}, \tag{104}$$

Equation of motion

$$\mu\Delta\vec{u} + (\lambda+\mu)\nabla\nabla.\vec{u} + b\nabla\phi - \beta\nabla T = \rho\frac{\partial^2\vec{u}}{\partial t^2}, \tag{105}$$

Equilibrated stress equations of motion

$$\alpha\Delta\phi - b\nabla.\vec{u} - \alpha_1\phi + \gamma_1 T = \chi_1\frac{\partial^2\phi}{\partial t^2}, \tag{106}$$

$$\begin{aligned} k^*\left(1+\frac{{\tau_v}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta T + {k_1}^*\left(1+\frac{{\tau_T}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}\right)\Delta\dot{T} \\ = \left(1+\frac{{\tau_q}^{\alpha^*}}{(\alpha^*)!}\frac{\partial^{\alpha^*}}{\partial t^{\alpha^*}}+\frac{{\tau_q}^{2\alpha^*}}{(2\alpha^*)!}\frac{\partial^{2\alpha^*}}{\partial t^{2\alpha^*}}\right)\left(\beta T_0\nabla.\ddot{\vec{u}} + \gamma_1 T_0\ddot{\phi} + \rho C^*\ddot{T}\right), \end{aligned} \tag{107}$$

The non-dimensional boundary conditions at the bounding surface $x_3 = 0$ will remain

$$\text{(i) } t_{33} = F_1\delta(x_3 - vt), \tag{108}$$

$$\text{(ii) } t_{31} = 0, \tag{109}$$

$$\text{(iii) } \sigma_3 = 0, \tag{110}$$

$$\text{(iv) } T = F_3\mathrm{H}(t - v_0)\,\delta(x_1), \tag{111}$$

Then, the problem will become the corresponding problem of a homogeneous, isotropic, primary porous thermoelastic medium having Three-Phase-Lag using fractional order half space medium. Following the same process of solution as described in the problem discussed above, the components of normal stress and equilibrated stress corresponding to pores and temperature distribution can be obtained in the frequency domain, and the same can be obtained directly from (97)-(101) by using (102).

7. Inversion of the transforms

The field variables of the problem are initially formulated within the transformed domain. To retrieve their representations in the physical domain, it is necessary to apply the inverse operations of the respective transforms. Specifically, the field expressions depend on the spatial variable x_3

and are expressed in terms of the Laplace and Fourier transform parameters, denoted by s and ξ, respectively. Consequently, the transformed field functions are represented as $\tilde{f}\ (\xi,\ x_3,\ s)$.

To recover the physical domain function $f\ (x_1,\ x_3,\ t)$, we must first perform the inverse Fourier transform, which is given by

$$\bar{f}\ (x_1, x_3, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\xi x_1}\, \tilde{f}(\xi, x_3, s)\ d\xi = \frac{1}{\pi}\int_{0}^{\infty}\{cos(\xi x) f_e - i\, sin(\xi x) f_0\}\ d\xi. \qquad (112)$$

Here, f_e and f_o corresponds to the even and odd components of the function $\tilde{f}\ (\xi,\ x_3,\ s)$, respectively. Therefore, Eq. (112) represents the Laplace transform of the physical domain function $f\ (x_1,\ x_3,\ t)$, and may be compactly expressed as $\bar{f}\ (x_1,\ x_3,\ s)$.

Assuming the values of x_1 and x_3 are held constant, the term $\bar{f}\ (x_1,\ x_3,\ s)$ in Eq. (112) can be interpreted as the Laplace transform $\bar{g}(s)$ of a time-dependent function $g(t)$. In such a case, the inversion can be executed following the methodology outlined by Honig and Hirdes (1984).

The final step requires evaluating the integral given in Eq. (112). This can be accomplished via Romberg's method of integration, employing an adaptive step size strategy. This approach includes iterative refinement of the extended trapezoidal rule and extrapolates the results to the limit as the step size approaches zero. The complete inversion process is supported by the numerical procedure described in Press *et al.* (1986), which is tailored for such integrals.

