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A Fourier sine series solution of static and dynamic response of nano/micro-scaled FG rod under torsional effect

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Abstract. In the current work, static and free torsional vibration of functionally graded (FG) nanorods are investigated using Fourier sine series. The boundary conditions are described by the two elastic torsional springs at the ends. The distribution of functionally graded material is considered using a power-law rule. The systems of equations of the mechanical response of nanorods subjected to deformable boundary conditions are achieved by using the modified couple stress theory (MCST) and taking the effects of torsional springs into account. The idea of the study is to construct an eigen value problem involving the torsional spring parameters with small scale parameter and functionally graded index. This article investigates the size dependent free torsional vibration based on the MCST of functionally graded nano/micro rods with deformable boundary conditions using a Fourier sine series solution for the first time. The eigen value problem is constructed using the Stokes' transform to deformable boundary conditions and also the convergence and accuracy of the present methodology are discussed in various numerical examples. The small size coefficient influence on the free torsional vibration characteristics is studied from the point of different parameters for both deformable and rigid boundary conditions. It shows that the torsional vibrational response of functionally graded nanorods are effected by geometry, small size effects, boundary conditions and material composition. Furthermore, for all deformable boundary conditions in the event of nano-sized FG nanorods, the incrementing of the small size parameters leads to increas the torsional frequencies.

Keywords: FG nanorods; fourier sine series; modified couple stress theory; stokes' transformation; vibration analysis

1. Introduction

Functionally graded nanorod and nanobeam type structures have excellent dynamical and electrical properties because of the to their high elastic moduli, energy absorption capacity, high strength/weight ratio and low density. Specifically speaking, the weight to strength ratio is considerably lower than other structural members and therefore, these type of structures are widely employed elements in demanding industrial facilities that are aircraft, aerospace engineering, architectural products and nuclear engineering. Functionally graded nano-structures are widely utilized in demanding technological areas because of to their excellent performances. Aerospace, aircraft machines always suffer from the simultaneous and severe mechanical, thermal loadings and internal aerodynamic increasing the rigidity of these systems to overcome the larger vibration amplitude have not always been practical because of to stringent performance and weight specialties in the project of this type structures.

In a great number of basic applications and problems, classical elasticity theories, have been employed by scholars (Ramteke *et al.* 2020, 2021a, b, c, 2022, Zeverdejani and Beni 2020, Mehar *et al.* 2017, Ramteke and Panda 2021) for last century. At the beginning, classical continuum solid mechanics theories have been enhanced for the scales from macro to micro and nano, to investigate deformation processes and phenomena which can be overcomed by the theoretical analysis but in the past decades, they are utilized to define atomistic scales phenomena, too. Theoretical analysis and experimental observations with developed devices like atomic force microscopes and nano-indenters have discussed that local (classical) elasticity models do not adequate for detailed and a correct definition of corresponding item in this regime of nano and micron-scales. Currently, the introduction of new non-classical elasticity theories for theoretical modeling nano-structures has gained the attention of many scholars. Most research efforts for theoretical modeling microstructures and nanostructures are on the basis of Eringen's nonlocal theory (Eringen and Suhubi 1964) of the long-range force between atoms. In fact, this non-classical elasticity theory says that the stress of a point in a body is defined by the strains of all points of the body. Because it has only one scaling parameter, the nonlocal elasticity theory presented by Eringen is frequently utilized to explore the mechanical behavior of nano-scaled structures.

To demonstrate the applicability, Lim *et al.* (2015) employed Euler beam theory and Timoshenko beam theory to examine wave propagation in nano-embeds. Contrary to

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the common nonlocal stress model, this new model predicts stiffening effect for very large wavelengths. Many studies have utilized this theory to show the mechanical properties of nanostructures. The dynamical response of these structures at micro-sized and nano-sized are discrepant from their response at macro-sized because of to the effects of small size parameters and surfaces strain or stress that are not available at macro scales. Consequently, a thorough theoretical investigation of the vibrational response of these structures is of significance in the theoretical investigation and plan of micro-scaled or nano-scaled devices and systems.

Because the conventional elastic theories have been proved to deficiency at micro/nano scale, various atomic simulations and new non-classical (higher-order) elastic models are essential. Over the last years, several non-classical elasticity theories like the couple stress, strain gradient, stress driven nonlocal elasticity, modified couple stress, strain driven nonlocal elasticity theories have been utilized to research the micro/ nanoscaled structures. These theories for instance strain gradient elastic theory (Yaylı 2011), micropolar elasticity theory (Eringen and Suhubi 1964, Chen *et al*. 2004, Ramezani *et al*. 2009), couple stress approach (Toupin 1962), nonlocal elasticity theory (Eringen 1983, Eringen and Edelen 1972, Numanoğlu *et al*. 2021, Yang *et al*. 2010, Civalek *et al*. 2022, Ali *et al*. 2021, Khosravi *et al*. 2020, Dastjerdi and Beni 2019, Khadimallah *et al*. 2021) and the modified couple stress theory (Ma *et al*. 2008, Park and Gao 2006, Mehralian and Beni 2016) have been employed in studying micro/nano structures. In the mentioned non-classical elasticity theories, one or more length parameters are considered in addition to elastic modulus, shear modulus and the Poisson ratio, that enable these theories (Wang *et al*. 2008, Civalek *et al*. 2020a, b, 2021a, b, Schadler *et al*. 1998, Ru 2001, Uzun and Yaylı 2020, Wagner *et al*. 1998, Qian 2000, Uzun *et al*. 2021, Danesh *et al*. 2012, Bower *et al*. 1999, Chang 2012, Sarparast *et al*. 2020, Şimşek 2012, Akgöz and Civalek 2014, Faghidian 2020, Murmu *et al*. 2014, Huang 2012, Demir and Civalek 2017, Numanoğlu *et al*. 2018, Akbaş *et al*. 2021, Ebrahimi *et al*. 2020, Jalaei and Civalek 2019, Yaylı *et al*. 2021, Özarpa and Esen 2020, Abouelregal 2020, Abouelregal and Mohammed 2020) to capable of size effects. Reddy and Pang (2008) have studied Euler-Bernoulli and Timoshenko beam theories via the Eringen's nonlocal elasticity theory. An analytical solutions on the vibrational behaviors of nano-sized tubes and nano-sized beams restrained with elastic boundary conditions have been given by Yayli (2016, 2017). Aydogdu and Elishakoff (2014) have explored the dynamics of nanotubes supported by a spring in-span. Several scientist (Murmu *et al*. 2011, Lim *et al*. 2012, Kiani 2013, Li 2014) have investigated torsional dynamics of nano-scaled tubes in the context of the Eringen's nonlocal theory of elasticity. Roostai and Haghpanahi (2014), Loya *et al*. (2014) have investigated the dynamic response of cracked nanotube. Classic mechanical theories are not effective of capturing the small scale effects; therefore, different size dependent elasticity theories are employed to research the nano or micro structures. These theories, mainly, attempt to enhance the model by using different number of size coefficients to overcome size effects (Ebrahimi and Beni 2016, Esmaeili and Tadi Beni 2019, Guo *et al*. 2016, Feng *et al*. 2021, Delfani 2017, Tadi Beni 2016, Nejadi *et al*. 2021, Akgöz and Civalek 2013, Swadener *et al*. 2002, Hadji and Avcar 2021, Lam *et al*. 2003, Luat *et al*. 2021, Liebold and Mller 2016, Kumar *et al*. 2021, Lei *et al*. 2016, Lazar 2021, Ansari *et al*. 2021, Civalek *et al*. 2020c, Madenci 2021, Forsat *et al*. 2021). With the demonstrated effects of the two alternative gradient theories i.e., stiffening by strain gradient theory and softening by the stress gradient (or Eringen's nonlocal elasticity theory), one can felt crucial to unify both these elasticity theories. It is apparent that the Eringen's higher order elasticity theory known as "nonlocal elasticity theory" has been utilized widely in order to investigate the size dependent vibration, post buckling, static, wave propagation and buckling response of nano-structures.

