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Longitudinal vibration of a nanorod embedded in viscoelastic medium considering nonlocal strain gradient theory

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Abstract. This article investigates the longitudinal vibration of a nanorod embedded in viscoelastic medium according to the nonlocal strain gradient theory. Viscoelastic medium is considered based on Kelvin-Voigt model. Governing partial differential equation is derived based on longitudinal equilibrium and analytical solution is obtained by adopting harmonic motion solution for the nanorod. Modal frequencies and corresponding damping ratios are presented to demonstrate the influences of nonlocal parameter, material length scale, elastic and damping parameters of the viscoelastic medium. It is observed that material length scale parameter is very influential on modal frequencies especially at lower values of nonlocal parameter whereas increase in length scale parameter has less effect at higher values of nonlocal parameter when the medium is purely elastic. Elastic stiffness and damping coefficient of the medium have considerable impacts on modal frequencies and damping ratios, and the highest impact of these parameters on frequency and damping ratio is seen in the first mode. Results calculated based on strain gradient theory are quite different from those calculated based on classical elasticity theory. Hence, nonlocal strain gradient theory including length scale parameter can be used to get more accurate estimations of frequency response of nanorods embedded in viscoelastic medium.
Keywords: longitudinal vibration; nanorod; nonlocal elasticity; strain gradient theory; viscoelastic medium

1. Introduction

Nanostructured materials have recently gained a considerable attraction due to their great potential to the possible applications of electronics, photonics, medicine and environment. Several types of nanostructures i.e., nanorods, nanotubes, nanobeams nanoplates and nanoshells have been developed to meet the demands of advanced technological devices. Micro- and Nano- electro mechanical systems (MEMS and NEMS) are some of the examples for these devices in which nanostructures can find several applications. Nanostructures are materials with different structures in the range of nanometers (typically 1-100 nm) (Bilal Tahir *et al.* 2019). Nanostructure materials exhibit novel and astonishing properties that are entirely different from their bulk material counterparts. The properties of nanostructures always depend on the size, shape and morphology of the nanostructures and hence they can be tuned (Rafique *et al.* 2020). Therefore, investigation of mechanical and vibration characteristics of nanostructures have a great importance in design and development of advanced technological devices.

The mechanical behavior of nanostructures can be examined by means of different analysis methods which are experimental method, atomistic method, i.e., the Molecular Dynamics (MD) simulation and continuum mechanics approach. Experimental studies at nanoscale level requires advanced equipment and dedicated instrumentation, which makes experimental studies expensive. MD simulation was originally developed in the early 1950s, soon afterward, this method acquired popularity in materials science. It was used to investigate the physical properties of the nano-technological devices. Hence, MD simulation can be regarded as one of the most precise and effective methods to analyze molecular scale phenomenon. This simulation tool is the leading tool for interpreting the experimental studies at atomic scale level (Allen 2004). However, MD simulation takes long times since it uses data from simulations in nanoseconds (10^{-9}s) to microseconds (10^{-6}s) interval (Cruz *et al.* 2014, Cruz *et al.* 2016), thus it is restricted to computation power. Hence, the use of MD simulation is not useful for the analysis of nanostructures especially including large number of atoms and molecules. Instead, continuum mechanics approach may be a sensible alternative in the analysis of nanostructures. However, experimental studies on nanostructures showed that size effect has a significant impact on the mechanical behavior of nanostructures (Namazu *et al.* 2000, Guz *et al.* 2017, Hsu *et al.* 2019, Tung *et al.* 2021). Therefore, classical continuum theory which is independent of the size effect is not suitable option to estimate mechanical behavior of nanostructures. Roudbari *et al.* (2022) reviewed the most frequently used size dependent continuum mechanics models for micro- and nano-structures. Several size dependent continuum theories were developed such as nonlocal elasticity theory (Eringen 1972), modified couple stress theory (Yang *et al.* 2002), micropolar theory (Eringen 1967), micromorphic theory (Wang and Lee 2010), strain gradient theory (Aifantis 1999). First size dependent continuum model is the nonlocal elasticity theory of Eringen (1972) which expressed stress component at any

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known reference point is not only related to the strains at the same reference point but also related to the strains at every location in the medium. Many studies were carried out to static (Peddieson *et al*. 2003, Reddy 2007), buckling (Adali 2008, Murmu and Pradhan 2009, Murmu and Adhikari 2011, Pradhan and Reddy 2011, Ansari *et al*. 2011, Wu and Li 2017), vibration and wave propagation (Aydogdu 2009, Murmu and Pradhan 2009, Simsek 2010, 2011a, b, 2012, Eltaher *et al.* 2012, Thai 2012, Ebrahimi *et al.* 2019a) behaviors of nanostructures using Eringen's nonlocal elasticity theory.

Although Eringen's nonlocal elasticity theory is frequently used in many studies, it considers only softening effect without accounting for stiffening enhancement which can be exhibited in the experimental studies of micro- and nano- structures. Modified couple stress theory (Yang *et al.* 2002) and the strain gradient theory (Aifantis 1999) takes into account the stiffness enhancement of nanostructures (Tsepoura 2002, Park *et al.* 2006, Ma *et al.* 2008, Simsek 2010, 2014, Reddy 2011, Simsek and Reddy 2013a, b). Since nonlocal elasticity theory and strain gradient theory deals with different characteristics of size dependency in the mechanical behavior of nanostructures, the need for the combination of these two theories emerges to get more accurate estimations. In this scope, Challamel (2013) developed a new hybrid theory called as nonlocal strain gradient theory to examine bending, buckling and vibration analysis of beams based on different beam theories. Lim *et al.* (2015) developed theoretical formulations of nonlocal strain gradient theory based on thermodynamics framework and authors applied this theory to the wave propagation in Euler-Bernoulli and Timoshenko beams. Li and Hu (2015) used nonlocal strain gradient theory for post buckling analysis of nonlinear Euler-Bernoulli beam. Recently, strain gradient theory has been used in many studies in the literature. Li *et al.* (2016a) examined longitudinal vibration of size-dependent rods using nonlocal strain gradient theory. Nonlinear bending of beams made of functionally graded materials (FGMs) were examined using strain gradient theory (Li and Hu 2016a). Li and Hu (2016b) investigated wave propagation in fluid conveying viscoelastic carbon nanotubes (CNTs) based on nonlocal strain gradient theory. Simsek (2016a) studied nonlinear free vibration of FGM nanobeam using nonlocal strain gradient theory and a novel Hamiltonian approach. Bending, buckling and vibration behaviors of different kinds of micro/nanobeams, nanorods and nanotubes were examined in the open literature (Li *et al.* 2016a, Lu *et al.* 2017, Tang *et al.* 2017, She *et al.* 2018, Wang *et al.* 2018, Sahmani *et al.* 2018, Malikan *et al.* 2018, Tang *et al.* 2019, Mohammadian *et al.* 2019, Ebrahimi *et al.* 2019b, Ebrahimi *et al.* 2019c, Fu *et al.* 2020, Esen 2020, Mirjavadi *et al.* 2020, Yan *et al.* 2021, El-Furjan *et al.* 2021, Alizadeh-Hamidi *et al.* 2021, Thang *et al.* 2021a). Thang *et al.* (2021c) examined bending and free vibration response of FGM carbon nanotube-reinforced composite nanoshells with double curvature based on nonlocal strain gradient theory. Later, Thang *et al.* (2021b) applied nonlocal strain gradient theory to obtain vibration response of functionally graded carbon nanotube-reinforced composite nanoplates considering the influence of nonlocal parameter and strain gradient coefficient.

Aydogdu (2012) investigated the influence of elastic medium stiffness on axial vibration of single walled carbon nanotubes (SWCNTs). Gul *et al.* (2017) examined axial dynamics of a nanorod embedded in elastic medium based on doublet mechanics. Bensaid *et al.* (2018) examined the free vibration characteristics of nanoscale-beams resting on elastic Pasternak's foundation considering nonlocal strain gradient theory and a higher order hyperbolic beam model which takes into account the shear deformation effect. Imboden and Mohanty (2014) thoroughly discussed dissipation mechanisms in nanoelectromechanical systems (NEMS), which results in damping. Kazemi-Lari *et al.* (2013) reported that embedded CNTs in polymer or metal media have been proposed for many applications. Significant toughening of polymer matrices through the incorporation of CNTs has been reported (Dalton *et al.* 2003, Ruan *et al.* 2003). Embedded CNTs may effectively prolong the formation of and/or bridge micro-cracking/ crazing that can propagate and lead to fatigue failure and CNT reinforced polymer composites are seen as a potentially fruitful area for new, tougher or fatigue-resistant materials (Andrews and Weisenberger 2003). Soltani *et al.* (2010) developed the transverse vibration model in biological soft tissue and authors used the nonlocal Euler-Bernoulli beam theory to examine fluid-induced vibration of the SWCNT while viscoelastic behavior of the surrounding tissue was simulated by Kelvin-Voigt model (Ghavanloo *et al.* 2011). Akbas (2020) studied the forced vibration analysis of viscoelastic nanorods under the effect of axially harmonic load. Karlicic *et al.* (2015) presented a theoretical study for a free vibration of a nonlocal viscoelastic double-nanorod system (VDNRS) including viscoelastic layer. Later, Karlicic *et al.* (2017) carried out a study on dynamic stability of a SWCNT embedded in a polymer matrix under the influence of the axially harmonic load, and authors modelled polymer matrix as viscoelastic medium. The transverse vibration of nonlocal viscoelastic orthotropic multi-nanoplate system (MNPS) embedded in a viscoelastic medium was studied (Karlicic *et al.* 2014). Dynamic behavior of a multi-layered viscoelastic nanobeam resting on a viscoelastic medium with a moving nanoparticle was studied (Hashemi and Khaniki 2017). Arda and Aydogdu (2015) investigated free torsional vibration in carbon CNTs embedded in a viscoelastic medium. Mohammadimehr *et al.* (2015) examined vibration of viscoelastic tapered micro-rod resting on visco-pasternak foundation based on strain gradient theory. Arda and Aydogdu (2019) examined torsional dynamics of coaxial nanotubes with different lengths in viscoelastic medium. Safeer *et al.* (2019) carried out a study to examine the influence of viscoelastic medium on wave propagation along protein microtubules. Khosravi *et al.* (2020) studied the forced and free dynamic vibrations of a SWCNT embedded in a viscoelastic medium under a harmonic external torque.