8. Numerical results and discussion

To discuss numerically the problem considered and solved above theoretically in the transformed domain, we take a specific model of a double porous thermoelastic medium having Three-Phase-Lag and fractional order derivatives to get the solution of the problem in the physical domain by using the numerical inversion technique described above. The numerical values of the physical parameters for the model considered are taken as under.

Sherief and Saleh (2005) have given the numerical values of physical parameters for this material as

$$\lambda = 7.76\times 10^{10} Nm^{-2},\ \mu = 3.86\times 10^{10} Nm^{-2},$$
$$\rho = 8.954\times 10^{3} Kgm^{-3},\ \alpha_t = 1.78\times 10^{-5} K^{-1},$$
$$T_0 = 0.293\times 10^{3} K,\ C^* = 3.831\times 10^{3} m^2 s^{-2} K^{-1},$$
$$K_1{}^* = 3.86\times 10^{3} Ns^{-1}K^{-1}.$$

The double porous parameters of the material are taken, following Khalili (2003), as

$$b = 0.9\times 10^{10} Nm^{-2},\ d = 0.1\times 10^{10} Nm^{-2},$$
$$\alpha = 1.3\times 10^{-5} N,\ b_1 = 0.12\times 10^{-5} N,$$
$$\gamma_0 = 1.1\times 10^{-5} N,\ \alpha_1 = 2.3\times 10^{10} Nm^{-2},$$
$$\alpha_2 = 2.4\times 10^{10} Nm^{-2},\ \alpha_3 = 2.5\times 10^{10} Nm^{-2},$$
$$\gamma_1 = 0.16\times 10^{5} Nm^{-2},\ \gamma_2 = 0.219\times 10^{5} Nm^{-2},$$
$$\chi_1 = 0.1456\times 10^{-12} Nm^{-2}s^2,\ \chi_2 = 0.1546\times 10^{-12} Nm^{-2}s^2.$$

Three-Phase-Lag parameters τ_q, τ_T, τ_ν fractional order parameter α^* are taken from Ezzat *et al.* (2012), Choudhuri (2007),

$$\tau_T = 1.0\times 10^{-7} s,\ \tau_q = 2.0\times 10^{-7} s\ ,$$
$$\tau_\nu = 3.0\times 10^{-7} s,\ v = 0.1\ ,\ v_0 = 0.0,$$
$$F_1 = 1.0,\ F_3 = 1.0,\ t = 0.1.$$