In this study, using the modified couple stress theory and Fourier sine series, static and free torsional vibration analysis of slender functionally graded nanorods of circular cross section, with non-rigid boundary conditions, is performed. The equations of motion and corresponding end conditions are computed by taking into account the torsional springs at the ends, but the effects of warping of the section is neglected due the section is considered to be circular. Stoke's transformation is an attractive and efficient tool for the construction of the eigen-value problem for torsional dynamic analysis with non-rigid boundary conditions. The deformable boundary conditions twisting direction are replaced with a massless torsional springs whose elastic flexibility are represented by the spring coefficients. Utilizing the proposed mathematical procedure, the systems of equations for the torsional vibrations of such functionally graded nanorods with any two boundary conditions can be calculated from a system of two equations. For the common boundary conditions, namely free-fixed, fixed-fixed and free-free, systems of equations are derived explicitly by giving special values to torsional spring coefficients. Several examples are shown to verify the presented solution for rigid and non-rigid boundary conditions. Moreover, the effects of the functionally graded index and torsional spring coefficients are explored and the numerical results are discussed via a number of figures and tables. It can be said that present method (Stokes' transformation with Fourier sine series) bridges the gap between rigid boundary conditions and deformable end conditions. The present analytical method provide an insight into the linear free torsional vibration of functionally graded nanorods with arbitrary boundary conditions and will be helpful for new researches on this issue.

2. Modified couple stress theory for torsional vibration

In this section, modified couple stress theory which developed by Lam *et al*. (2003) will be reviewed at first. Based on this theory, the strain energy U may be expressed as (Gheshlaghi *et al*. 2010):

$$U = \frac{1}{2}\int_V (\sigma\epsilon + m\chi)\, dV \tag{1}$$

in the above equations, V represents the volume of the body, ϵ denotes the strain, σ represents the stress, χ expresses the curvature and m denotes the deviatoric part of the couple stress.

$$\sigma = \lambda tr(\epsilon)I + 2G\epsilon \tag{2}$$

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \tag{3}$$

$$m = 2l^2 G\chi \tag{4}$$

$$\chi = \frac{1}{2}(\nabla\Theta + (\nabla\Theta)^T) \tag{5}$$

where *tr(ϵ)* is the sum of diagonal parts of the linear strain tensor, λ and G express the Lame's constants. I represents the identity matrix. u represents the displacement field, l denotes the scale parameter, θ is the rotation vector and it can be formulated as:

$$\theta = \frac{1}{2}curl(u) \tag{6}$$

This paper addresses the problem of torsional free vibration of functionally graded nanorod undergoing small rotation about the center of twist. The boundaries of nanorod is assumed to be non-movable in the twisting direction and elastically restrained against torsional rotation. The displacement field is expressed as:

$$u_x(x,t) = 0 \tag{7}$$

$$u_y(x,t) = -z\phi(x,t) \tag{8}$$

$$u_z(x,t) = y\phi(x,t) \tag{9}$$

here u_z, u_y and u_x are the z, y and x constituents of the displacement field. Φ(x,t) is the rotation function. By using Eqs. (3), (7)-(9), the strain relations can be obtained as:

$$\epsilon_{xy} = \epsilon_{yx} = -\frac{z}{2}\frac{\partial\phi(x,t)}{\partial x} \tag{10}$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{y}{2}\frac{\partial\phi(x,t)}{\partial x} \tag{11}$$

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon_{yz} = \epsilon_{zy} = 0 \tag{12}$$

Similarly, the Eqs. (13) and (15) are derived from Eqs. (6)-(9)

$$\Theta_x = \phi(x,t) \tag{13}$$

$$\Theta_y = -\frac{y}{2}\frac{\partial\phi(x,t)}{\partial x} \tag{14}$$

$$\Theta_z = -\frac{z}{2}\frac{\partial\phi(x,t)}{\partial x} \tag{15}$$

By using the relations Eqs. (2), (10)-(12), one can found the stress tensor as follows (Gheshlaghi *et al.* 2010):

$$\sigma = G\begin{bmatrix} 0 & -z\frac{\partial\phi(x,t)}{\partial x} & y\frac{\partial\phi(x,t)}{\partial x} \\ -z\frac{\partial\phi(x,t)}{\partial x} & 0 & 0 \\ y\frac{\partial\phi(x,t)}{\partial x} & 0 & 0 \end{bmatrix} \tag{16}$$

Here, G denotes the shear modulus. Similarly, using Eqs. (5), (13)-(15), deviatoric couple stress part found as (Gheshlaghi *et al.* 2010):

$$m = Gl^2\begin{bmatrix} 2\frac{\partial\phi(x,t)}{\partial x} & -\frac{y}{2}\frac{\partial^2\phi(x,t)}{\partial x^2} & -\frac{z}{2}\frac{\partial^2\phi(x,t)}{\partial x^2} \\ -\frac{y}{2}\frac{\partial^2\phi(x,t)}{\partial x^2} & -\frac{\partial\phi(x,t)}{\partial x} & 0 \\ -\frac{z}{2}\frac{\partial^2\phi(x,t)}{\partial x^2} & 0 & -\frac{\partial\phi(x,t)}{\partial x} \end{bmatrix} \tag{17}$$

Approaches to the problem of free vibration of nanorods have gained renewed interest due to the development of functionally graded materials that can withstand deformable boundary conditions and small-scale effects. Functionally graded materials are formed by combining at least two different component phases whose volume content varies continuously and smoothly in one, two or more directions according to coefficient of a distribution law. Hamilton's principle can be expressed as:

$$\delta\int_{t_1}^{t_2}(U - T - W)dt = 0 \tag{18}$$

In Eq. (18), U represents the strain energy, T specifies the kinetic energy and W expresses the work done by external force, respectively.

$$U = \frac{1}{2}\int_V \left[G(y^2 + z^2 + 3l^2)\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2 + \frac{1}{4}Gl^2(y^2 + z^2)\left(\frac{\partial^2\phi(x,t)}{\partial x^2}\right)^2\right]dV \tag{19}$$

The following relation is obtained by using the polar inertia moment $I_p = \int_A (y^2 + z^2)dA$

$$U = \frac{1}{2}\int_0^L \left[G\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2 (I_p + 3l^2 A) + \frac{1}{4}Gl^2\left(\frac{\partial^2\phi(x,t)}{\partial x^2}\right)^2 I_p\right]dx \tag{20}$$

where, L is the length of nanorod. The task computed here seems to be difficult in the presence of functionally graded nanorod where the boundary conditions are deformable, the kinetic energy can be formulated as:

$$T=\frac{1}{2}\int_{0}^{L}\rho I_p\left(\frac{\partial^2\phi(x,t)}{\partial x^2}\right)^2 dx \tag{21}$$

and the work done can be expressed as:

$$W=\int_{0}^{L} m_t\phi(x,t)\,dx \tag{22}$$

In the above equation, ρ and m_t indicate the mass density of the nanorod and torsional moment, respectively. Inserting the strain energy, the kinetic energy and work done by the torsional moment in Eqs. (20)-(22), respectively, into the Hamilton's principle in Eq. (18), the equation of motion has been obtained. Taking the variation of this equation $\left(\delta\int_{t_1}^{t_2}(U-T-W)dt\right)$

$$\begin{aligned}&\delta\int_{t_1}^{t_2}\int_{0}^{L}\left[\begin{array}{l}\frac{1}{4}\frac{\partial^2}{\partial x^2}\left(Gl^2I_p\frac{\partial^2\phi(x,t)}{\partial x^2}\right)\\ -\frac{\partial}{\partial x}\left(GI_p+3GAl^2\right)\frac{\partial\phi(x,t)}{\partial x}\\ +\frac{\partial}{\partial t}\left(\rho I_p\frac{\partial\phi(x,t)}{\partial t}\right)-m_t\end{array}\right]\delta\phi dxdt\\ &\qquad+\int_{t_1}^{t_2}\Bigg[\left(GI_p+3GAl^2\right)\frac{\partial\phi(x,t)}{\partial x}\\ &\qquad-\frac{1}{4}\frac{\partial}{\partial x}\left(Gl^2I_p\frac{\partial^2\phi(x,t)}{\partial x^2}\right)\delta\phi\\ &\qquad+\frac{1}{4}Gl^2I_p\frac{\partial^2\phi(x,t)}{\partial x^2}\delta\left(\frac{\partial\phi}{\partial t}\right)\Bigg]_0^L dt\\ &\qquad+\int_{0}^{L}\left[\rho I_p\frac{\partial\phi(x,t)}{\partial t}\delta\phi\right]_{t_1}^{t_2}dx=0\end{aligned} \tag{23}$$

Then the Euler Lagrange equation is found as:

$$\varGamma_1\frac{\partial^4\phi(x,t)}{\partial x^4}-\varGamma_2\frac{\partial^2\phi(x,t)}{\partial x^2}+\varGamma_3\frac{\partial^2\phi(x,t)}{\partial t^2}=m_t \tag{24}$$