The literature provided in the foregoing paragraphs indicated that number of studies have been conducted regarding static and dynamic mechanical behavior of nanostructures based on nonlocal elasticity and strain

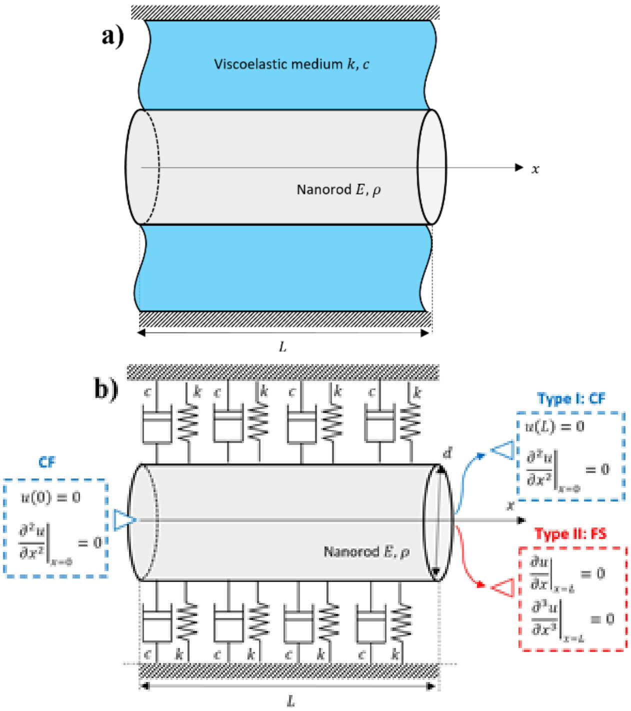

Fig. 1 (a) Nanorod embedded in a viscoelastic medium, (b) Nanorod with CF/CF and CF/FS boundary conditions and the Kelvin-Voigt model for the viscoelastic medium

gradient theories. Studies related to the influence of viscoelastic medium on dynamic response of nanostructures have been limited in the literature. However, to the best of author's knowledge, longitudinal vibration problem of a nanorod embedded in a viscoelastic medium based on nonlocal strain gradient theory has not been researched yet. The main purpose of this study is to fill this gap in the literature. Viscoleastic medium around the nanorod is constructed according to the Kelvin-Voigt model. Analytical solution is obtained for nanorods having different type boundary conditions and obtained results show the influences of stiffness and viscous damping properties of the medium on longitudinal vibration characteristics of the nanorod considering length scale parameter. Parametric analyses are carried out to examine the influence of key parameters on vibration response of a nanorod/viscoelastic medium system. Following sections are organized as problem description, formulation and solution methodology, numerical results and conclusion parts, respectively.

2. GOP-based composites

The nanorods embedded in mediums have a wide range of application in industry especially in fluid storage, nanosensors and nanocomposites. For instance, mass nanosensors can be utilized in calorimetric gas detection, drug screening and metabolic measurements (Khosravi and Hosseini 2020). A schematic view of a nanorod with viscoelastic medium is depicted in Fig. 1(a). The structure of viscoelastic medium is shown in Fig. 1(b), and it is modelled as a continuously distributed spring-damper system of Kelvin-Voigt type with neglected mass (Karlicic *et al.* 2014, 2015). It is assumed that the top and bottom surfaces of the nanorod are connected to the fixed base by a viscoelastic medium with a stiffness coefficient k and damping coefficient c. Longitudinal vibration problem is examined considering clamped and free boundaries. Li *et al.* (2016a) reported that the common type expressions used in the classical theory (such as clamped and free ends) are no longer meaningful for the size dependent rod based on the nonlocal strain gradient theory. Therefore, it is required to introduce new boundary conditions called as Clamped Forcing (CF) and Free Strained (FS) types. These boundary types satisfy the requirements emerging due to the nonlocal strain gradient theory for size dependent rods and they are depicted in Fig. 1(b). Type I shows the CF end and Type II indicates the FS end at $x = L$. Section 3 is designated to express the necessary formulation steps.

3. Formulation and solution methodology

The strain energy U in a linear elastic isotropic material within the volume-V of structure based on nonlocal strain gradient theory can be written as (Li *et al.* 2016a):

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + \sigma_{ijm}{}^{(1)}\varepsilon_{ij,m}\right) dV, \tag{1}$$

where σ_{ij} and $\sigma_{ijm}{}^{(1)}$ respectively show the nonlocal stress tensor and the higher order nonlocal stress tensor which reads:

$$\sigma_{ij} = C_{ijkl}\int_V \alpha_0\left(|\text{x-x}'|, e_0 a\right)\varepsilon_{kl}'dV, \tag{2}$$

$$\sigma_{ijm}{}^{(1)} = l_m{}^2 C_{ijkl}\int_V \alpha_1\left(|\text{x-x}'|, e_1 a\right)\varepsilon_{kl,m}'dV, \tag{3}$$

Based on the nonlocal strain gradient theory, total stress tensor is a function of nonlocal stress tensor and strain gradient tensor as (Li *et al.* 2016a):

$$t_{ij} = \sigma_{ij} - \nabla\sigma_{ijm}{}^{(1)}, \tag{4}$$

where ∇ is the Laplacian operator. In Eqs. (2)-(3), C_{ijkl} is the elastic moduli tensor of the material. The material length scale parameter is introduced to consider the significance of strain gradient stress field. For the wave propagation of CNTs using Euler-Bernoulli beam model, the strain energy density function based on the nonlocal strain gradient theory is expressed as (Lim *et al.* 2015):

$$\begin{aligned} F &= \frac{1}{2}E\alpha_0\left(|\text{x,x}'|, e_0 a\right)\varepsilon_{xx}'(x')\varepsilon_{xx}(x) \\ &+ \frac{1}{2}l_m{}^2 E\alpha_1\left(|\text{x,x}'|, e_1 a\right)\varepsilon_{xx,x}'(x')\varepsilon_{xx,x}(x), \end{aligned} \tag{5}$$

where E is the elastic modulus and parameters $\alpha_0(|\text{x-x}'|, e_0 a)$ and $\alpha_1(|\text{x-x}'|, e_1 a)$ indicate the nonlocal attenuation functions associated with the strain ε_{xx} and the first order strain gradient $\varepsilon_{xx,x} = \frac{d\varepsilon_{xx}}{dx}$, respectively. l_m denotes the strain gradient length scale parameter and it should be noted that l_m and higher order strain gradient attenuation function $\alpha_1(|x-x'|, e_1a)$ are not existing in the Eringen's nonlocal elasticity theory. In one-dimensional (1D) case, classical stress, higher order stress and the total stress are respectively expressed as:

$$\sigma_{xx} = \int_0^L E\alpha_0\left(|x-x'|, e_0a\right)\varepsilon'_{xx}(x')dx', \tag{6}$$

$$\sigma_{xx}{}^{(1)} = l_m{}^2\int_0^L E\alpha_1\left(|x-x'|, e_1a\right)\varepsilon'_{xx,x}(x')dx', \tag{7}$$

$$t_{xx} = \sigma_{xx} - \frac{d\sigma_{xx}{}^{(1)}}{dx}, \tag{8}$$

where L is the length of the rod. The nonlocal parameters e_0a and e_1a are significant to examine the nonlocal elastic stress field. The generalized higher order nonlocal constitutive equation based on the nonlocal strain gradient theory is obtained as follows:

$$\begin{aligned}&[1-(e_0a)^2\nabla^2]\ [1-(e_1a)^2\nabla^2]t_{xx}\\&= E[1-(e_1a)^2\nabla^2]\ \varepsilon_{xx} - E_{lm}{}^2[1-(e_0a)^2\nabla^2]\varepsilon_{xx}\end{aligned} \tag{9}$$

where $\nabla^2 = \frac{d^2}{dx^2}$ is defined as one-dimensional differential operator. Bishop rod theory takes into account the contribution of both shear stiffness and inertia of the lateral motion (Rao 2007). However, while deriving analytical formulation shear stiffness and inertia of the lateral motion are not taken into account since aspect ratio is assumed high enough to neglect the shear stiffness and the inertia of the lateral deformation within the nanorod. When results of Li *et al*. (2017) are examined, the difference between results obtained by simple rod and Bishop rod theories diminishes for higher aspect ratio. The Poisson's ratio of the material is not utilized in formulation steps. In the rod type structure, the size dependent behavior in the width and thickness directions can be ignored (Li *et al*. 2016b). Hence, the simple rod theory is used and nanorod is assumed to be made of elastic material and have no intrinsic defects in the bulk and the surface. Hence, dissipation effects in the nanorod is ignored. When $e_0 = e_1 = e$, and terms of order $O(\nabla^2)$ are retained, the generalized constitutive equation given by Eq. (9) is simplified to Eq. (10) as follows (Challamel *et al.* 2009, Challamel 2013, Lim *et al.* 2015, Li and Hu 2015, 2016, Li *et al.* 2016b):

$$[1-(e_0a)^2\nabla^2]\ t_{xx} = E\left[1-l_m{}^2\nabla^2\right]\ \varepsilon_{xx}, \tag{10}$$

Generalized nonlocal constitutive equation provided by Eq. (10) can easily be reduced to the lower order nonlocal stress model when length scale parameter is taken as $l_m = 0$ (Eringen 1972, 1983, 2002).

$$[1-(e_0a)^2\nabla^2]\ t_{xx} = E\varepsilon_{xx} \tag{11}$$

If the nonlocal parameter is assumed as $e_0 = 0$ and if it is substituted into Eq. (10), the formulation for a pure strain gradient model is obtained as follows (Mindlin 1964, 1965):

$$t_{xx} = E\left[1-l_m{}^2\nabla^2\right]\varepsilon_{xx}, \tag{12}$$

The explicit form of the nonlocal strain gradient constitutive equation for one-dimensional rod takes the following form:

$$t_{xx} - (e_0a)^2\frac{\partial^2 t_{xx}}{\partial x^2} = E\varepsilon_{xx} - El_m{}^2\frac{\partial^2\varepsilon_{xx}}{\partial x^2}, \tag{13}$$

Using Hamilton's principle, the equation of motion for the longitudinally vibrating rod embedded in a viscoelastic medium can be obtained as:

$$\begin{aligned}&\frac{\partial N}{\partial x} + f(x,t) - k_u u(x,t) - c_u\frac{\partial u(x,t)}{\partial t}\\&\qquad = \rho A\frac{\partial^2 u(x,t)}{\partial t^2},\end{aligned} \tag{14}$$

where k is the elastic stiffness and c is the damping coefficient of the viscoelastic medium.

$$N = \int_A \sigma_{xx}dA, \tag{15}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \tag{16}$$