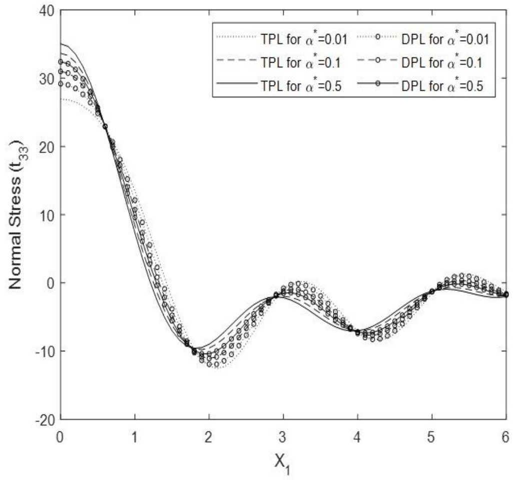

Fig. 1 Variation of normal stress t_{33} with respect to distance x_1

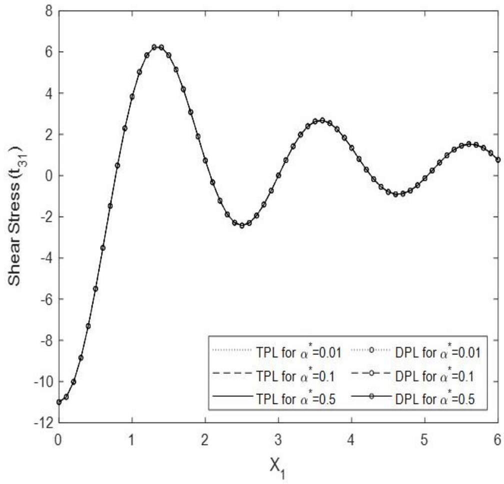

Fig. 2 Variation of shear stress t_{31} with respect to distance x_1

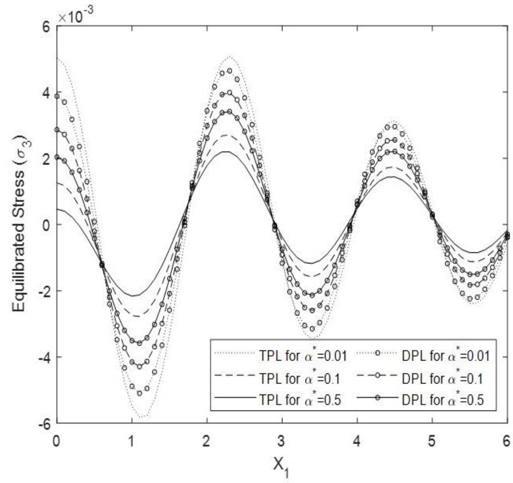

Fig. 3 Variation of equilibrated stress σ_3 with respect to distance x_1

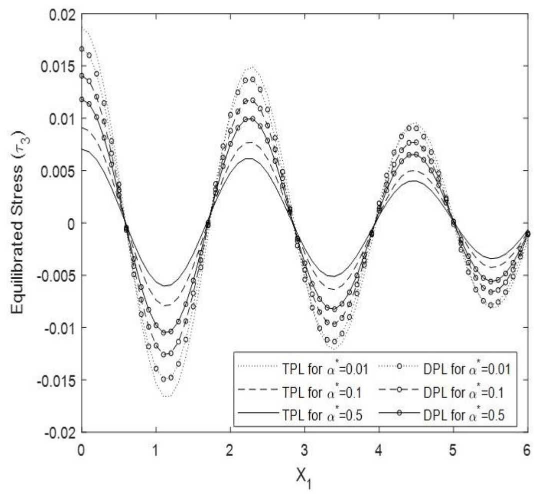

Fig. 4 Variation of equilibrated stress τ_3 with respect to distance x_1

The variations in stresses and temperature distributions in the half space medium against the distance 'x_1' are calculated numerically using a code in MATLAB for the numerical inversion technique stated above and taking the data given above for the numerical constants.

8.1 Case 1

For the fixed value of the depth parameter x_3=0.05. The graphs for the field variables are shown in Figs. 1-5 for the fractional order parameters $\alpha^* = 0.01, 0\ .1, 0.5$. We have taken dotted curve (…….), dashed curve (- - - -), and solid curve (—) for representing the impact of Three-Phase-

Lag (TPL) and dotted curve (…….) with ‘o’ symbol, dashed curve (- - - -) with ‘o’ symbol, and solid curve (—) with ‘o’ symbol for showing the impact of Dual-Phase-Lag (DPL) corresponding to $\alpha^* = 0.01, 0\ .1, 0.5$ respectively.

Fig. 1 depicts the variation of normal stress t_{33} with distance x_1 for three different values of fractional order parameter, i.e., $\alpha^* = 0.01,0\ .1,0.5$ for both Three-Phase-Lag (TPL) and Dual-Phase-Lag (DPL) cases. The values obtained in TPL are small as compared to DPL, which also shows the impact of TPL over DPL. It is noticed that for all the values of α^* the magnitude values of normal stress are large initially for both the cases of TPL and DPL, which goes on decreasing with the increase in the value of x_1 and ultimately tends to zero value following the oscillatory pattern with further increase in the value of ‘x_1’. Also, the curves show that the magnitude values of the normal stress change with the change in the value of the fractional order parameter and the trend of curves are almost similar for three different values of fractional order parameter for both the cases, which shows that the impact of fractional order parameter on the normal stress component for TPL and DPL cases is mainly quantitative.

Fig. 2 shows the variation of shear stress t_{31} with distance ‘x_1’ for three different values of fractional order parameters, i.e., $\alpha^* = 0.01,0\ .1,0.5$ for both TPL and DPL cases. It is observed that for all the values of α^*, the magnitude values of shear stress are large initially, i.e., near $x_1=0$ for both the cases of TPL and DPL, which goes on decreasing with the increase in the value of ‘x_1’ and ultimately approaches the zero-value following oscillatory pattern. Further, it is observed that with the change in the value of fractional order parameter, there is only negligible change in the values of shear stress for TPL and DPL cases for all the values of fractional order parameters. Thus, fractional order parameter has no significant effect on shear stress component, and TPL has no significant impact on DPL.