Eq. (24) is the free torsional vibration equation of a homogeneous nanoscale / microscale rod according to the modified stress couple theory. Here,

$$\varGamma_1=\frac{l^2GI_p}{4} \tag{25}$$

$$\varGamma_2=GI_p+3Gl^2A \tag{26}$$

$$\varGamma_3=\rho I_p \tag{27}$$

Considering the above expressions, it should be highlighted that if the expressions containing the length scale parameter are removed from the equation, the equation is simplified to the torsional vibration equation of a classical homogeneous nanorod. The initial conditions are expressed as Eq. (28);

$$\left(\frac{\partial\phi(x,t_2)}{\partial t}\right)\delta\phi(x,t_2)-\left(\frac{\partial\phi(x,t_1)}{\partial t}\right)\delta\phi(x,t_1)=0 \tag{28}$$

In the current analytical model, there are two end points with one degree of freedom per node for torsion of elastic spring parameter. Consequently, the boundary conditions are also formulated as:

$$\varGamma_2\left(\frac{\partial\phi(x,t)}{\partial x}\right)-\varGamma_1\left(\frac{\partial^3\phi(x,t)}{\partial x^3}\right)=0\ \ or\ \ \phi(x,t)=0 \tag{29}$$

$$\frac{\partial^2\phi(x,t)}{\partial x^2}=0\ \ or\ \ \frac{\partial\phi(x,t)}{\partial x}=0 \tag{30}$$

By reviewing the above two expressions, it is important to note that it can be possible to construct the eigen value analysis for linear free torsional vibration analysis of functionally graded nanorod by given proper values to deformable boundary conditions. Eq. (24) defines the torsional motion of homogeneous nanorod related to rotation function via the modified couple stress theory (Gheshlaghi *et al*. 2010). Provided that the nanorod is circular, the above relation is independent of the geometry.

3. Formulations of FG nanorod

Functionally graded materials have been extensively used in automotive, aerospace and naval engineering areas due to stiffness to weight and high strength to weight ratios. As it is said before, the material properties of the functionally graded nanorod alter in the direction of the radius. Therefore, the Young's modulus (E), shear modulus and mass density of the functionally graded nanorod need to be defined as a function dependent on the radius, i.e.r. In this work, following relations are used to represent the functionally graded material:

$$E(r)=(E_c-E_m)\left(\frac{r}{R}\right)^p+E_m \tag{31a}$$

$$\rho(r)=(\rho_c-\rho_m)\left(\frac{r}{R}\right)^p+\rho_m \tag{31b}$$

$$G(r)=\frac{E(r)}{2(1+\nu)} \tag{31c}$$

It is known that the relation between the shear modulus and the Young's modulus is dependent on the Poisson's ratio (ν). In this study, the shear modulus of functionally graded nanorod which the material properties change in the radius direction, is calculated via the changing Young's modulus and Poisson's ratio, as in Eq. (31c). Here, the indices (c and m) given in the equations, represent the ceramic and metal components, respectively. In addition, the Poisson's ratio is not given with a indice (c or m). Because in this study, the Poisson's ratios of the components that form the functionally graded nanorod are equal. Also, p represents the power law index and R denotes the radius of functionally graded nanorod. The power law index is a parameter that affects the variation of the material distribution. And this parameter can be given numerically positive values from 0 to infinity. Whether the power law index is small or large determines the response of the functionally graded nanorod. Using the functionally graded

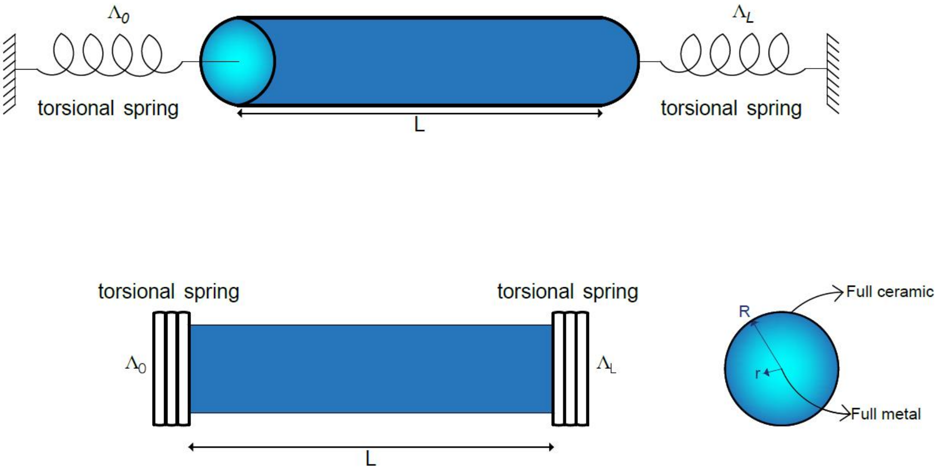


Fig. 1 A functionally graded nanorod restrained with torsional springs at both ends

material changes frequently used in the literature above; The torsional stiffnesses can be written as:

$$G(r)I_p = \frac{\pi}{1+\nu}\int_0^R ((E_c - E_m)\left(\frac{r}{R}\right)^p + E_m)r^3dr \tag{32a}$$

$$G(r)A = \frac{\pi}{1+\nu}\int_0^R ((E_c - E_m)\left(\frac{r}{R}\right)^p + E_m)rdr \tag{32b}$$

$$\rho(r)I_p = 2\pi\int_0^R ((\rho_c - \rho_m)\left(\frac{r}{R}\right)^p + \rho_m)r^3dr \tag{32c}$$

This section of the study is dedicated to presenting the solution equations for torsional vibration based on the modified couple stress theory of a functionally graded nanorod with elastic torsional constraints at both ends (see Fig. 1). The concept of the presented method is to gain a coefficient matrix for eigen-value solution including the torsional spring parameters. In order to define the coefficients in Eqs. (32a)-(32c) by using functionally graded formulations, following parameters are defined:

$$\bar{\Gamma}_1 = \frac{l^2G(r)I_p}{4} \tag{33a}$$

$$\bar{\Gamma}_2 = G(r)I_p + 3G(r)l^2A \tag{33b}$$

$$\bar{\Gamma}_3 = \rho(r)I_p \tag{33c}$$

3.1 Modal displacement function

Surprisingly, dynamical response of a functionally graded nanorod with deformable boundary conditions subjected to free vibration has received much less attention. Results on free vibration behavior of nanorods reported in the literature are confined to free-fixed support and fixed-fixed boundary condition. To derive the explicit terms of the unknown constants, namely Fourier coefficients, require to be defined with the equation of motion of functionally graded nano-scaled rod. In the present study, Fourier sine series is used to obtain the torsional vibration frequencies of functionally graded nanorod. Thanks to method separation of variables, $\phi(x,t)$ may be reformulated as the following form:

$$\phi(x,t) = \psi(x)e^{i\omega t} \tag{34}$$

here, $\psi(x)$ denotes the rotation function about the center of twist and ω expresses the torsional frequency of FG nanorod. To obtain the free torsional vibration equation representing the behavior of the functionally graded nanorod within the context of the modified couple stress theory, the expressions Γ_1, Γ_2, Γ_3 in Eq. (24) are replaced by Eqs. (33a)-(33c). By substituting the above equation into the free torsional vibration equation representing the behavior of the functionally graded nanorod below equation is obtained

$$\bar{\Gamma}_1\frac{\partial^4\psi(x)}{\partial x^4} - \bar{\Gamma}_2\frac{\partial^2\psi(x)}{\partial x^2} + \bar{\Gamma}_3\omega^2\psi(x) = m_t \tag{35}$$

In order to design functionally graded nanoscale rods restrained with deformable boundaries correctly, it is necessary to have sufficient and consistent knowledge about the torsional vibration behavior of these structures. The angular rotation around the center of twist, denoted by $\psi(x)$, is defined in the study as follows:

$$\psi(x) = \left[\begin{array}{ll} \psi_0 & x = 0 \\ \psi_L & x = L \\ \sum_{n=1}^{\infty} C_n sin(\alpha_n x) & 0 < x < L \end{array}\right] \tag{36}$$

in which

$$\alpha_n = \frac{n\pi}{L} \tag{37}$$

3.2 Stokes' transformation

The torsional vibrational behaviors of nano structures have been widely investigated by scholars. But, these investigations are about the homogeneous nano structures and/or rigid boundary conditions. When the scientific literature is scanned, it is understood that there are few studies on the torsional vibration behavior of nanoscale structures with torsion springs. In this study, as mentioned before, the effects of torsion spring coefficients are investigated. For this reason, the Stokes transform is employed to the governing equation of the problem and boundary conditions. The unknown Fourier coefficients (C_n) seen in Eq. (36) can be specified as follows.