Using Eqs. (13), (15)-(16), the longitudinal force for the nonlocal strain gradient model is:

$$N - (e_0a)^2\frac{\partial^2 N}{\partial x^2} = EA\frac{\partial u}{\partial x} - EAl_m{}^2\frac{\partial^3 u}{\partial x^3}, \tag{17}$$

$$\begin{aligned}N &= EA\frac{\partial u}{\partial x} - EAl_m{}^2\frac{\partial^3 u}{\partial x^3}\\&+(e_0a)^2\left\{\rho A\frac{\partial^3 u}{\partial x\partial t^2} - \frac{\partial f}{\partial x} + k\frac{\partial u}{\partial x} + c\frac{\partial^2 u}{\partial x\partial t}\right\},\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial N}{\partial x} &= EA\frac{\partial^2 u}{\partial x^2} - EAl_m{}^2\frac{\partial^4 u}{\partial x^4}\\&+(e_0a)^2\left\{\rho A\frac{\partial^4 u}{\partial x^2\partial t^2} - \frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^3 u}{\partial x^2\partial t}\right\}.\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (14) yields:

$$\begin{aligned}&EA\frac{\partial^2 u}{\partial x^2} - EAl_m{}^2\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2}{\partial t^2}\left((e_0a)^2\frac{\partial^2 u}{\partial x^2} - u\right)\\&+k\left((e_0a)^2\frac{\partial^2 u}{\partial x^2} - u\right) + c\frac{\partial}{\partial t}\left((e_0a)^2\frac{\partial^2 u}{\partial x^2} - u\right)\\&\quad = (e_0a)^2\frac{\partial^2 f}{\partial x^2} - f, \qquad 0 < x < L, \quad 0 < t.\end{aligned} \tag{20}$$

Eq. (20) is a fourth order partial differential equation. In order to solve this equation, four boundary conditions are required in total. Analytical solution is developed considering Type I and Type II boundary conditions that results in problem models of the nanorod with CF/CF and CF/FS boundaries. Two of boundary conditions are determined from nonlocal strain gradient elasticity theory and the remaining two are specified according to the classical elasticity theory. The following equations are valid for the fixed-fixed nanorod system considering non-classical boundary conditions (Challamel 2013):

$$\left(EA\frac{\partial u}{\partial x} - EAl_m{}^2\frac{\partial^3 u}{\partial x^3}\right)\delta u\Big|_0^L = 0, \tag{21}$$

$$\left(EAl_m{}^2\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial \delta u}{\partial x}\bigg|_0^L = 0, \tag{22}$$

Classical elasticity theory requires that there is no displacement at clamped ends, henc eu=0 at the corresponding boundary. For the CF/CF boundaries of the nanorod, Eqs. (21)-(22) can be reduced to the following non-classical boundary conditions. For the CF/CF ends, boundary conditions can be determined as (Tsepoura 2002, Lie *et al*. 2016a):

$$u(0) = 0,\ \frac{\partial^2 u}{\partial x^2}\bigg|_{x=0} = 0, \tag{23}$$

$$u(L) = 0,\ \frac{\partial^2 u}{\partial x^2}\bigg|_{x=L} = 0, \tag{24}$$

On the other hand, following boundary conditions can be written for the CF/FS ends based on classical and nonlocal strain gradient elasticity theories (Lie *et al*. 2016a):

$$u(0) = 0,\ \frac{\partial^2 u}{\partial x^2}\bigg|_{x=0} = 0, \tag{25}$$

$$\frac{\partial u}{\partial x}\bigg|_{x=L} = 0, \frac{\partial^3 u}{\partial x^3}\bigg|_{x=L} = 0. \tag{26}$$

The harmonic vibration is assumed in longitudinal direction and analytical solution can be assumed as:

$$u(x,t) = U(x)e^{i\omega t}, \tag{27}$$

When harmonic solution provided by Eq. (27) is substituted into Eq. (20), the following ordinary differential equation is obtained.

$$\frac{d^4U}{dx^4} + \gamma_n\frac{d^2U}{dx^2} + \lambda_n U(x) = 0, \tag{28}$$

Then, roots of Eq. (28) are found as:

$$r_{1n} = +\frac{1}{\sqrt{2}}\sqrt{-\sqrt{\gamma_n{}^2 - 4\lambda_n} - \gamma_n}, \tag{29}$$

$$r_{2n} = +\frac{1}{\sqrt{2}}\sqrt{+\sqrt{\gamma_n{}^2 - 4\lambda_n} - \gamma_n}, \tag{30}$$

$$r_{3n} = -\frac{1}{\sqrt{2}}\sqrt{-\sqrt{\gamma_n{}^2 - 4\lambda_n} - \gamma_n}, \tag{31}$$

$$r_{4n} = -\frac{1}{\sqrt{2}}\sqrt{+\sqrt{\gamma_n{}^2 - 4\lambda_n} - \gamma_n}, \tag{32}$$

where

$$\gamma_n = \left(\frac{\rho A(e_0a)^2\omega_n{}^2 - c(e_0a)^2\omega_n - k(e_0a)^2 - EA}{EAl_m{}^2}\right), \tag{33}$$

$$\lambda_n = \left(\frac{k_u + ci\omega_n - \rho A\omega_n{}^2}{EAl_m{}^2}\right). \tag{34}$$

The analytical solution can be written using the roots given in Eq. (29)-(32) as follows:

$$U(x) = C_1\sin(r_{1n}x) + C_2\cos(r_{1n}x) + C_3\sinh(r_{2n}x) + C_4\cosh(r_{2n}x), \tag{35}$$

where C_1, C_2, C_3 and C_4 indicate unknown constants to be determined. Since addressed problem involves two types of boundaries such as Type I: CF/CF and Type II: CF/FS, unknown constants and matrix elements are marked with superscript $()^I$ and $()^{II}$ to point these problems clearly without any confusion. For the problem with CF/CF boundaries (Type I), when boundary conditions described by Eqs. (23)-(24) are applied to the solution, the unknown constants $C_1{}^I$, $C_2{}^I$, $C_3{}^I$ and $C_4{}^I$ are determined. The following matrix expresses the constants of required equations to get unknown constants.

$$\begin{bmatrix} A_{11}{}^I & A_{12}{}^I & A_{13}{}^I & A_{14}{}^I \\ A_{21}{}^I & A_{22}{}^I & A_{23}{}^I & A_{24}{}^I \\ A_{31}{}^I & A_{32}{}^I & A_{33}{}^I & A_{34}{}^I \\ A_{41}{}^I & A_{42}{}^I & A_{43}{}^I & A_{44}{}^I \end{bmatrix}\begin{Bmatrix} C_1{}^I \\ C_2{}^I \\ C_3{}^I \\ C_4{}^I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{36}$$

where

$$A_{11}{}^I = 0,\ A_{12}{}^I = 1,\ A_{13}{}^I = 0, A_{14}{}^I = 1, \tag{37}$$

$$A_{21}{}^I = \sin(r_{1n}L),\ A_{22}{}^I = \cos(r_{1n}L), \quad A_{23}{}^I = \sinh(r_{2n}L), A_{24}{}^I = \cosh(r_{2n}L), \tag{38}$$

$$A_{31}{}^I = 0, A_{32}{}^I = -r_{1n}{}^2,\ A_{33}{}^I = 0, A_{34}{}^I = r_{2n}{}^2, \tag{39}$$

$$A_{41}{}^I = -r_{1n}{}^2\sin(r_{1n}L), A_{42}{}^I = -r_{1n}{}^2\cos(r_{1n}L), \quad A_{43}{}^I = r_{2n}{}^2\sinh(r_{2n}L), A_{44}{}^I = r_{2n}{}^2\cosh(r_{2n}L). \tag{40}$$

In order to get the nontrivial solution, the determinant of the matrix $[A^I]$ should be zero. Then, the solution of determinant equation yields:

$$C_2{}^I = C_3{}^I = C_4{}^I = 0, \tag{41}$$

$$C_1{}^I\sin(r_{1n}L) = 0, \tag{42}$$

$$r_{1n} = \frac{n\pi}{L},\ n = 1,2,\dots,\infty. \tag{43}$$

The longitudinal natural frequencies of the nanorod embedded in a viscoelastic medium with CF/CF boundaries can be obtained by equating Eq. (43) to Eq. (29). For the problem with CF/FS boundaries (Type II), when boundary conditions provided by Eqs. (25)-(26) are applied to the solution, the unknown constants appearing in Eq. (35) are determined. The following linear algebraic equation system is useful to get constants $C_1{}^{II}$, $C_2{}^{II}$, $C_3{}^{II}$ and $C_4{}^{II}$.

$$\begin{bmatrix} A_{11}{}^{II} & A_{12}{}^{II} & A_{13}{}^{II} & A_{14}{}^{II} \\ A_{21}{}^{II} & A_{22}{}^{II} & A_{23}{}^{II} & A_{24}{}^{II} \\ A_{31}{}^{II} & A_{32}{}^{II} & A_{33}{}^{II} & A_{34}{}^{II} \\ A_{41}{}^{II} & A_{42}{}^{II} & A_{43}{}^{II} & A_{44}{}^{II} \end{bmatrix}\begin{Bmatrix} C_1{}^{II} \\ C_2{}^{II} \\ C_3{}^{II} \\ C_4{}^{II} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{44}$$

where

$$A_{11}{}^{II} = 0,\ A_{12}{}^{II} = 1,\ A_{13}{}^{II} = 0, A_{14}{}^{II} = 1, \quad (45)$$

$$A_{21}{}^{II} = r_{1n}{}^{3}\cos(r_{1n}L), A_{22}{}^{II} = -r_{1n}{}^{3}\, sin(r_{1n}L),$$
$$A_{23}{}^{II} = r_{2n}{}^{3}\cosh(r_{2n}L), A_{24}{}^{II} = r_{2n}{}^{3}\, sinh(r_{2n}L), \quad (46)$$

$$A_{31}{}^{II} = 0, A_{32}{}^{II} = -r_{1n}{}^{2},\ A_{33}{}^{II} = 0, A_{34}{}^{II} = r_{2n}{}^{2}, \quad (47)$$

$$A_{41}{}^{II} = r_{1n}\, cos(r_{1n}L),\ A_{42}{}^{II} = -r_{1n}\, sin(r_{1n}L),$$
$$A_{43}{}^{II} = r_{2n}\cosh(r_{2n}L), A_{44}{}^{II} = r_{2n}\, sinh(r_{2n}L). \quad (48)$$

In order to get the nontrivial solution, the determinant of the matrix $[A^{II}]$ should be zero. Then, the solution of determinant equation yields:

$$C_2{}^{II} = C_3{}^{II} = C_4{}^{II} = 0, \quad (49)$$

$$C_1{}^{II}\, cos(r_{1n}L) = 0, \quad (50)$$

$$r_{1n} = \frac{(2n-1)\pi}{2L},\ n = 1,2,\ldots,\infty. \quad (51)$$

The longitudinal natural frequencies of the nanorod embedded in a viscoelastic medium with CF/FS boundaries can be obtained by equating Eq. (51) to Eq. (29). Alternatively, analytical solution for the longitudinal displacement can be written in the following form:

$$u(x,t) = \sum_{n=1}^{\infty} q_n\,(t) U_n(x), \quad (52)$$

where $q_n(t)$ is the unknown time dependent function and $U_n(x)$ denotes the mode shape function. Mode shapes of nanorod with CF/CF or CF/FS boundaries has a sinusoidal form and solution given in Eq. (52) can be expressed as follows:

$$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} \overline{q_n}\, sin(\omega_n t)\, sin\left(\frac{n\pi x}{L}\right), & \text{Type I: CF/CF,} \\ \sum_{n=1}^{\infty} \overline{q_n}\, sin(\omega_n t)\, sin\left(\frac{(2n-1)\pi x}{2L}\right), & \text{Type II: CF/FS.} \end{cases} \quad (53)$$