Fig. 3 represents the variation of equilibrated stress (σ_3) corresponding to pores with distance x_1 for both TPL and DPL cases for three different values of fractional order parameter, i.e., $\alpha^* = 0.01,\ 0.1, 0.5$. It is noticed that for all the values of α^* the magnitude values of equilibrated stress (σ_3) are large initially for both the cases of TPL and DPL, which goes on decreasing with the increase in the value of x_1 and ultimately tends to zero value following the oscillatory pattern. Also, the curves show that the magnitude values of the component change with the change in the value of the fractional order parameter and the trend of curves are almost similar for different values of fractional order parameter for all the cases, which shows that the impact of fractional order parameter on the equilibrated stress (σ_3) component for TPL and DPL cases is in terms of magnitude. Further, the peak values are decreasing with the increase in the value of fractional order parameter for both the cases of TPL and DPL. Thus, fractional order parameter and three-phase- lag have a significant impact on equilibrated stress (σ_3) component.

Fig. 4 shows the variation of equilibrated stress (τ_3) corresponds to fissures with distance x_1 for TPL and DPL. The maximum modulus values of equilibrated stress (τ_3) exist near the value $x_1=0$ and approaches to zero value for all the values of fractional order parameter, i.e., $\alpha^* = 0.01,\ 0.1$, 0.5 following oscillatory pattern. Further, peak values of curves decreasing with the increase in the value of the fractional order parameter. Also, the values obtained for TPL cases corresponding to fractional order parameters are larger than the values obtained for DPL cases. That means there is a significant impact of Fractional order parameters on equilibrated stress (τ_3) and TPL has a noticeable impact on DPL.

Further, Fig. 5 showing the variation of absolute temperature distribution T with distance ‘x_1’ for three different values of fractional order parameters, i.e., $\alpha^* = 0.01,\ 0.1, 0.5$, for TPL and