$$C_n = \frac{2}{L}\int_0^L \psi(x) sin(\alpha_n x)dx \tag{38}$$

A series approach based on the Fourier sine series expansions will be presented here to exactly calculate the torsional dynamic and static response for the linear functionally graded nanorods with the deformable boundaries. First derivative of Eq. (36) may be calculated as below:

$$\psi'(x) = \sum_{n=1}^{\infty} \alpha_n C_n cos(\alpha_n x) \tag{39}$$

Eq. (39) can be given by a cosine series as follows:

$$\psi'(x) = \frac{f_0}{L} + \sum_{n=1}^{\infty} f_n cos(\alpha_n x) \tag{40}$$

The explicit expressions of two Fourier constants (f_0 and f_n) given in Eq. (40) are defined as follows.

$$f_0 = \frac{2}{L}\int_0^L \psi'(x)dx = \frac{2}{L}\big(\psi(L) - \psi(0)\big) \tag{41}$$

$$f_n = \frac{2}{L}\int_0^L \psi'(x) cos(\alpha_n x)dx \qquad (n = 1,2,\dots) \tag{42}$$

the f_n coefficient is derived by integrating by parts of using Eq. (42)

$$f_n = \frac{2}{L}[\psi(x) cos(\alpha_n x)]_0^L + \frac{2}{L}\left(\alpha_n \int_0^L \psi(x) sin(\alpha_n x)dx\right) \tag{43}$$

$$f_n = \frac{2}{L}\big((-1)^n \psi(L) - \psi(0)\big) + \alpha_n C_n \tag{44}$$

To determine the torsional natural frequencies, a linear eigenvalue problem is formulated by separating the equation of motion about the the center of twist and solved using Fourier series expansions.

3.3 Torsional static analysis

The present eigen-value solution procedure (Stokes' transformation and Fourier series) is sufficient when dealing with functionally graded nanorods with deformable boundary conditions (torsional elastic springs) at both ends. The first-fourth derivatives of $\psi(x)$ may be obtained as:

$$\frac{d\psi(x)}{dx} = \frac{\psi_L - \psi_0}{L} + \sum_{n=1}^{\infty} cos(\alpha_n x)\left(\frac{2((-1)^n \psi_L - \psi_0)}{L} + \alpha_n C_n\right) \tag{45}$$

$$\frac{d^2\psi(x)}{dx^2} = -\sum_{n=1}^{\infty} \alpha_n sin(\alpha_n x)\left(\frac{2((-1)^n \psi_L - \psi_0)}{L} + \alpha_n C_n\right) \tag{46}$$

$$\frac{d^3\psi(x)}{dx^3} = \frac{\psi_L'' - \psi_0''}{L} + \sum_{n=1}^{\infty} cos(\alpha_n x)\left(\frac{2((-1)^n \psi_L'' - \psi_0'')}{L} - \alpha_n^2\left(\frac{2((-1)^n \psi_L - \psi_0)}{L} + \alpha_n C_n\right)\right) \tag{47}$$

$$\frac{d^4\psi(x)}{dx^4} = -\sum_{n=1}^{\infty} \alpha_n sin(\alpha_n x)\left(\frac{2((-1)^n \psi_L'' - \psi_0'')}{L} - \alpha_n^2\left(\frac{2((-1)^n \psi_L - \psi_0)}{L} + \alpha_n C_n\right)\right) \tag{48}$$

One of the steps to realize the solution involves determining the Fourier coefficients which simultaneously meet the governing equation. Therefore, substituting Eqs. (36), (46) and (48) into Eqs. (35), the Fourier coefficient C_n: written in terms of ψ_0, ψ_L, ψ_0'' and ψ_L'' as follows:

$$C_n = \frac{2n\pi(\bar{\Gamma}_2 L^2(-1)^{n+1}\psi_L + \psi_0)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} + \frac{2n\pi\left(\bar{\Gamma}_1\left(n^2\pi^2((-1)^{n+1}\psi_L + \psi_0)\right)\right)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} - \frac{2n\pi L^2(\bar{\Gamma}_1(-1)^{n+1}\psi_L'' + \psi_0'')}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} \tag{49}$$

It has to be notified that Fourier coefficients in Eq. (49) are functions of the torsional rotations of degree of freedom and the angular rotation about the center of twist is formulated as:

$$\psi(x) = \sum_{n=1}^{\infty}\left(\frac{2n\pi(\bar{\Gamma}_2 L^2(-1)^{n+1}\psi_L + \psi_0)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} + \frac{2n\pi\left(\bar{\Gamma}_1\left(n^2\pi^2((-1)^{n+1}\psi_L + \psi_0)\right)\right)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} - \frac{2n\pi L^2(\bar{\Gamma}_1(-1)^{n+1}\psi_L'' + \psi_0'')}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1}\right) sin(\alpha_n x) \tag{50}$$

The expression (50) of the angle of rotation about the center of twist may be therefore obtained as a rotation function of the torsion. These mentioned Fourier coefficients can be expressed by enforcing the left and right ends torsional springs. Fourier coefficients are received based on the modified couple stress theory here, which may be simplified to those for the local elasticity theory. Actually, the Fourier coefficients of classical elasticity theory may be obtained with $l = 0$. Using the above equation and the exponential material change function of FG material in Eqs. (31a)-(32c), a static analysis of the torsion problem can be done depending on the scale parameter. In addition, it is possible to make static calculations with the given formulation in Eq. (50) without any torsion moment by giving different rotational values to the displacement parameters in the boundaries. In the literature, there is no algorithm capable of performing static analysis by giving deformable values to the boundary conditions. This is just one of the features that show the originality of this work.

3.4 Torsional vibration formulation according to deformable boundary conditions

From the literature review, it is seen that there is no research regarding the effects of deformable boundary conditions on the vibrational response of functionally graded nanorods based modified couple stress theory. In this study, a circular functionally graded nanorod restrained with torsional elastic springs at both ends seen in Fig. 1 is examined. The purpose of presented study is to obtain a general solution to the torsional vibration problem of the functionally graded nano scale circular rod. For this, the force boundary conditions given below should be used.

$$\bar{\Gamma}_2 \frac{d\psi}{dx} - \bar{\Gamma}_1 \frac{d^3\psi}{dx^3}\bigg|_{x=0} = \Lambda_0 \psi_0 \tag{51}$$

$$\bar{\Gamma}_2 \frac{d\psi}{dx} - \bar{\Gamma}_1 \frac{d^3\psi}{dx^3}\bigg|_{x=L} = \Lambda_L \psi_L \tag{52}$$

$$\psi_0''\big|_{x=0} = 0 \tag{53}$$

$$\psi_L''\big|_{x=L} = 0 \tag{54}$$

here Λ_0 and Λ_L denote the torsional stiffnesses of the torsional springs attached to both ends of the functionally graded nano rod. Inserting Eqs. (45), (47) and (49) into Eqs. (51)-(54) causes to the two simultaneous homogeneous equations as follows:

$$\left(-\frac{\bar{\Gamma}_2}{L} - \Lambda_0 + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1}\right)\psi_0 + \left(\frac{\bar{\Gamma}_2}{L} + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L(-1)^{n+1}\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1}\right)\psi_L = 0 \tag{55a}$$

$$\left(\frac{\bar{\Gamma}_2}{L} + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L(-1)^{n+1}\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1}\right)\psi_0 + \left(-\frac{\bar{\Gamma}_2}{L} - \Lambda_L + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1}\right)\psi_L = 0 \tag{55b}$$

With the above homogeneous equations, an eigenvalue problem involving both the size effect and the torsional spring constants can be established.