Solution given in Eq. (53) is substituted into Eq. (20) and following relation is obtained. The longitudinal natural frequencies of the nanorod embedded in a viscoelastic medium can be found by solving the following equations for undetermined ω_n:

$$\left\{\frac{n^4\pi^4}{L^4} - \gamma_n \frac{n^2\pi^2}{L^2} + \lambda_n\right\} = 0, \quad \text{Type I: CF/CF}, n = 1,2,\ldots,\infty. \quad (54)$$

$$\left\{\left(\frac{(2n-1)\pi}{2L}\right)^4 - \gamma_n \left(\frac{(2n-1)\pi}{2L}\right)^4 + \lambda_n\right\} = 0, \quad \text{Type II: CF/FS},\ n = 1,2,\ldots,\infty. \quad (55)$$

where γ_n and λ_n are functions of circular frequency ω_n which is in the units of [rad/s]. In order to perform parametric analyses, non-dimensional quantities are introduced as follows:

$$\alpha = \left(\frac{e_0 a}{L}\right), \quad (56)$$

$$\beta = \frac{l_m}{L}, \quad (57)$$

$$K_U = \frac{kL^2}{EA}, \quad (58)$$

$$C_U = \frac{cLc_L}{EA}, \quad (59)$$

$$\Omega_n = \omega_n L\sqrt{\frac{\rho}{E}}. \quad (60)$$

where α is the non-dimensional nonlocal parameter, β is the non-dimensional length scale parameter, K_U is the non-dimensional stiffness parameter, C_U is the non-dimensional damping parameter and Ω_n is the non-dimensional frequency. c_L denotes the longitudinal wave propagation speed in the solid which can be calculated by $c_L = \sqrt{\frac{E}{\rho}}$. Constants γ_n and λ_n can be expressed based on non-dimensional quantities as:

$$\gamma_n = \frac{1}{c_L{}^2}\left(\frac{\alpha}{\beta}\right)^2 \omega_n{}^2 - \frac{1}{Lc_L}\left(\frac{\alpha}{\beta}\right)^2 C_u\omega_n - \left(\frac{\alpha}{\beta}\right)^2 \frac{1}{L^2} K_U - \frac{1}{l_m{}^2}, \quad (61)$$

$$\lambda_n = \frac{K_U}{L^2 l_m{}^2} + \frac{C_U}{L l_m{}^2 c_L} i\omega_n - \frac{1}{c_L{}^2}\frac{1}{l_m{}^2}\omega_n{}^2. \quad (62)$$

4. Numerical results

In this section, parametric analyses are carried out to examine the free vibration of a nanorod embedded in a viscoelastic medium based on nonlocal strain gradient theory. Numerical results are obtained for nanorods with viscoelastic medium considering CF/CF and CF/FS boundaries according to different values of non-dimensional quantities provided in Eqs. (56)-(59). Non-dimensional vibration frequencies (Ω_n) are calculated and they are provided in Tables 1-4. Table 1 shows the first vibration frequency (Ω_1) of the nanorod with CF/CF and CF/FS boundaries considering length scale parameter $\beta = 0$ for various values of nonlocal parameter. In the existing of an elastic medium, results of the CF/CF type nanorod are compared with those found by Simsek (2016b) and a very good agreement is achieved between results. Moreover, results generated for a nanorod with CF/FS boundaries are compared with those obtained by Liu *et al.* (2016) for various values of nonlocal parameter and again, results of present study display a high degree of accuracy. It should be remarked that increase in the nonlocal parameter leads to decrease in first vibration frequency for all values of elastic stiffness parameter (K_u).

Table 2 provides the first vibration frequency (Ω_1) of the nanorod with CF/CF and CF/FS boundaries for different values of length scale parameter (β) assuming nonlocal

Table 1 The first vibration frequency-Ω_1 of the nanorod with CF/CF and CF/FS boundaries for various values of the nonlocal and elastic medium parameters,$\beta = 0.0,\ C_U = 0.0$

CF/CF	Ω_1Simsek (2016b)				Ω_1Present study			
K_U	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$
0	3.1415	2.4706	1.6871	0.9528	3.1416	2.4706	1.6871	0.9529
1	3.2969	2.6653	1.9612	1.3813	3.2969	2.6654	1.9612	1.3813
5	3.8561	3.3322	2.8011	2.4306	3.8561	3.3323	2.8011	2.4306
10	4.4575	4.0130	3.5841	3.3027	4.4575	4.0130	3.5842	3.3027
CF/FS	Ω_1Liu *et al.* (2016a)				Ω_1Present study			
K_U	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$
0	1.5708	1.4621	1.2353	0.8436	1.5708	1.4621	1.2353	0.8436

Table 2 The first vibration frequency-Ω_1 of the nanorod with CF/CF and CF/FS boundaries for various values of the length scale and elastic medium parameters, $\alpha = 0.0,\ C_U = 0.0$

CF/CF	Ω_1Simsek (2016b)				Ω_1Present study			
K_U	$\beta = 0.0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 1.0$	$\beta = 0.0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 1.0$
0	3.1415	3.9947	5.8499	10.357	3.1416	3.9953	5.8499	10.3576
1	3.2969	4.1179	5.9348	10.405	3.2969	4.1180	5.9348	10.4057
5	3.8561	4.5779	6.2627	10.596	3.8561	4.5780	6.2727	10.5962
10	4.4575	5.0948	6.6499	10.829	4.4575	5.0949	6.6499	10.8295
CF/FS	Ω_1Liu *et al.* (2016a)				Ω_1Present study			
K_U	$\beta = 0.0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 1.0$	$\beta = 0.0$	$\beta = 0.25$	$\beta = 0.50$	$\beta = 1.0$
0	1.5708	1.6876	1.9973	2.9250	1.5708	1.6876	1.9973	2.9250

parameter as zero $(\alpha = 0)$. As β is increased, first vibration frequency is increased. Again, a very good agreement is accomplished between the results of the present study and those given by Simsek (2016b) and Liu *et al.* (2016a) for various values of β. In Tables 3-4, the first three rows are allocated for the vibration frequency of a nanorod with CF/CF boundaries while the last three rows indicate those of a nanorod with CF/FS boundaries. Table 3 shows the first vibration frequency of the nanorod embedded in the viscoelastic medium for various values of nonlocal parameter α considering $\beta = 0.0$. Due to the viscous effect of the medium when $C_u \neq 0$, vibration frequency is a complex number in the form $\Omega_n = a + jb$, and it is no longer a real number. As damping parameter (C_U) of the viscoelastic medium is increased, imaginary part of the vibration frequency is increased. The same behavior is observed in the study conducted by Karlicic *et al.* (2015). If there is no viscous effect in the medium $(C_u = 0)$, the real part of the vibration frequency is decreased as nonlocal parameter is increased. However, this trend tends to change at larger values of damping parameters. If viscoelastic medium with parameters $K_U = 10,\ C_U = 5$ is considered, increase in the nonlocal parameter leads to increase in the real part of the frequency while it results in decrease in the imaginary part, which implies the alleviation of viscous effect on damping response at greater values of nonlocal parameter.

As K_U is increased, the first vibration frequency for both CF/CF and CF/FS type nanorods is increased. Obtained vibration frequencies for a CF/FS nanorod are always lower than those calculated for CF/CF nanorod. Increase in the elastic medium parameter produces greater values of the vibration frequency for all values of nonlocal parameter. When nonlocal parameter is zero i.e., $\alpha = 0.0$, although increase in the stiffness (K_U) leads to larger values of undamped frequency, increase in the damping parameter (C_U) leads to decrease in the undamped frequencies for both CF/CF and CF/FS type nanorods. However, at larger values of nonlocal parameters such as$\alpha = 0.5,\ 1.0$, the increase in the damping parameter results in occurrence of larger frequency values for both type of nanorods. As the stiffness of the elastic medium is increased in the absence of viscous effect, the difference between vibration frequencies of CF/CF and CF/FS nanorods is reduced. For instance, $\frac{{\Omega_1}^{\mathrm{CF/CF}}}{{\Omega_1}^{\mathrm{CF/FS}}} = 2$ at $K_U = 0$, $C_U = 0$, this ratio becomes 1.262 at $K_U = 10, C_U = 0$. In all the viscoelastic cases, increase in the damping parameter leads to increase in the imaginary part of the frequency and this increase is relatively greater when $\alpha = 0.0$. For instance, while the ratio $\frac{Im(\Omega_1)}{Re(\Omega_1)}$ is 0.677 when $\alpha = 0.0$, this ratio becomes 0.057 for a CF/CF nanorod at $\alpha = 1.0, K_U = 5,\ C_U = 5$. At the same parameters, the ratio $\frac{Im(\Omega_1)}{Re(\Omega_1)}$ is 1.00 for a CF/FS nanorod at $\alpha = 0.0$ and this ratio becomes 0.196 at $\alpha = 1.0$. It is worthy to say that viscous effect is more pronounced at smaller values of α. The similar conclusion was drawn from the study conducted by Karlicic *et al.* (2015) that increase in the nonlocal

Table 3 The first vibration frequency-$\Omega_1$of the nanorod with CF/CF and CF/FS boundaries for various values of the nonlocal, elastic medium and damping parameters, $\beta = 0.0$

	C_U	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$
CF/CF $K_U = 0$	0	3.1416+0.0000j	2.4706+0.0000j	1.6871+0.0000j	0.9529+0.0000j
	1	3.1015+0.5000j	2.6495+0.3323j	2.0742+0.1741j	1.5087+0.0658j
	5	1.9025+2.5000j	3.2018+2.2022j	4.1824+1.2547j	4.7303+0.4422j
CF/CF $K_U = 5$	0	3.8561+0.0000j	3.3323+0.0000j	2.4306+0.0000j	2.4306+0.0000j
	1	3.8236+0.5000j	3.5142+0.3270j	2.8011+0.0000j	2.9263+0.0545j
	5	2.9359+2.5000j	4.0914+2.0162j	3.1758+0.1624j	5.5916+0.3871j
CF/CF $K_U = 10$	0	4.4575+0.0000j	4.0129+0.0000j	3.5842+0.0000j	3.3027+0.0000j
	1	4.4294+0.5000j	4.1964+0.3239j	3.9547+0.1585j	3.7875+0.0522j
	5	3.6905+2.5000j	4.7971+1.9299j	5.7283+1.0458j	6.2731+0.3604j
CF/FS $K_U = 0$	0	1.5708+0.0000j	1.4621+0.0000j	1.2353+0.0000j	0.8436+0.0000j
	1	1.4891+0.5000j	1.4650+0.4539j	1.4029+0.3579j	1.2617+0.2008j
	5	0.0000+4.4449j	0.7793+3.7909j	2.1775+2.7514j	3.7259+1.3798j
CF/FS $K_U = 5$	0	2.7327+0.0000j	2.6717+0.0000j	2.5546+0.0000j	2.3898+0.0000j
	1	2.6865+0.5000j	2.7039+0.4441j	2.7338+0.3324j	2.7678+0.1655j
	5	1.1034+2.5000j	1.9919+2.6024j	3.2872+2.1782j	4.7029+1.1597j
CF/FS $K_U = 10$	0	3.5309+0.0000j	3.4839+0.0000j	3.3950+0.0000j	3.2728+0.0000j
	1	3.4953+0.5000j	3.5244+0.4415j	3.5771+0.3267j	3.6448+0.1598j
	5	2.4935+2.5000j	3.0957+2.4279j	4.1565+2.0067j	5.4503+1.0704j