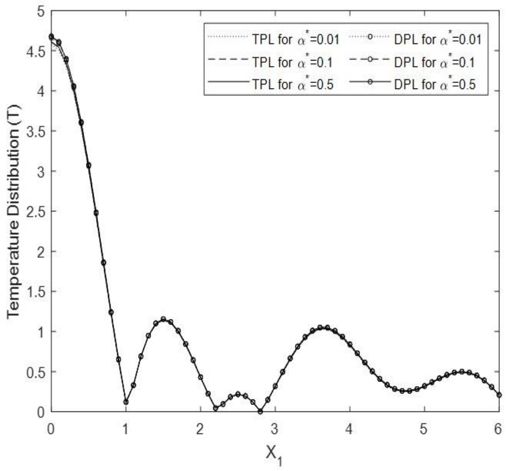

Fig. 5 Variation of temperature distribution T with respect to distance x_1

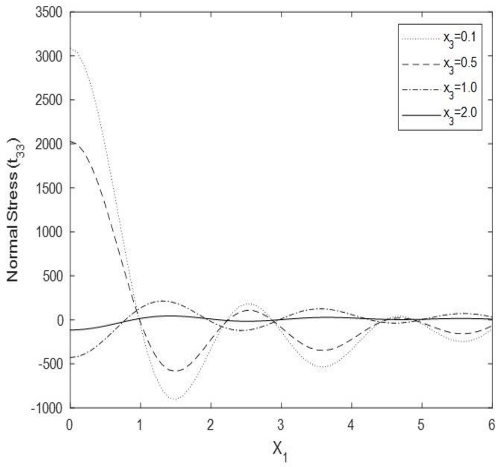

Fig. 6 Variation of normal stress t_{33} with respect to distance x_1

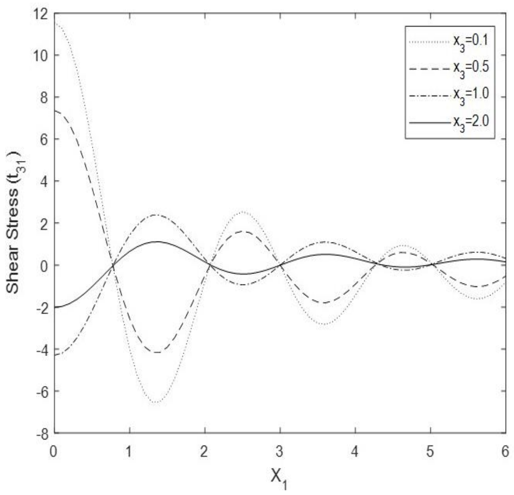

Fig. 7 Variation of shear stress t_{31} with respect to distance x_1

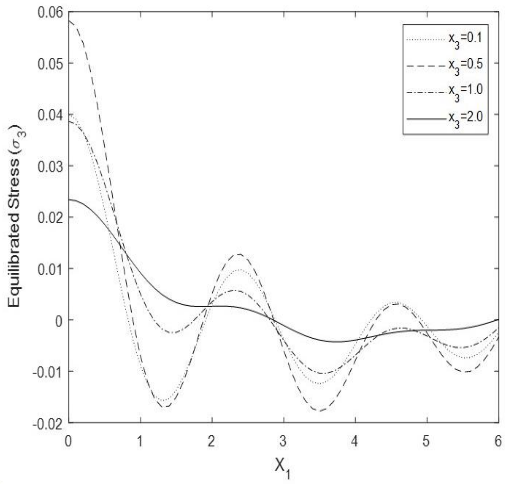

Fig. 8 Variation of equilibrated stress σ_3 with respect to distance x_1

DPL cases. It is noticed that the values are large initially for all the cases of TPL and DPL and tend to zero value with the increase in the values of x_1.

8.2 Case 2

The graphs for the field variables are shown in Figs. 6-10 for the depth parameter x_3=0.1, 0.5, 1.0 and 2.0 with dotted curve (……), small-dashed curve (- - - -), dash-dot curve (-.-.-.-.-.) and solid curve (—) respectively. Here we have fixed fractional order parameter as $\alpha^* = 0.01$.

Fig. 6 depicts the deviation of normal stress t_{33} with space coordinate x_1. It is detected that the numerical values obtained are large for the first two cases and small for the other cases of the