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_L \end{bmatrix} = 0 \tag{56}$$

In this work, the Stokes' transformation and a Fourier series are superposed to overcome the angular rotation around the center of twist and torsion constraints at both ends of the functionally graded nano/micro rod. With this presented solution method, all boundaries between free and clamped ends can be examined. In addition, rigid boundary conditions may be achieved by giving appropriate stiffness values to the torsional spring parameters. Such that it would be convenient to set $\Lambda_0 = 0$ and $\Lambda_L = 0$ to obtain free boundary conditions at x = 0 and x = L. Or, at x = 0 and x = L, the clamped boundary conditions can be calculated with $\Lambda_0 = \infty$ and $\Lambda_L = \infty$. The torsional vibration frequencies of the functionally graded micro/nano rod are calculated by setting the determinant of the coefficient matrix to zero after adjusting the torsional spring parameters.

$$\left|\Psi_{ij} = 0\right| \text{ (i,j = 1,2)}, \tag{57}$$

in which

$$\Psi_{11} = -\frac{\bar{\Gamma}_2}{L} - \Lambda_0 + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} \tag{58a}$$

$$\Psi_{12} = \frac{\bar{\Gamma}_2}{L} + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L(-1)^{n+1}\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} \tag{58b}$$

$$\Psi_{21} = \frac{\bar{\Gamma}_2}{L} + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L(-1)^{n+1}\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} \tag{58c}$$

$$\Psi_{22} = -\frac{\bar{\Gamma}_2}{L} - \Lambda_L + \sum_{n=1}^{\infty} \frac{2\bar{\Gamma}_3 L\omega^2(\bar{\Gamma}_2 L^2 + n^2\pi^2\bar{\Gamma}_1)}{\bar{\Gamma}_3 L^4(-\omega^2) + n^2\pi^2\bar{\Gamma}_2 L^2 + n^4\pi^4\bar{\Gamma}_1} \tag{58d}$$

In this section, all the necessary formulations to provide the solution are extracted and finally a coefficients matrix is obtained. As can be seen, this matrix of coefficients includes the length scale parameter to include the small size effect in the analysis, the power law index expressing the material change and the torsion spring parameters expressing the boundary conditions. There is a need to highlight a few important points here regarding the coefficients matrix. If we set the length scale parameter to zero in the coefficients matrix and perform the analysis, we obtain the torsional vibration frequencies of a classical functionally graded rod. In addition to this, if the power law index is zero or infinitely large, the problem is simplified to the torsional vibration of a homogeneous nano/micro rod. If these two points are considered together on the coefficients matrix, the torsional vibration behavior of a classical homogeneous rod is given. Another important point for the solution method based on this presented eigenvalue problem is that the stiffnesses of the torsional springs at both ends of the functionally graded nanorod do not have to be the same values. Both torsional spring stiffnesses can be taken in different values and included in the solution. As it can be understood from here, a very large tolerance is provided regarding the boundary conditions and the solution range is greatly expanded. Solutions can be provided by giving desired values from 0 to infinity for any or both of the torsional spring parameters.

4. Numerical applications and validations

Via the mathematical steps and eigen value procedure derived in the previous section, a number of representative numerical examples are solved to discussed the sufficieny of the developed solution method. Free torsional frequencies of the functionally graded nano/micro rod are calculated by solving Eq. (57). In Eq. (57), ω values give the torsional frequencies. Firstly, validity and the efficiency of the present solution method is checked. Then, the influences of material length scale parameter, power law index and torsional springs on the free torsional vibration behaviors of the FG nano rods are examined.

4.1 Comparison study

The free torsional vibrational frequencies based on the local elasticity theory for a fixed-fixed homogeneous rod is formulated as Eq. (59) (Gorman 1975):

$$\varpi\_n = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}} \tag{59}$$

in which, ϖ_n represents the free torsional vibration frequency in classical elasticity, the index (n) denotes the mode number. Polar moment of inertia and area of a circular hollow section are formulated as follows

$$A = \pi(R_2^2 - R_1^2) \qquad I_p = \frac{\pi}{2}(R_2^4 - R_1^4) \tag{60}$$

In order to compare the results in the Eq. (59) the homogeneous material properties are taken as ρ = 7800 kg/m^3 and G = 79.3 GPa. Inner radius R_1 = 0.0 mm, outer radius is R_2 = 50.0 mm. As can be understood from the numerical values of the inner and outer radius, the section has transformed into a solid circular section. The length of rod is taken as L = 1000 mm. Also the length scale parameter l^2 and functionally graded index are assumed to be zero. Thus, the element to be examined has come to the form of a classical rod element, which is homogeneous and independent of the size effect.

The mathematical results obtained for clamped boundary conditions at both ends by taking the torsional spring parameters ($\Lambda_0 = 10\times10^{10}$ N/mm and $\Lambda_L = 10\times10^{10}$ N/mm). The accuracy of the present method is tested by comparing the literature (Gorman 1975, Tabassian 2013). By using different truncated term, present formulation is compared with those predicted by prismatic bar theory ($\varpi = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}}$). The competency and precision of this method are assessed in Table 1 examples of homogeneous nanorod with clamped-clamped ends with ($\Lambda_0 = 10 \times 10^{10}$ N/mm and $\Lambda_L = 10 \times 10^{10}$ N/mm), which indicates the correctness of the proposed analytical procedure.

Similarly, the frequency response in modified couple stress theory is calculated exactly for clamped-clamped boundary condition from the following formulation (Gheshlaghi *et al.* 2010):

$$\omega\_n = \frac{\pi n}{L}\sqrt{\frac{Gl^2n^2\pi^2}{4\rho L^2} + \frac{3GAl^2}{\rho I_p} + \frac{G}{\rho}} \tag{61}$$

Table 1 First three torsional frequencies of the clamped-clamped boundary conditions

Number of terms	$\varpi_1 x \frac{L}{\sqrt{\frac{G}{\rho}}}$	$\varpi_2 x \frac{L}{\sqrt{\frac{G}{\rho}}}$	$\varpi_3 x \frac{L}{\sqrt{\frac{G}{\rho}}}$
10	3.141587	6.283175	9.424763
100	3.141587	6.283175	9.424763
180	3.141587	6.283175	9.424763
200	3.141587	6.283175	9.424763
Tabassian (2013)	3.141592	6.283185	9.424777
Gorman (1975)	3.141592	6.283185	9.424777

Eq. (61) can be rewritten for clamped-free ends (Gheshlaghi *et al.* 2010).

$$\omega\_n = \frac{\pi(2n-1)}{2L}\sqrt{\left(\frac{(2n-1)}{2}\right)^2\frac{Gl^2\pi^2}{4\rho L^2} + \frac{3GAl^2}{\rho I_p} + \frac{G}{\rho}} \tag{62}$$

in which, the free torsional vibration frequencies of a homogeneous nanorod can be defined as the parameter ω_n, (n = 1,2,3,4, ...). The subscript n demonstrates the mode number. Table 2 shows a convergence study for a homogeneous classical rod with one end fixed (clamped) and one end free (CF). The material properties of the functionally graded nanorod are used for this comparison. In this presented method, it is important how many terms the solution is realised. The higher number of terms in the calculations, the greater the convergence of the result. Since this comparison study is performed for a homogeneous rod, the power law index p and length scale parameter l are taken as zero. The value of the torsional spring stiffness at the left end of the nanorod is chosen as 10^{14} nN/nm to meet the fixed support condition. The value of the torsional spring stiffness at the right end of the nanorod is chosen as 10^{-14} nN/nm to provide the free end condition. The number of terms is considered in various values from 100 to 150. As can be seen from the table, the values presented are in very good agreement with the exact result. Here again, it should be noted that with the solution made in higher terms, the results can be equalized one-to-one. In order to better understand the convergence, the number of various terms and percentage difference changes are shown in Fig. 5. As can be seen from Fig. 5, the difference decreases below 0.2 percent from the solution with 110 term. This difference is almost negligible and is sufficient for a solution.