Table 4 The first vibration frequency-$\Omega_1$of the nanorod with CF/CF and CF/FS boundaries for various values of the nonlocal, elastic medium and damping parameters, $\beta = 0.5$

	C_U	$\alpha = 0.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$
CF/CF $K_U = 0$	0	5.8499+0.0000j	4.6006+0.0000j	3.1416+0.0000j	1.7744+0.0000j
	1	5.8285+0.5000j	4.7849+0.3221j	3.5142+0.1604j	2.2850+0.0570j
	5	5.2889+2.5000j	5.4029+1.8777j	5.3349+1.0817j	5.1477+0.4114j
CF/CF $K_U = 5$	0	6.2627+0.0000j	5.1152+0.0000j	3.8561+0.0000j	2.8545+0.0000j
	1	6.2428+0.5000j	5.3002+0.3208j	4.2256+0.1575j	3.3441+0.0532j
	5	5.7421+2.5000j	5.9310+1.8425j	5.9752+1.0266j	5.9127+0.3733j
CF/CF $K_U = 10$	0	6.6499+0.0000j	5.5826+0.0000j	4.4575+0.0000j	3.6261+0.0000j
	1	6.6311+0.5000j	5.7680+0.3198j	4.8252+0.1557j	4.1081+0.0517j
	5	6.1621+2.5000j	6.4089+1.8165j	6.5316+0.9909j	6.5436+0.3522j
CF/FS $K_U = 0$	0	1.9974+0.0000j	1.8591+0.0000j	1.5708+0.0000j	1.0726+0.0000j
	1	1.9338+0.5000j	1.8761+0.4492j	1.7431+0.3472j	1.4776+0.1899j
	5	0.0000+4.0035j	0.9293+3.3815j	2.3850+2.5766j	3.8256+1.3477j
CF/FS $K_U = 5$	0	2.9982+0.0000j	2.9080+0.0000j	2.7327+0.0000j	2.4800+0.0000j
	1	2.9563+0.5000j	2.9431+0.4433j	2.9126+0.3309j	2.8572+0.1647j
	5	1.6551+2.5000j	2.3359+2.5274j	3.4698+2.1324j	4.7755+1.1490j
CF/FS $K_U = 10$	0	3.7403+0.0000j	3.6683+0.0000j	3.5309+0.0000j	3.3392+0.0000j
	1	3.7067+0.5000j	3.7101+0.4411j	3.7133+0.3260j	3.7109+0.1595j
	5	2.7819+2.5000j	3.3232+2.4080j	4.2976+1.9872j	5.5091+1.0649j

parameter reduces the imaginary part of the complex frequency. Table 4 indicates the first vibration frequency of the nanorod embedded in viscoelastic medium for various values of nonlocal parameter α when length scale parameter β is assumed as 0.5.

When Table 3 and Table 4 are compared, increase in the

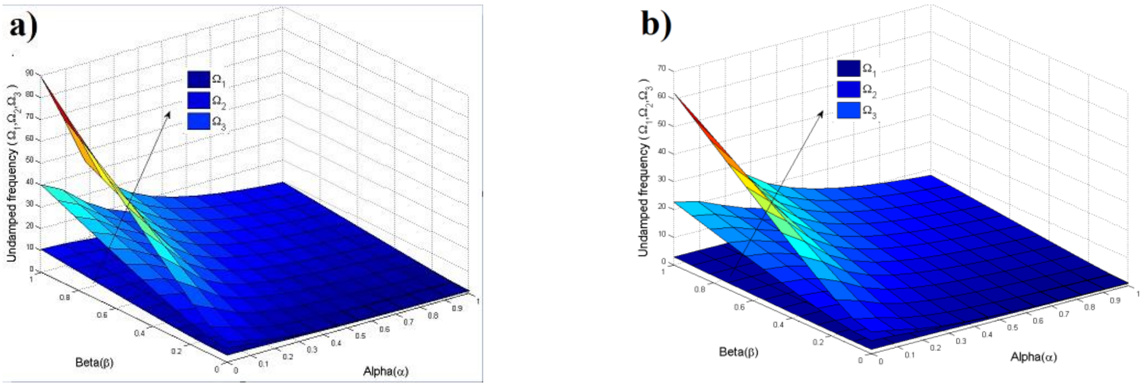

Fig. 2 First three vibration frequencies $(\Omega_1, \Omega_2, \Omega_3)$ of the nanorod with respect to different nonlocal (α) and length scale (β) parameters, (a) CF/CF boundaries, (b) CF/FS boundaries $K_U = 0$, $C_U = 05$

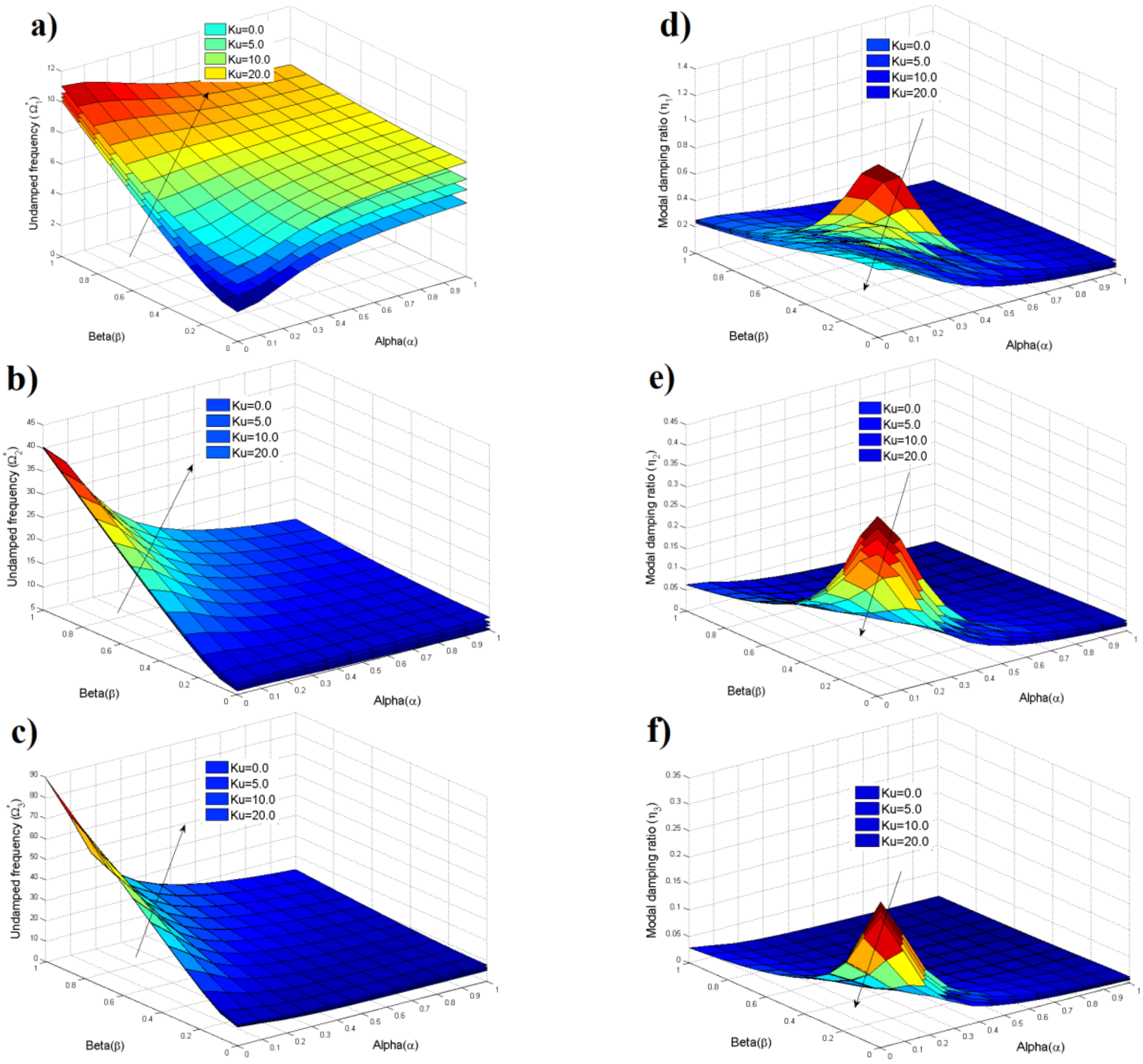

Fig. 3 The influence of stiffness parameter on undamped frequency and modal damping ratio for a nanorod with CF/CF boundaries (a)-(c) First three vibration frequencies $(\Omega_1^*, \Omega_2^*, \Omega_3^*)$, (d)-(f) First three modal damping ratios (η_1, η_2, η_3) with respect to different α and β; $C_U = 5$

length scale parameter (β) leads to increase in the first vibration frequency. It is obvious that increase in the damping parameter (C_U) induces increase in the imaginary part of the frequency in all cases and the level of this increase seems less than or equal to that observed in Table 3. However, real part of the frequency values are greater at $\beta = 0.5$. Therefore, the influence of the viscous effect on frequency response becomes less prominent at larger values of length scale parameter such as $\beta = 0.5$.