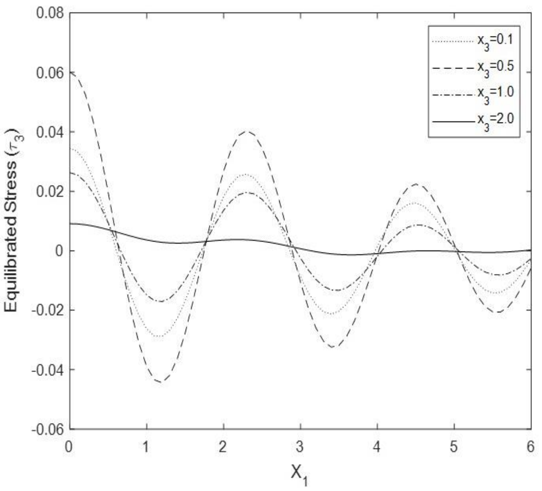

Fig. 9 Variation of equilibrated stress τ_3 with respect to distance x_1

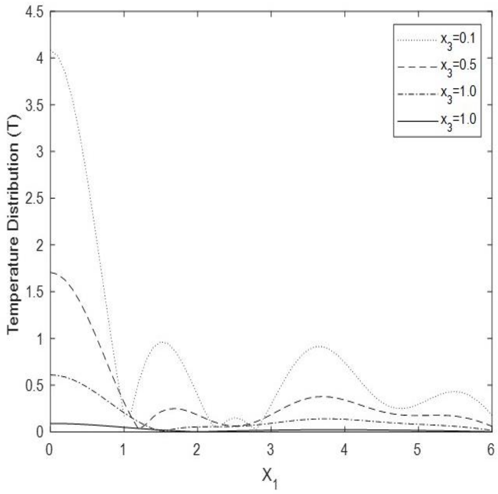

Fig. 10 Variation of temperature distribution T with respect to distance x_1

depth parameter. Further, to show clearly the variation of normal stress for all the cases taken in a single figure, the values for the normal stress for x_3=1.0 and 2.0 are taken after multiplied by 10 as the values obtained for x_3=1.0 and 2.0 cases are small as compared to the corresponding values obtained for x_3=0.1 and 0.5. The variation of curves for all the four cases follows oscillatory pattern and tends to zero value, if the values of x_1 increases. Moreover, the scale value is largest for x_3=0.1. Further, if the value of the depth parameter increases, the scale values are decreasing and becomes zero for large values of depth parameter, which is clear from the values obtained corresponding to $x_3 = 2.0$.

Fig. 7 displays the deviation of shear stress t_{31} verses x_1. It is perceived that the deviations follow almost the similar trends as were seen in Fig. 6 for the case of normal stress however the numerical values obtained are small as compared to the values obtained for the case of normal stress. Further, to show clearly the variation of normal stress for all the cases taken in a single figure, the values for the normal stress for x_3=1.0 and 2.0 are taken after multiplied by 10 as the values obtained for x_3=1.0 and 2.0 cases are small as compared to the corresponding values obtained for x_3=0.1 and 0.5. The variation of curves for all the four cases follows oscillatory pattern and tends to zero value, if the values of x_1 increases.

Figs. 8 and 9 demonstrate the variations of equilibrated stresses σ_3 and τ_3. Both these cases show the oscillatory pattern of curves and ultimately tend to zero value for large values of x_1. It is seen that the scale values are growing through the escalation in the value of x_3 for lower values of x_3 and then decreases for $x_3 = 2.0$ near the origin and follows the same pattern. Also, the number of wave patterns is large for smaller values of x_3. Almost similar trends are seen for the two cases of equilibrated stresses σ_3 and τ_3.

Fig. 10 exhibits the variation of temperature distribution T with respect to x_1. It is seen that the scale values are decreasing with the increase in the values of depth x_3 The magnitude of temperature is highest near the origin and diminishes gradually with increase in x_1.

9. Conclusions

The present study investigates the thermoelastic deformation behaviour of a double-porous thermoelastic half-space medium characterized by fractional-order heat conduction and governed by the Three-Phase-Lag (TPL) model, under the influence of a coupled thermo-mechanical loading. To address this complex problem, a transformation methodology is employed to derive the solution in the transformed domain. Subsequently, numerical inversion techniques are utilized to retrieve the physical domain responses, which are then illustrated graphically across various parametric scenarios. The analysis emphasizes the influence of both the fractional-order parameter and the depth parameter on key deformation-related field variables. Additionally, the study explores how the incorporation of the TPL model alters the response in comparison to the conventional Dual-Phase-Lag (DPL) framework.

The physical configuration under consideration comprises an infinite, homogeneous, isotropic, double-porous thermoelastic medium subjected to a fractional-order TPL-based heat conduction law. The external excitation consists of a normal line load traversing the boundary and a point heat step source located at the origin. The problem is modelled under plane strain assumptions within the x_1x_3-plane. The governing equations are first reduced to their Laplace- and Fourier-transformed counterparts, facilitating the derivation of closed-form solutions in the transform domain. These are then numerically inverted via computational algorithms to recover the time- and space-dependent behaviour of temperature and stress fields within the medium.

The results clearly demonstrate that the fractional-order parameter and depth parameter exert a pronounced effect on the thermoelastic response, particularly in terms of stress distribution and thermal gradients. Moreover, the TPL model is shown to significantly modify the system's behaviour relative to the DPL model, underscoring the added complexity introduced by the third phase lag. To assess the limiting behaviour of the system, a degenerate case is examined wherein the double-porosity parameters are nullified, effectively simplifying the model to a single-porosity configuration. This investigation offers a robust theoretical framework that may serve as a foundation for future research in generalized thermoelasticity, porous media mechanics, and fractional-order modelling.