4.2 Numerical examples

This sub-section of the study presents the analysis provided by the solution method derived from mathematical expressions in the previous sections. For this, two components that combine the functionally graded nanorod, one ceramic and the other metal, are chosen. The outer surface of the functionally graded nanorod is considered as full-ceramic, while the inner surface is assumed to be full-metal. The region between the inner and outer surfaces may be called the transition zone. In this region, the nanorod cannot be said to be fully ceramic or fully metal. The nanorod shows both metal and ceramic properties here. The properties of ceramic and metal components to be used in

Table 2 Convergence study for clamped-free rod for $p = 0$ and $l = 0$

Mode Number	Number of terms 100	110	120	130	140	150	Gorman (1975)
1	1.57397	1.57368	1.57344	1.57324	1.57307	1.57291	1.570796327
2	4.72191	4.72105	4.72033	4.71972	4.7192	4.71874	4.712388980
3	7.86985	7.86841	7.86721	7.8662	7.86533	7.86457	7.853981634
4	11.0178	11.0158	11.0141	11.0127	11.0115	11.0104	10.99557429
5	14.1657	14.1631	14.1610	14.1592	14.1576	14.1562	14.13716694

Table 3 Convergence study for clamped-clamped rod for $p = 0$ and $l = 0$

Mode Number	Torsional spring stiffness 10^4	10^5	$5*10^6$	10^7	10^8	10^{10}	Gorman (1975)
1	3.05256	3.13245	3.14141	3.1415	3.14158	3.14159	3.141592654
2	6.10560	6.2649	6.28282	6.2830	6.28317	6.28319	6.283185307
3	9.15961	9.39735	9.42423	9.4245	9.4275	9.42478	9.424777961
4	12.2150	12.5298	12.5656	12.566	12.5663	12.5664	12.56637061
5	15.2722	15.6623	15.7070	15.7075	15.7079	15.7080	15.70796327

Table 4 Variation of dimensionless torsional frequencies of CC FG nanorod for various lengths and length scale parameter (p = 1.0) ($\Lambda_0 = \Lambda_L = 10^{14}$)

Scale parameter	Length of FG nanorod 4R	6R	8R	10R	12R
0.0	2.99771	2.99771	2.99771	2.99771	2.99771
0.5	3.48895	3.48205	3.47963	3.47851	3.47790
1.0	4.66178	4.64109	4.63383	4.63046	4.62863
1.5	6.13715	6.10176	6.08933	6.08357	6.08043
2.0	7.74403	7.69416	7.67663	7.66850	7.66408

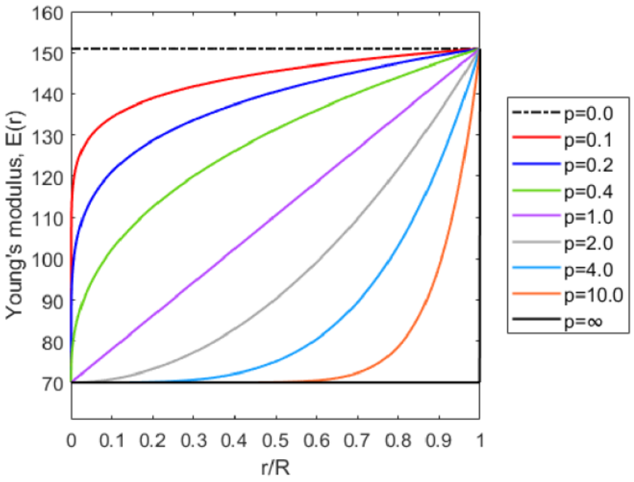


Fig. 2 Variation of Young's modulus $E(r)$ versus r/R ratio

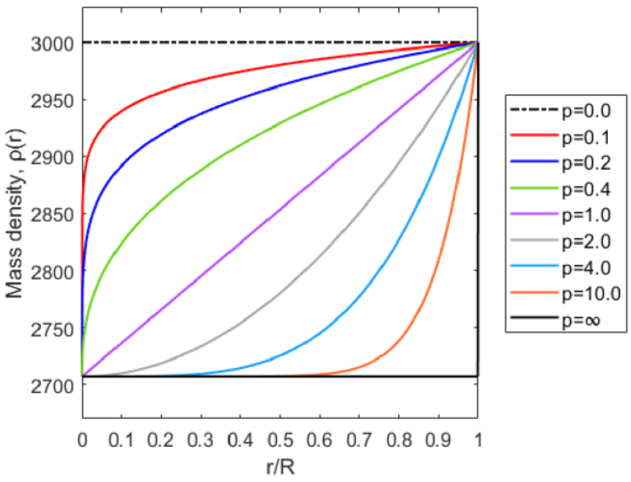


Fig. 3 Variation of mass density $p(r)$ versus r/R ratio

the analysis are as follows: The properties of ceramic and metal components to be used in the analysis are as follows: $E_c = 151$ GPa, $E_m = 70$ GPa, $\rho_c = 3000$ kg/m^3, $\rho_m = 2707$ kg/m^3, $\nu_c = \nu_m = 0.3$.

In this study, as mentioned earlier, the material change release along the radius direction of the FG nanorod according to the power law distribution. Figs. 2, 3 and 4 are plotted to show the effect of this variation rule on material properties. Figs. 2, 3, 4 show the Young's modulus, density and shear modulus variation of the FG nanorod, respectively, versus the r/R ratio for various power law indexes. Power law indexes are chosen as 0, 0.1, 0.2, 0.4, 1, 2, 4, 10, ∞. The variation of the shear modulus given in Fig. 4 is plotted using Eq. (31c). One of the most notable points in

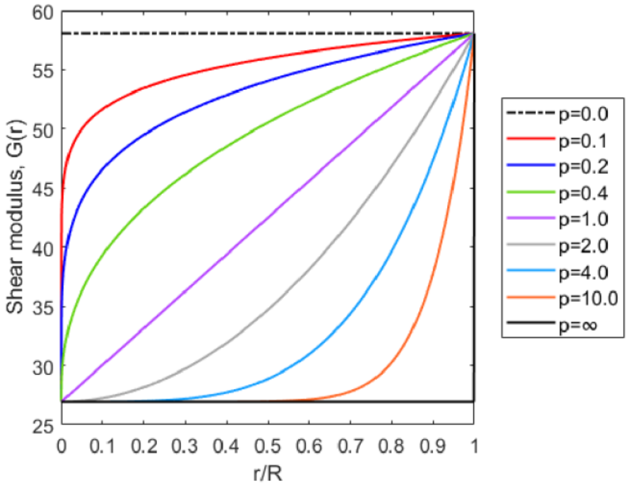
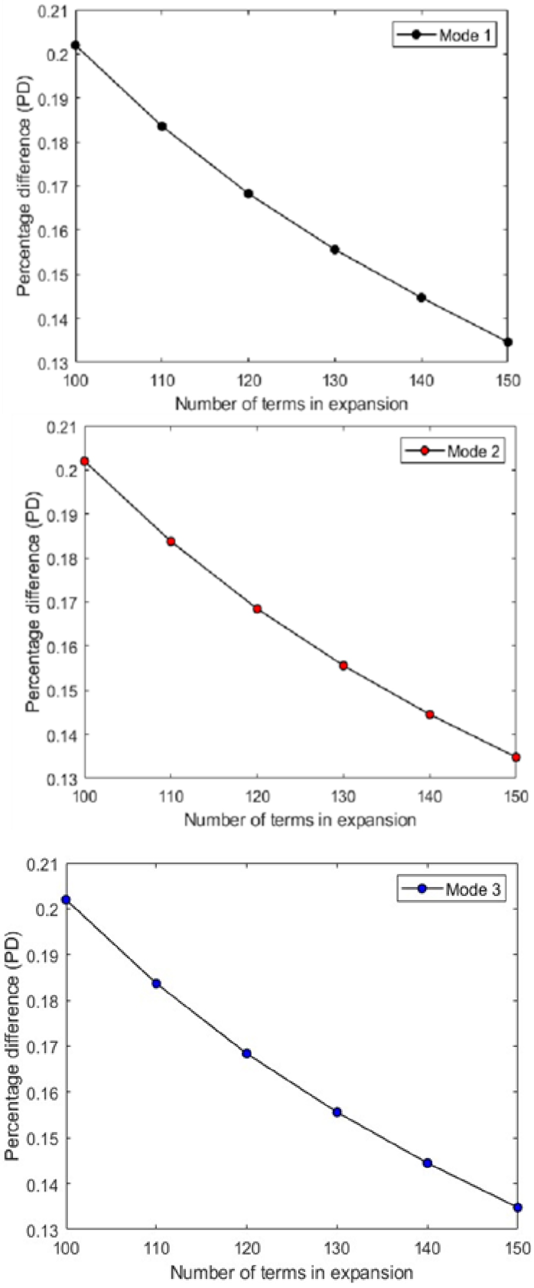


Fig. 4 Variation of shear modulus $G(r)$ versus r/R ratio



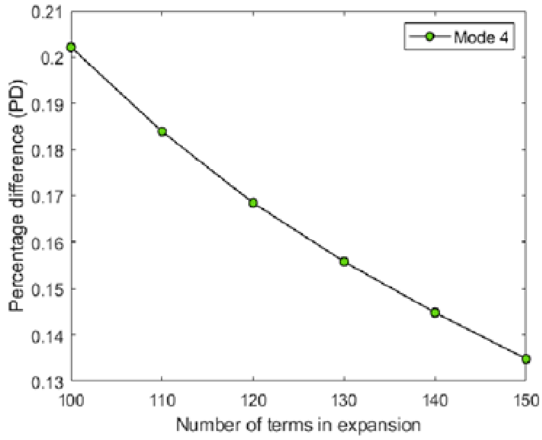


Fig. 5 A convergence check study for variation of dimensionless torsional frequencies