Due to the viscous effect of the medium, obtained non-

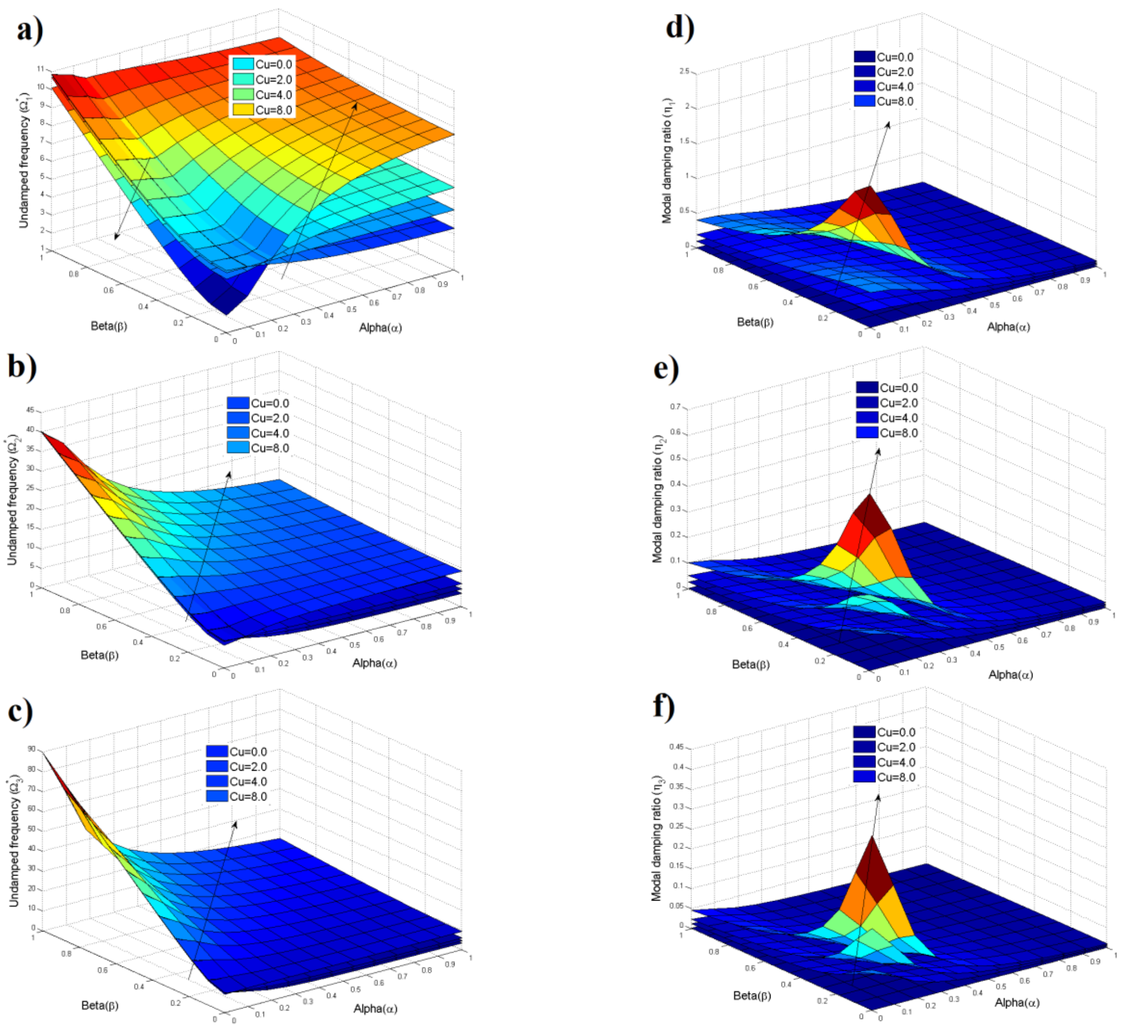

Fig. 4 The influence of damping parameter on undamped frequency and modal damping ratio for a nanorod with CF/CF boundaries (a)-(c) First three vibration frequencies $(\Omega_1^*, \Omega_2^*, \Omega_3^*)$, (d)-(f) First three modal damping ratios (η_1, η_2, η_3) with respect to different α and β; $K_U = 10$

dimensional natural frequencies have real and imaginary part. Hence, it is worthy to introduce the following parameters as (Kung and Singh 1998a, b, Fu *et al.* 2018):

$$\Omega_n^* = Re(\Omega_n), n = 1,2,3,\ldots \quad (63)$$

$$\eta_n = \frac{Im(\Omega_n)}{Re(\Omega_n)}, \; n = 1,2,3,\ldots \quad (64)$$

where Ω_n^* is the undamped natural frequency and η_n denotes the modal damping ratio of the fundamental vibration frequency. Fig. 2 shows the first three vibration frequencies of the nanorod having CF/CF and CF/FS boundaries when there is no viscoelastic medium exists($C_U = 0$). Since there is no viscous effect, vibration frequencies have no imaginary parts, which results in $\Omega_n^* = \Omega_n$, $n = 1,2,3,\ldots$ Additionally, surface plots are constructed to observe the influence of two-different parameters, i.e., non-dimensional nonlocal parameter (α) and non-dimensional length scale parameter (β) on non-dimensional vibration frequency (Ω_n) and modal damping ratio (η_n). These surface plots can be seen in Figs. 2-6. Figs. 2(a) and 2(b) respectively show the vibration frequencies of the nanorod with CF/CF and CF/FS boundaries where there is no viscoelastic effects. In both cases, $\Omega_3 > \Omega_2 > \Omega_1$.

Frequency level of a nanorod with CF/FS type is always lower than that obtained for a CF/CF type. Surface plot for the frequency depicted in Fig. 2(a) is the same distribution obtained by Simsek (2016b) for the CF/CF type nanorod. The highest value of the frequency is obtained when nonlocal parameter $\alpha = 0.0$ and length scale parameter $\beta = 1.0$. Although vibration frequencies are close to each other at $\beta = 0.0$, difference between three frequencies (Ω_1, Ω_2 and Ω_3) becomes gradually higher while β is increased from 0 to 1. Moreover, increase in the length scale parameter is more influential at lower values of nonlocal parameter since these three vibration frequencies converges to Ω_1 at larger values of α. The similar frequency distributions are obtained for a CF/FS type nanorod as inferred from Fig. 2(b). Fig. 3 shows the influence of nonlocal parameter α and length scale parameter β on vibration frequencies and modal damping ratios of the nanorod with CF/CF boundaries for increasing values of stiffness parameter K_U. Figs. 3(a)-3(c) respectively

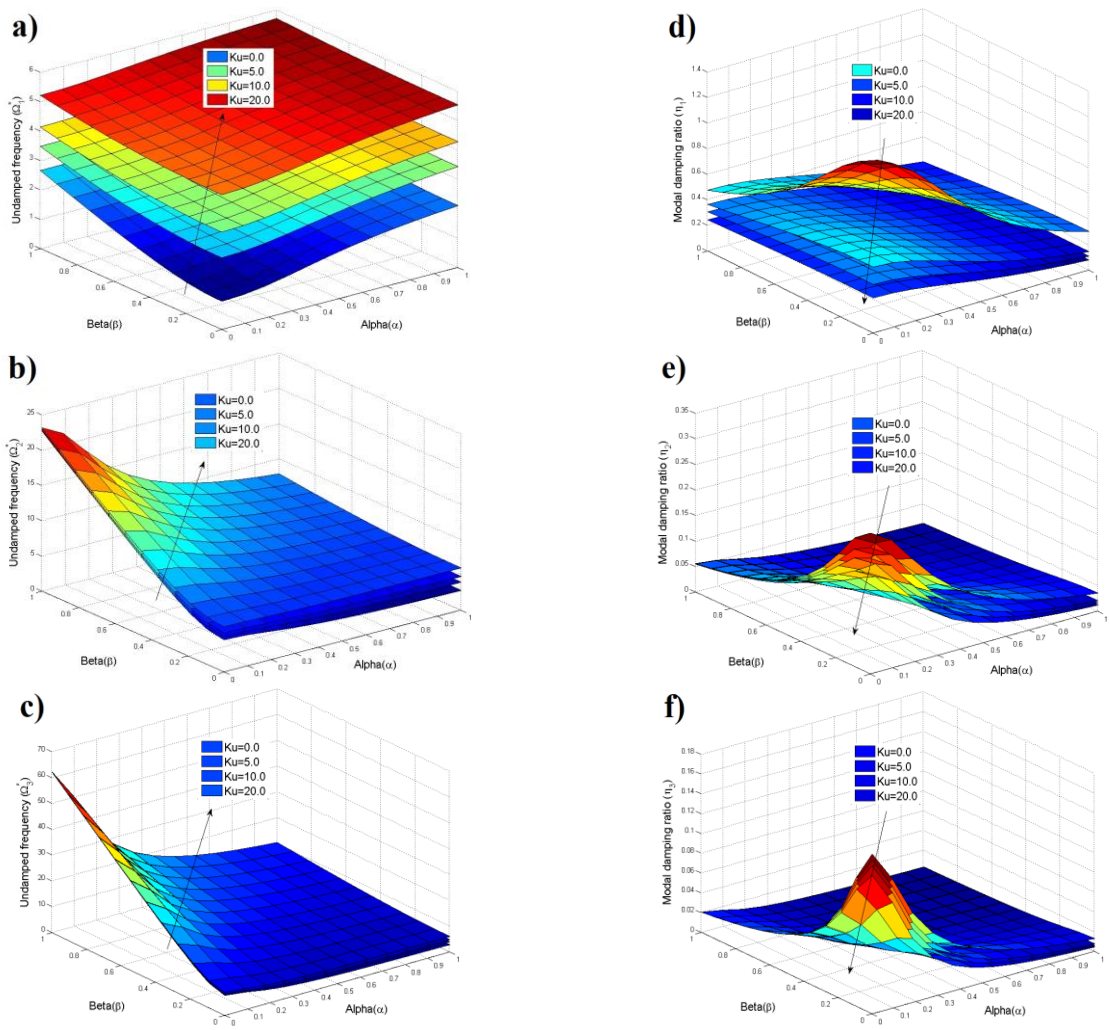

Fig. 5 The influence of stiffness parameter on undamped frequency and modal damping ratio for a nanorod with CF/FS boundaries (a)-(c) First three vibration frequencies $(\Omega_1{}^*, \Omega_2{}^*, \Omega_3{}^*)$, (d)-(f) First three modal damping ratios (η_1, η_2, η_3) with respect to different α and β; $C_U = 2.5$

show the undamped vibration frequencies of the nanorod $\Omega_n{}^*$ while K_U is increased. Figs. 3(d)-3(f) depict the modal damping ratios, respectively. As nonlocal parameter (α) is increased, the undamped vibration frequency is decreased when there is no viscous effect. Results presented in Fig. 2 and Tables 3-4 indicate this fact, and the same conclusion was reported in several studies such as Aydogdu (2009), 2012), Simsek (2016b), Karlicic et. al. (2015) and Liu *et al*. (2016). However in this case, vibration frequency is slightly increased in Figs. 3(a) and 3(b) due to viscous effect of the medium when damping parameter is adjusted as $C_U = 5$.

As stiffness parameter is increased from 0 to 20, first vibration frequency $\Omega_1{}^*$is increased and reaches the maximum level at $\alpha = 0.0$ and $\beta = 1.0$ case as shown in Fig. 3(a). Additionally, increase in αleads to increase in $\Omega_1{}^*$at small values of β, whereas it leads to decrease in $\Omega_1{}^*$at larger values of β. Fig. 3(b) depicts the second vibration frequency $\Omega_2{}^*$for different values of stiffness parameterK_U. Increase in $K_U$leads to increase in $\Omega_2{}^*$but the amount of this increase is smaller than that seen Fig. 3(a). Fig. 3(c) demonstrates the third undamped vibration frequency $\Omega_3{}^*$. Increase in K_U induces increase in the third frequency but this increase is very small and even is not clearly observed. Hence, it can be said that the increase in the stiffness parameter is the most influential at first vibration frequency $\Omega_1{}^*$rather than other frequencies $\Omega_2{}^*$, $\Omega_3{}^*$. As observed from Figs. 3(d)-3(f), while$K_U$is increased from 0 to 20, all modal damping ratios η_n are decreased. The highest decrease in η_n occurs when $\alpha = 0$ and $\beta = 0$. It is seen that the levels of damping ratios are in the order such that $\eta_1 > \eta_2 > \eta_3$. Moreover, the amount of decrease in damping ratios for η_2 and η_3 is smaller than that appear in η_1.