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Appendix

Coefficients of Eq. (58)

$$
\begin{gathered}
E_1 = \delta_{24}(\delta_4\delta_{11} - \delta_5\delta_{10}),\\
E_2 = q_9(\delta_4\delta_{11} - \delta_5\delta_{10}) + \delta_{24}(q_7\delta_4 - q_6\delta_5 - \delta_1\delta_5\delta_{12} + \delta_2\delta_4\delta_{12} - \delta_2\delta_6\delta_{10} + q_1\delta_4\delta_{11} + \\
\delta_1\delta_6\delta_{11} - q_1\delta_5\delta_{10} + q_3\delta_{11} - q_4\delta_{10}) + \delta_3\delta_{21}(\delta_4\delta_{11} - \delta_5\delta_{10}),\\
E_3 = q_7q_9\delta_4 - \delta_4\delta_{15}\delta_{23} + q_3q_9\delta_{11} + q_3q_7\delta_{24} - q_6q_9\delta_5 + \delta_5\delta_{15}\delta_{22} - q_4q_9\delta_{10} - \\
q_4q_6\delta_{24} + \delta_9\delta_{10}\delta_{23} - \delta_9\delta_{11}\delta_{22} + q_1q_9\delta_4\delta_{11} + q_1q_7\delta_4\delta_{24} + q_1q_3\delta_{11}\delta_{24} - q_1q_9\delta_5\delta_{10} - \\
q_1q_6\delta_5\delta_{24} - q_1q_4\delta_{10}\delta_{24} + q_9\delta_1\delta_6\delta_{11} + q_7\delta_1\delta_6\delta_{24} - q_2\delta_1\delta_{11}\delta_{24} - q_9\delta_1\delta_5\delta_{12} + \\
q_5\delta_1\delta_5\delta_{24} - \delta_1\delta_5\delta_{15}\delta_{21} - q_4\delta_1\delta_{12}\delta_{24} + \delta_1\delta_9\delta_{11}\delta_{21} - q_9\delta_2\delta_6\delta_{10} - q_6\delta_2\delta_6\delta_{24} + \\
q_2\delta_2\delta_{10}\delta_{24} + q_9\delta_2\delta_4\delta_{12} - q_5\delta_2\delta_4\delta_{24} + \delta_2\delta_4\delta_{15}\delta_{21} + q_3\delta_2\delta_{12}\delta_{24} - \delta_2\delta_9\delta_{10}\delta_{21} - \\
\delta_3\delta_6\delta_{10}\delta_{23} + \delta_3\delta_6\delta_{11}\delta_{22} + \delta_3\delta_4\delta_{12}\delta_{23} + q_7\delta_3\delta_4\delta_{21} + q_8\delta_3\delta_4\delta_{11} + q_3\delta_3\delta_{11}\delta_{21} - \\
\delta_3\delta_5\delta_{12}\delta_{22} - q_6\delta_3\delta_5\delta_{21} - q_8\delta_3\delta_5\delta_{10} - q_4\delta_3\delta_{10}\delta_{21},\\
E_4 = q_3q_7q_9 - q_3\delta_{15}\delta_{23} - q_4q_6q_9 + q_4\delta_{15}\delta_{22} + q_6\delta_9\delta_{23} - q_7\delta_9\delta_{22} + q_1q_7q_9\delta_4 - \\
q_1\delta_4\delta_{15}\delta_{23} + q_1q_3q_9\delta_{11} + q_1q_3q_7\delta_{24} - q_1q_6q_9\delta_5 + q_1\delta_5\delta_{15}\delta_{22} - q_1q_4q_9\delta_{10} - \\
q_1q_4q_6\delta_{24} + q_1\delta_{10}\delta_9\delta_{23} - q_1\delta_{11}\delta_9\delta_{22} + \delta_1\delta_6q_9q_7 - \delta_1\delta_6\delta_{15}\delta_{23} - q_9\delta_{11}q_2\delta_1 - \\
q_2q_7\delta_1\delta_{24} + q_5q_9\delta_1\delta_5 + \delta_1\delta_5\delta_{15}q_8 - q_4q_9\delta_1\delta_{12} + q_4q_5\delta_1\delta_{24} - \delta_1q_4\delta_{15}\delta_{21} + \\
\delta_1\delta_9\delta_{12}\delta_{23} + q_7\delta_1\delta_9\delta_{21} + \delta_9\delta_{11}q_8\delta_1 - q_6q_9\delta_2\delta_6 + \delta_2\delta_6\delta_{15}\delta_{22} + q_2q_9\delta_2\delta_{10} + \\
q_2q_6\delta_2\delta_{24} - q_5q_9\delta_2\delta_4 + q_8\delta_2\delta_4\delta_{15} + q_3q_9\delta_2\delta_{12} - q_3q_5\delta_2\delta_{24} + q_3\delta_2\delta_{21}\delta_{15} - \\
\delta_2\delta_9\delta_{12}\delta_{22} - q_8\delta_2\delta_9\delta_{10} - q_6\delta_2\delta_9\delta_{21} - q_6\delta_3\delta_6\delta_{23} + q_7\delta_3\delta_6\delta_{22} + q_2\delta_3\delta_{10}\delta_{23} - \\
q_2\delta_3\delta_{11}\delta_{22} - q_5\delta_3\delta_4\delta_{23} + q_7q_8\delta_3\delta_4 + q_3\delta_3\delta_{12}\delta_{23} + q_3q_7\delta_3\delta_{21} + q_3q_8\delta_3\delta_{11} + \\
q_5\delta_3\delta_5\delta_{22} - q_6q_8\delta_3\delta_5 - q_4\delta_3\delta_{12}\delta_{22} - q_4q_6\delta_3\delta_{21} - q_4q_8\delta_3\delta_{10},\\
E_5 = q_1q_3q_7q_9 - q_1q_3\delta_{15}\delta_{23} - q_1q_4q_6q_9 + q_1q_4\delta_{15}\delta_{22} + q_1q_6\delta_9\delta_{23} - q_1q_7\delta_9\delta_{22} - \\
q_2q_7q_9\delta_1 + q_2\delta_1\delta_{15}\delta_{23} + q_4q_5q_9\delta_1 - q_4q_8\delta_1\delta_{15} - q_5\delta_1\delta_9\delta_{23} + q_7q_8\delta_1\delta_9 + q_2q_6q_9\delta_2 - \\
q_2\delta_2\delta_{15}\delta_{22} - q_3q_5q_9\delta_2 + q_3q_8\delta_2\delta_{15} + q_5\delta_2\delta_9\delta_{22} - q_6q_8\delta_2\delta_9 + q_2q_6\delta_3\delta_{23} - \\
q_2q_7\delta_3\delta_{22} - q_3q_5\delta_3\delta_{23} + q_3q_7q_8\delta_3 + q_4q_5\delta_3\delta_{22} - q_4q_6q_8\delta_3.
\end{gathered}
\tag{A.1}
$$