Table 5 Variation of dimensionless torsional frequencies of CF FG rod for various lengths and length scale parameter (p = 1.0) ($\Lambda_0 = 10^{14}$, $\Lambda_L = 10^{-14}$)

Scale parameter	Length of FG nanorod				
	4R	6R	8R	10R	12R
0.0	1.50102	1.50102	1.50102	1.50102	1.50102
0.5	1.74225	1.74144	1.74115	1.74101	1.74094
1.0	2.31991	2.31756	2.31671	2.31631	2.31609
1.5	3.04836	3.04441	3.04297	3.04228	3.04191
2.0	3.84281	3.83727	3.83525	3.83429	3.83377

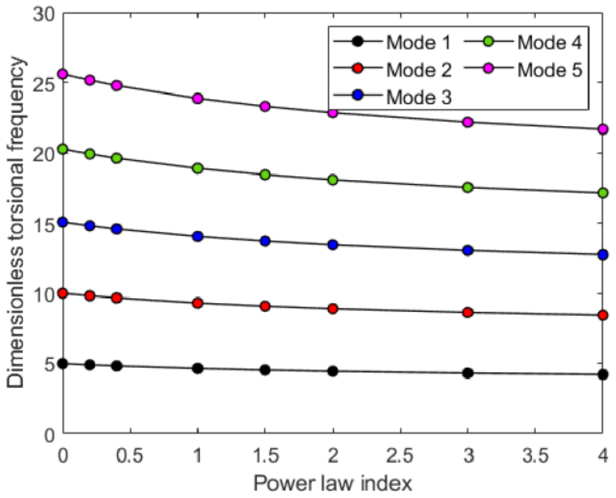


Fig. 6 Variation of dimensionless torsional frequencies of FG nanorod versus power law index (CC)

these figures is that giving zero or infinity values to the power law index reduces the properties of the FG nanorod to a single component. Based on the figures., it can be observed that; If 0 is selected for the power law index, the nanorod shows fully ceramic properties, while if the power law index is selected as infinite, the FG nanorod displays fully metal properties. When we look at the curves plotted in the figures, it is understood that while the functionally graded nanorod exhibits ceramic dominant properties at

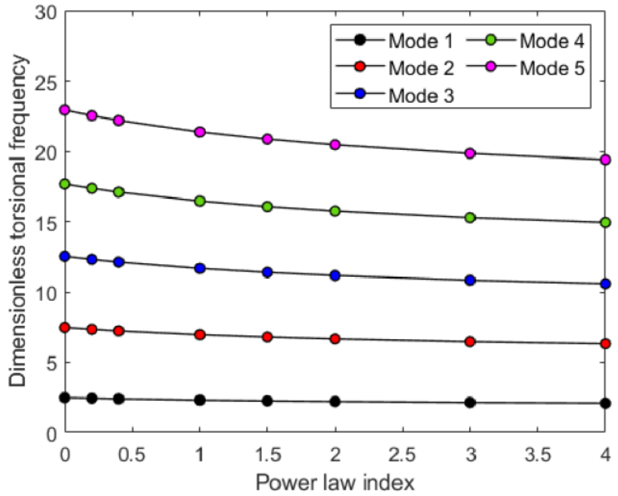


Fig. 7 Variation of dimensionless torsional frequencies of FG nanorod versus power law index (CF)

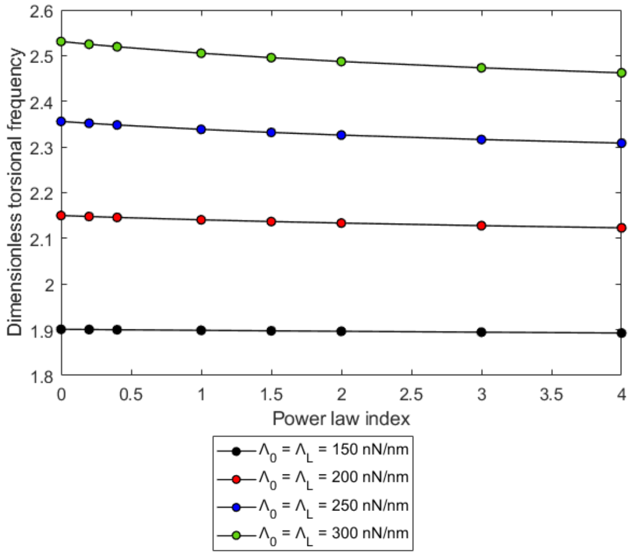


Fig. 8 Variation of dimensionless torsional frequencies of FG nanorod versus power law index for various spring parameters

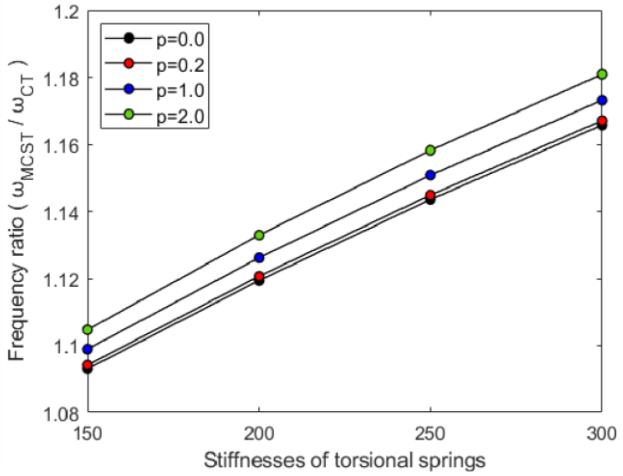


Fig. 9 Variation of frequency ratio of FG nanorod versus stiffnesses of torsional springs

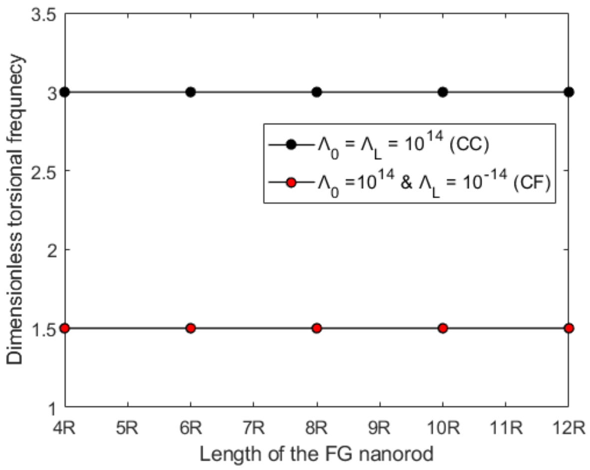


Fig. 10 Variation of dimensionless frequency of FG nanorod versus length for clamped-clamped and clamped-free boundary conditions

lower power law index values, metal dominant features at higher power law index values.

In Table 3, the various values given to the torsional spring parameters and the dimensionless torsional vibration frequencies of the nanorod with fixed (clamped) support at both ends (CC) are compared. The purpose of this study is to show the selection of the correct spring stiffness in order to provide the fixed boundary condition. Various spring stiffness values ranging from 10^4 to 10^{10} are selected for this comparison. As can be seen from the table, when it reaches 10^{10}, the clamped support condition is met. If we give the torsion spring stiffness values of 10^{10} and above, that is, infinitely large values, we will meet the fixed support condition.