Figs. 4(a)-4(c) and Figs. 4(d)-4(f) indicate the undamped vibration frequencies and modal damping ratios of the nanorod with CF/CF boundaries for different values of damping parameter C_U, respectively. As observed in Fig. 4(a), increase in damping parameter C_U leads to decrease in $\Omega_1{}^*$ for small values of α while it induces increase in $\Omega_1{}^*$for larger values of α. As C_U is increased from 0 to 8, frequencies $\Omega_2{}^*$ and $\Omega_3{}^*$ are increased as illustrated in Figs. 4(b) and 4(c). However, the amount of this increase in second and third mode is smaller than that observed in the

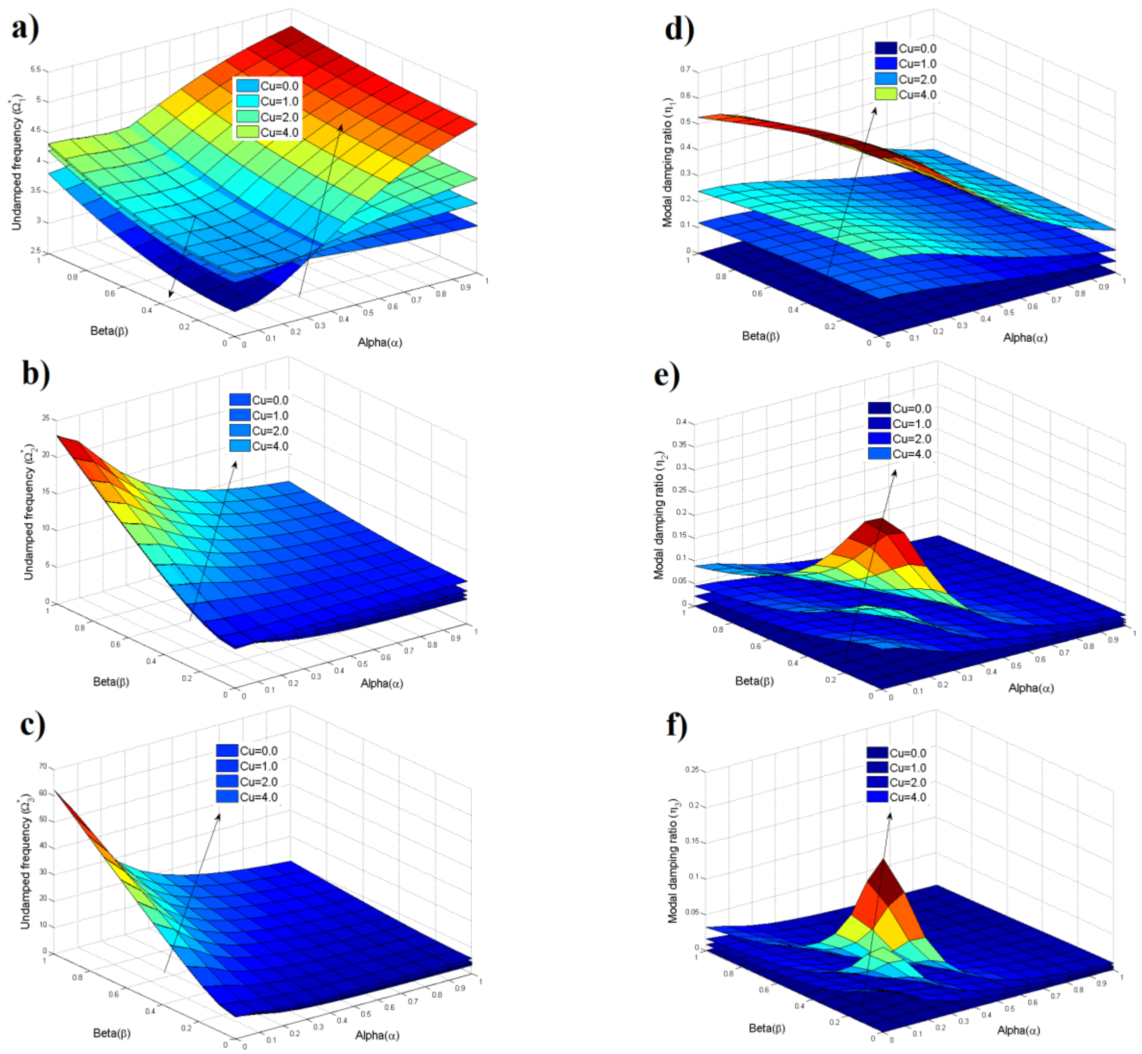

Fig. 6 The influence of damping parameter on undamped frequency and modal damping ratio for a nanorod with CF/FS boundaries (a)-(c) First three vibration frequencies$(\Omega_1^*, \Omega_2^*, \Omega_3^*)$, (d)-(f) First three modal damping ratios (η_1, η_2, η_3) with respect to different α and β; $K_U = 10$

first mode (Ω_1^*). When C_U is increased, modal damping ratios η_1, η_2 and η_3 are increased as observed in Fig. 4(d)-4(f). The highest increase in damping ratio occurs at the first mode. Besides, the effect of increasing C_U on damping parameter observed highest at $\alpha = 0$ and $\beta = 0$. As α and β are increased, modal damping ratio for a CF/CF type nanorod is decreased in all cases.

Figs. 5(a)-5(c) and Figs. 5(d)-5(f) depict the undamped vibration frequencies and modal damping ratios of the nanorod with CF/FS boundaries as functions of stiffness parameter K_U. Undamped frequency levels generated for CF/FS type nanorod are always less than those obtained for CF/CF type nanorod. The highest impact of stiffness parameter on vibration frequency is observed in the first mode. The influence of stiffness parameter on frequency gradually diminishes at second and third modes, respectively. As stiffness of the medium is increased, all modal damping ratios η_1, η_2 and η_3 are decreased, and the highest decrease in modal damping ratio occurs in the first mode as illustrated in Fig. 5(d). The level of this decrease gradually reduces in second and third modes, respectively. Again, increase in nonlocal parameter and length scale parameter leads to decrease in modal damping ratio in all modes.

The influence of viscoelastic medium's damping parameter on vibration frequencies and modal damping ratios of CF/FS type nanorod are shown in Figs. 6(a)-6(c) and Figs. 6(d)-6(f), respectively. In the first mode presented in Fig. 6(a), although increase in the damping parameter (C_U) leads to decrease in the vibration frequency at small values of α, it leads to considerable increase in frequency at larger values α. In the second and third modes, increase in damping parameter results in a slight increase in the vibration frequency as observed in Figs. 6(b) and 6(c). As damping parameter is increased, modal damping ratio increases in all three modes and this result can clearly be seen in Figs. 6(d)-6(f). However, the highest impact of damping parameter on modal damping ratio occurs in the first mode. In viscoelastic case, increase in nonlocal parameter (α) and length scale parameter (β) results in a decrease in the modal damping ratio for a CF/FS type nanorod. Hence, it is worthy to say that, the influence of viscous effect on modal damping ratio of the nanorod

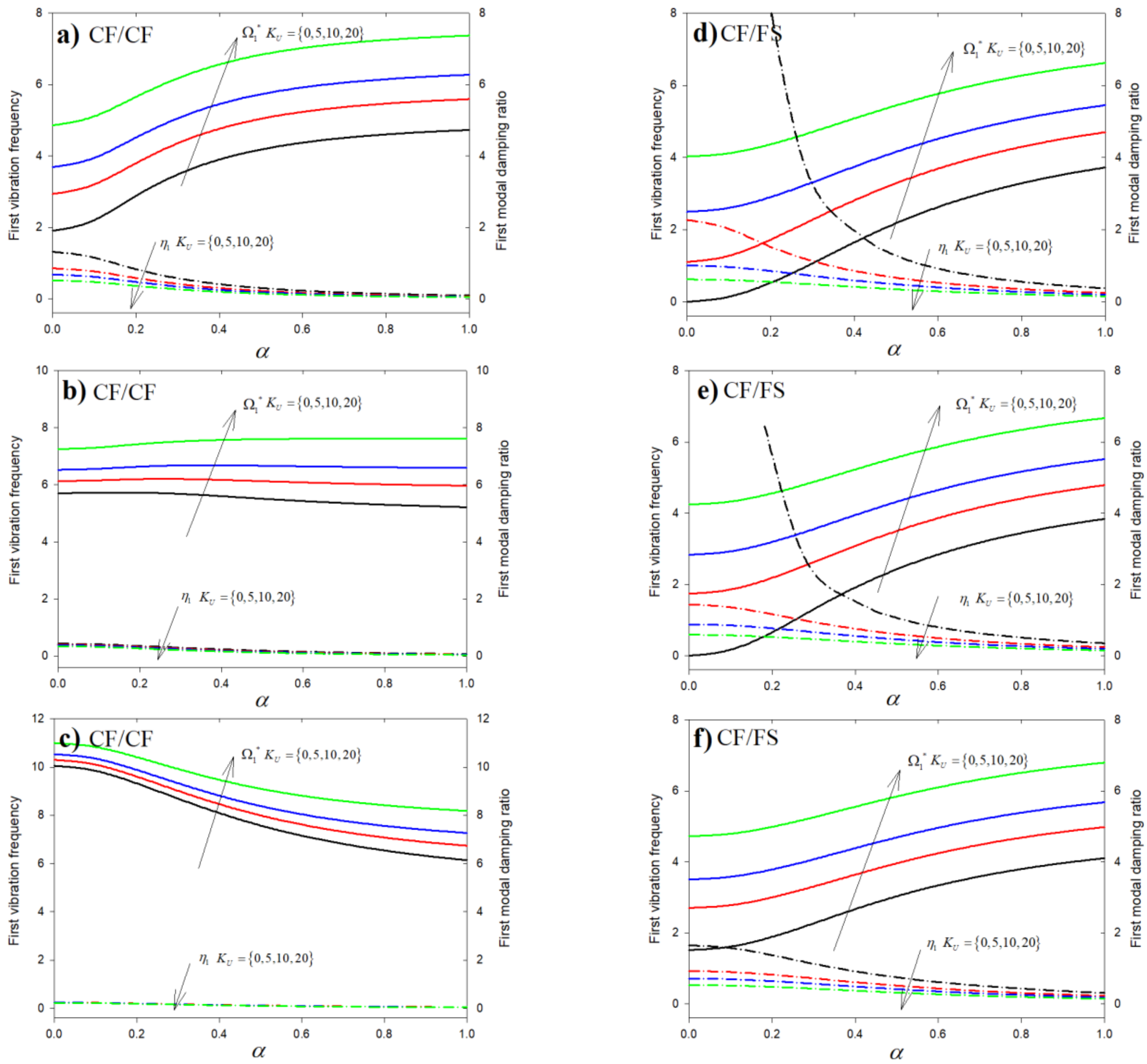

Fig. 7 First vibration frequency and modal damping ratio of a nanorod different values of stiffness parameter $C_U = 5.0$, (a) CF/CF$\beta = 0.0$, (b) CF/CF$\beta = 0.5455$, (c) CF/CF$\beta = 1.0$, (d) CF/FS $\beta = 0.0$, (e) CF/FS $\beta = 0.5455$, (f) CF/FS$\beta = 1.0$.

gradually diminishes at larger values of α and β.