and

$$
\begin{gathered}
\delta_{21} = -s^2\delta_{18}\left(1 + \frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right),\\
\delta_{22} = -s^2\delta_{19}\left(1 + \frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right),\\
\delta_{23} = -s^2\delta_{20}\left(1 + \frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right),\\
\delta_{24} = \delta_{16}\left(1 + \frac{\tau_v{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}\right) + s\delta_{17}\left(1 + \frac{\tau_T{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*}\right),\\
q_1 = -(\xi^2 + s^2),\quad q_2 = \delta_6\xi^2,\\
q_3 = -(\delta_4\xi^2 + s^2 + \delta_7),\ q_4 = -(\delta_5\xi^2 + \delta_8,\\
q_5 = \delta_{12}\xi^2,\quad q_6 = -(\delta_{10}\xi^2 + \delta_{13}),\\
q_7 = -(\delta_{11}\xi^2 + s^2 + \delta_{14}),\ q_8 = -\delta_{21}\xi^2,\\
q_9 = -(\delta_{24}\xi^2 + \left(1 + \frac{\tau_q{}^{\alpha^*}}{(\alpha^*)!}s^{\alpha^*} + \frac{\tau_q{}^{2\alpha^*}}{(2\alpha^*)!}s^{2\alpha^*}\right)s^2).
\end{gathered}
\tag{A.2}
$$