Figs. 6 and 7 show the dimensionless vibrational frequencies of FG nanorods with CC and CF boundary conditions, respectively, for the first 5 modes. These figures are drawn to examine the effect of the power rule exponent on torsional frequencies. For this numerical study, the small scale parameter value is selected as 1 nm, while the nanorod length is chosen as L = 10R. In Figure 6, spring stiffnesses of 10^{14} at both ends are chosen to obtain the frequencies of the CC FG nanorod. In Fig. 7, the frequencies of the CF FG nanorod are obtained by giving the stiffness of the spring on the left 10^{14} and the stiffness of the right spring 10^{-14}. In addition, while obtaining the results in these figures, the material length scale parameter is considered as $l = 1$ nm. It is clear from the examined figures that the increase in the power law index causes a decrease in the dimensionless frequency values of the CC and CF FG nanorods. In other words, the nanorod properties begin to show metal dominant properties, resulting in a decrease in frequencies. Fig. 8 demonstrates the dimensionless torsional frequencies of FG nanorods with torsional springs have various stiffnesses for the first vibrational mode. To investigate the influence of the power law exponent on torsional spring stiffness, this figure is plotted. For this figure, the small scale parameter value is selected as 1 nm, while the nanorod

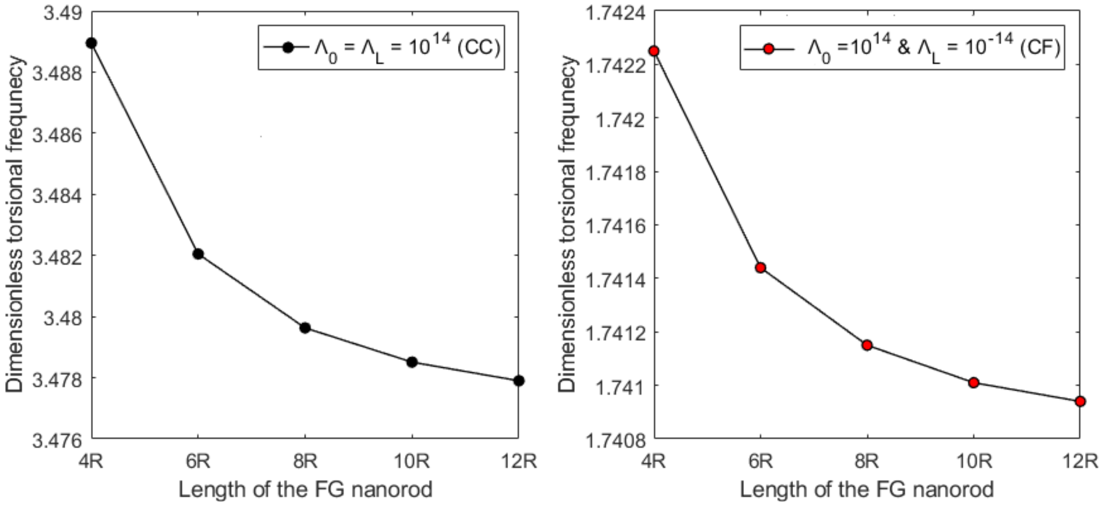


Fig. 11 Variation of dimensionless frequency of FG nanorod versus length for l = 0.5 nm

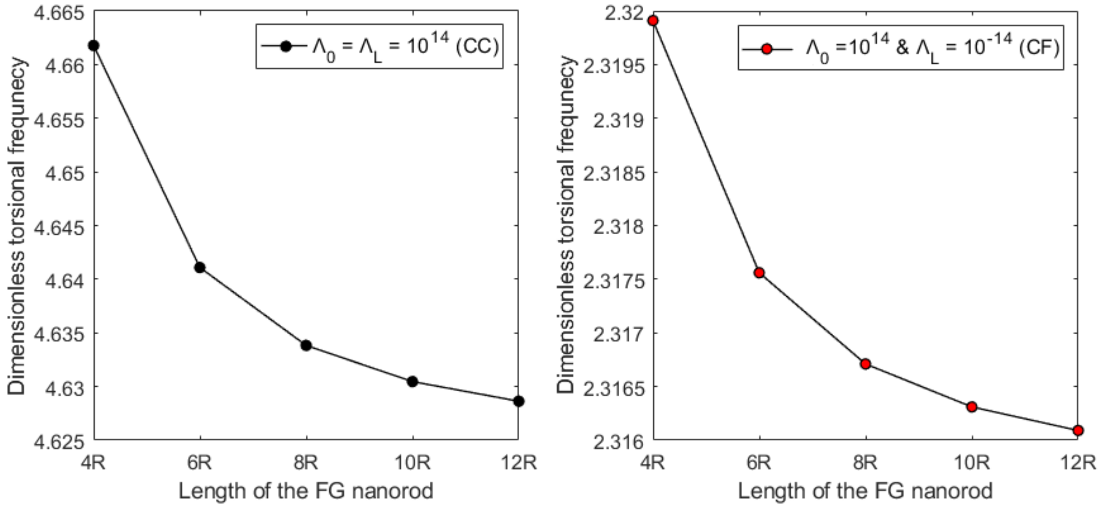


Fig. 12 Variation of dimensionless frequency of FG nanorod versus length for l = 1.0 nm

length is chosen as $L = 10R$. The reducing effect of the power law index on torsional frequencies can be easily seen in this figure also. The main point to be emphasized in this Figure is that the effect of the power law index is more efficient at higher spring parameter values.

In Fig. 9, the frequency ratio variation of the 1st mode of the functionally graded nanorod is plotted against the stiffness of the torsion springs. The frequency ratio is obtained by dividing the dimensionless frequency due to size effect based on the modified couple stress theory by dimensionless frequency due to the classical theory which is independent from the size effect. While obtaining the frequency values due to the size effect, the material scale parameter is chosen as 1 nm. On the other hand, it can be emphasized again here that by setting the material scale parameter to 0 nm, frequencies independent of the size effect are obtained. It is clear from the graph that the frequency ratio increases as the spring stiffness and power law index increase. From this, it can be concluded that the size effect based on the modified couple stress theory is more effective at higher spring stiffnesses and power law index. Figs. 10-12 are plotted to demonstrate the influence of nanorod length on free torsional vibration frequencies. While choosing p=1 for this examination, the length of the nanorod is considered in various values from 4R to 12R. Fig. 10 is drawn according to classical theory (l = 0 nm) while Figs. 11 and 12 are drawn on the basis of the modified couple stress theory. As can be seen, the dimensionless frequencies obtained with the classical theory are not affected by the length change. The dimensionless frequency values are constant for both boundary conditions and for all length values studied. However, when we give the material length scale parameter a value other than zero, that is, when we include the modified couple stress theory in the calculations, it is seen that the dimensionless frequencies do not remain constant. As can be seen from Figs. 11 and 12, the increase in nanorod length results in a decrease in torsional frequency values. In addition to those in Figs. 10, 11 and 12, values for $l = 1.5$ nm and $l = 2$ nm are also presented in Tables 4 and 5. It is concluded that this decrease, which occurs from the numerical values in the tables, is evident at high material scale parameter values. In addition, it can be inferred that the length variation of the nanorod is more important for the CC FG nanorod.

5. Conclusions

Because of to the superior dynamical and static responses of functionally graded nanorods, various types of these structures have been extensively used and developed in various engineering areas, such as aerospace vehicles, aircrafts, nuclear reactors, offshore and marine structures. Consequently, much attention of the scientific community has been given to the buckling, static and dynamic analysis of functionally graded material nanorod, nanoplate and nanoshell. In this paper, free torsional vibration characteristic of functionally graded nanorod considering the deformable boundary conditions is investigated based on the modified couple stress theory. Stokes' transformation with Fourier sine series is used to obtain the general eigen value analysis and the elastic torsional springs are introduced based on the non-rigid boundary conditions. The equations of motion and corresponding end conditions are computed by taking into account the torsional springs at the ends, but the effects of warping is neglected due the section is assumed to be circular. By using eigenvalue analysis, analytical results of torsional vibration response of functionally graded nanorod with deformable boundary conditions is presented and detailed numerical studies are given to display the relationship between small scale parameter, length of nanorod, the spring parameters at the ends and power-law index. Furthermore, the effects of deformable boundary conditions on the vibrational response of functionally graded nanorod are also discussed. Even though these numerical analyses have been confined to functionally graded nanorods with deformable boundary conditions, the proposed examples can be extended to rigid boundary conditions by giving proper values to the torsional spring parameters. The competency and precision of the proposed analytical method in free torsional vibration analysis are assessed through several examples of functionally graded nanorods with clamped-free and clamped-clamped ends, which indicates the correctness of the proposed analytical procedure. Elastic boundary conditions imposed on the circumferential displacement (hard support) at the nanorod boundaries edge has important effect on the free natural torsional frequencies. For lower values of the spring stiffnesss torsional restraints, including the limiting case of free boundary, the nanorod undergoing large torsional deformations exhibits free vibrational response under small scale effects. The novelty of this paper is both modified couple stress theory with deformable boundary conditions and functionally graded configurations are taken into account and an eigen-value matrix is constructed with arbitrary boundary conditions for the first time.

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