Figs. 7(a)-7(c) illustrate the influence of elastic stiffness parameter(K_U)on first vibration frequency and damping parameter together in the same figure for various values of length scale parameter β for a nanorod with CF/CF boundaries. Solid lines show the first vibration frequency while dashed lines indicate first modal damping ratio. Increase in$K_U$leads to remarkable increase in $\Omega_1{}^*$ while it results in formation of lower levels of damping ratio curves. The impact of increasing elastic stiffness parameter on first vibration frequency and damping parameter is the highest in the case where length scale parameter is $\beta = 0.0$ as shown in Fig. 7(a). This effect on first vibration frequency and damping ratio becomes gradually smaller in cases where $\beta = 0.5455$ and $\beta = 1.0$. While the level of decrease in damping ratio due to stiffness is quite obvious at $\beta = 0.0$case, this decrease becomes minimal and distributions converge to the same level at larger values β.

Figs. 7(d)-7(f) show the effect of elastic stiffness parameter on first vibration frequency and modal damping ratio of the nanorod with CF/FS boundaries. The first vibration frequency of CF/FS type nanorod is always lower than that of CF/CF type nanorod whereas the first modal damping ratio of CF/FS type nanorod is always greater than that of CF/CF one. As stiffness parameter is increased, first vibration frequency is increased. However, distributions for the first vibration frequency seems similar even length scale parameter is changed from 0 to 1.0 for a CF/FS type nanorod. Hence, vibration frequencies of CF/FS nanorods are said to be less sensitive to the change in length scale parameter β.

It is obvious that increase in the stiffness parameter leads to significant decrease in the first modal damping ratio and the level of this decrease is always more obvious for CF/FS type nanorods. Moreover, the variation of damping ratio with respect to nonlocal parameter is greater for CF/FS type nanorods, which indicates nonlocal parameter is more pronounced in damping characteristics of CF/FS nanorods.

Damping ratio for the purely viscous case in which $K_U = 0$ at smaller values of$\alpha$is not displayed in Figs. 7(d)

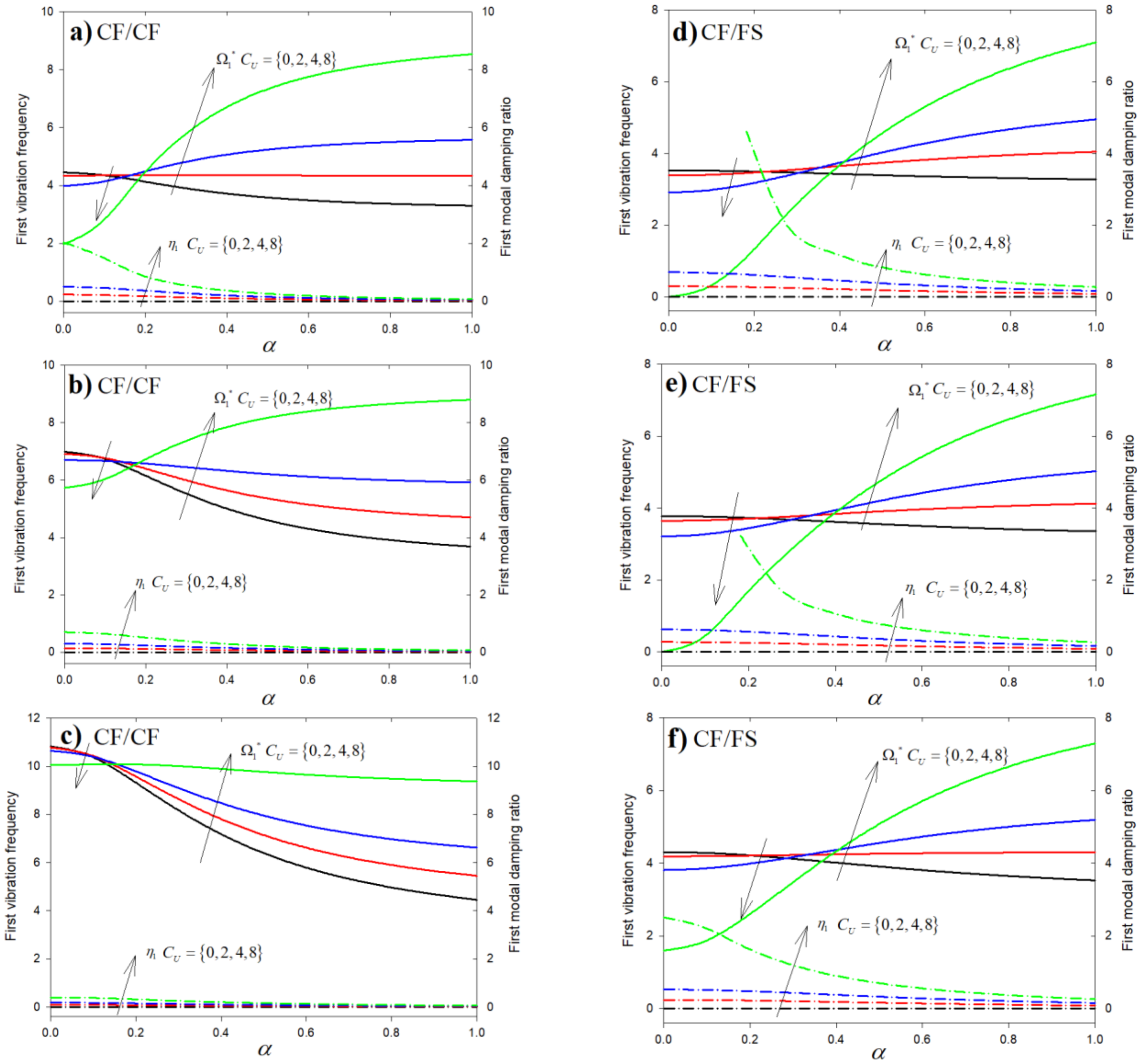

Fig. 8 First vibration frequency and modal damping ratio of a nanorod for different values of damping parameter $K_U = 10$, (a) CF/CF $\beta = 0.0$, (b) CF/CF $\beta = 0.5455$, (c) CF/CF $\beta = 1.0$, (d) CF/FS $\beta = 0.0$, (e) CF/FS $\beta = 0.5455$, (f) CF/FS $\beta = 1.0$

and 7(e) since they approach asymptotic infinity. Figs. 8(a)-8(c) demonstrate the influence of damping parameter C_U on first vibration frequency and damping ratio of the nanorod with CF/CF boundaries for different values of length scale parameter β. Increase in C_U results in the highest change in the first vibration frequency and modal damping ratio when $\beta = 0.0$, as seen in Fig. 8(a). Although increase in C_U induces a decrease in vibration frequency for lower values of α, it results in an increase in frequency for larger values α for both CF/CF and CF/FS type nanorods. Distributions of first vibration frequency and modal damping ratio changes significantly as length scale parameter is increased from $\beta = 0.0$ to $\beta = 1.0$ for CF/CF type nanorod. As opposed to CF/CF type nanorod, frequency and damping ratio distributions for a CF/FS type nanorod are not altered so much while β is increased as depicted in Figs. 8(d)-8(f). Increase in C_U results in relatively greater change in first vibration frequency and modal damping ratio for a CF/FS type nanorods.

5. Conclusions

The longitudinal free vibration analysis of a nanorod embedded in a viscoelastic medium was examined considering nonlocal strain gradient theory. Viscoelastic medium was constructed based on the Kelvin-Voigt model which involves elastic medium stiffness and damping parameters. Governing fourth order partial differential equation was derived incorporating the elastic stifness and viscous damping parameters of the viscoelastic medium. The analytical solution with unknown boundary conditions was obtained based on harmonic motion of the nanorod. Boundary conditions were determined for the nanorods with CF/CF and CF/FS boundaries using classical and nonlocal strain gradient elasticity theories. The influences of stiffness, damping, nonlocal and length scale parameters on vibration frequencies and modal damping ratios were determined. The following concluding remarks can be written:

• Increase in the length scale parameter (β) results in increase in vibration frequencies, which implies the stiffening behavior. The impact of β is greater in higher modes (i.e., second and third modes).

• Increase in the length scale parameter causes increase in vibration frequencies while it triggers decrease in modal damping ratio, which implies viscous effect is diminished at larger length scale parameter.

• As length scale parameter is increased, vibration frequencies are increased, and modal damping ratios are decreased. The level of this increase/decrease is higher for a nanorod with CF/CF boundaries than that observed for a nanorod with CF/FS boundaries. Hence, frequency behavior of nanorods with CF/CF boundaries are said to be more sensitive to the change in the length scale parameter.

• As nonlocal parameter is increased, vibration frequencies always decrease in purely elastic case $(C_U = 0)$, which indicates the softening behavior. However, for larger values of damping parameter where viscous effect is influential, increase in the nonlocal parameter leads to increase in the vibration frequencies at smaller values of β. Increase in nonlocal parameter leads to decrease in the damping ratio in all cases, which indicates viscous effect is less influential at larger α.

• Increase inαinduces greater change in frequency and damping ratio distributions for a CF/FS type nanorod when compared to CF/CF type, hence nonlocal parameter is more influential on vibration behavior of CF/FS type nanorod.

• Increase in the stiffness of the viscoelastic medium (K_U) leads to increase in vibration frequencies and decrease in the modal damping ratios in all modes. The highest impact of the stiffness parameter is observed in the first mode.

• As damping ratio C_U is increased, first vibration frequency $\Omega_1{}^*$ increases at larger values of α whereas it decreases for smaller values of α. The influence of C_U on vibration frequency and modal damping ratio found the highest in first mode when compared to other modes.

• It can be said that change in viscoelastic medium parameters results in relatively greater change in first vibration frequency and modal damping ratio for a nanorod with CF/FS boundaries. Consequently, selection of viscoelastic material properties is crucial to perform reliable design of composite mediums involving nanorods with CF/FS boundaries.

As a futurework, the vibration problem of the nanorod/viscoelastic medium system will be examined based on Bishop rod model which takes into account the shear stiffness and the inertia of the lateral deformation.

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