

Static deflection of nonlocal Euler Bernoulli and Timoshenko beams by Castigliano's theorem

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Abstract. This paper presents sets of explicit analytical equations that compute the static displacements of nanobeams by adopting the nonlocal elasticity theory of Eringen within the framework of Euler Bernoulli and Timoshenko beam theories. Castigliano's theorem is applied to an equivalent Virtual Local Beam (VLB) made up of linear elastic material to compute the displacements. The first derivative of the complementary energy of the VLB with respect to a virtual point load provides displacements. The displacements of the VLB are assumed equal to those of the nonlocal beam if nonlocal effects are superposed as additional stress resultants on the VLB. The illustrative equations of displacements are relevant to a few types of loadings combined with a few common boundary conditions. Several equations of displacements, thus derived, matched precisely in similar cases with the equations obtained by other analytical methods found in the literature. Furthermore, magnitudes of maximum displacements are also in excellent agreement with those computed by other numerical methods. These validated the superposition of nonlocal effects on the VLB and the accuracy of the derived equations.

Keywords: analytical solution; Castigliano's theorem; Eringen's nonlocal elasticity theory; Euler-Bernoulli beam theory; static displacements; Timoshenko beam theory

1. Introduction

There has been a proliferation of research works on applying Eringen's (Eringen and Wegner 2003, Eringen 1983) nonlocal elasticity theory to analyze beam-like elements of nanoelectron-mechanical devices, atomic force microscopes, nano actuators, and nanosensors. The theory incorporates small size effects in continuum mechanics by assuming stress to depend on strains of all surrounding points. In the last two decades, the Euler-Bernoulli beam theory (EBBT), the Timoshenko Beam Theory (TBT), and higher-order beam theories were extended to the nonlocal elasticity by various methods. The governing differential equations for static bending of nonlocal beams by the EBBT and the TBT, respectively, were reported by Peddieson *et al.* (2003) and Wang *et al.* (2008). Both narrated analytical methods of solutions of the governing differential equations under various boundary conditions and presented equations of displacements under transverse loads. Reddy and Pang (2008) adopted the direct integration method to deduce equations of displacements for both the EBBT and the TBT. Akbas also employed the direct integration method to obtain static deflection expressions of

edge cracked microbeams by the Couple Stress theory (Akbas 2016a). Another set of explicit equations was developed by Wang and Liew (2007) for the nonlocal EB beam and the Timoshenko beam. Akbas implemented the method of separation of variables for analytical solutions to the differential equations of axial vibration of cracked nanorods (Akbas 2019a), porous nanorods (Akbas 2019b) and viscoelastic nanorods (Akbas 2020) based on the nonlocal theory. Bensaid proposed a refined nonlocal higher-order hyperbolic shear deformation theory and presented a Navier type solution for static deflection and natural frequency of simply supported nanobeams (Bensaid 2017). Bensattalah *et al.* (2020) analytically derived nonlocal critical buckling loads for triple-walled carbon nanotubes by following the Timoshenko model. Nonlocal analytical solutions were reported by Ebrahimi *et al.* for bending of magneto-electro-elastic nanobeams under hygro-thermal loading embedded in Winkler-Pasternak foundation in the framework of parabolic third order beam theory (Ebrahimi *et al.* 2020). Finite element formulations of multilayered magneto-electro-elastic beams based on coupled constitutive equations revealed that stacking sequence had significant effects on static displacements but insignificant effects on the variation of shear stress along the beam length (Vinyas and Kattimani 2017a, b). Furthermore, greater hygro-thermal loads yielded higher static parameters (Vinyas *et al.* 2018a), uniform temperature profiles produced significant effects (Vinyas *et al.* 2018b),

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and pyro-effects exhibited negligible influence on displacements and stresses (Vinyas and Kattimani 2017c). By Adopting higher-order shear deformation beam theories for porous functionally graded nanobeams, Ahmed *et al.* (2019) obtained nonlinear buckling paths and Gafour *et al.* (2020) determined fundamental frequency through the Navier solution procedure.

Reddy (2007) formulated variational statements regarding the generalized displacements of nonlocal EB, Timoshenko, Reddy, and Levinson beams and followed the Navier solution technique for bending, buckling and vibration results. The same technique was adopted by Thai (2012) where the bending component of the axial strain was independent of the shear component; by Aydogdu (2009) where transverse shear deformation was a general function; and by Nikam and Sayyad (2018) where several polynomials and non-polynomial type strain functions were checked. Variational formulation, followed by the Galerkin technique for bending, buckling, and vibration solutions of nonlocal EB beams and Kirchhoff plates, were presented by Phadikar and Pradhan (2010); further extended to nonlocal Timoshenko beams by Pradhan (2012). Ritz's solution to the derived weak form was obtained by Ghannadpour *et al.* (2013). EBBT finite element analyses for bending, buckling and vibration of nanobeams by the nonlocal elasticity were presented by Civalek *et al.* (2020). A mixed finite element method was proposed for nonlocal EBBT by Nguen *et al.* (2015). Akbas presented EBBT finite element analyses based on modified couple stress theory for bending, vibration, and buckling of nanobeams (Akbas 2017c) and vibration and static bending of edge cracked functionally graded microscale beams (Akbas 2017b, 2018a). TBT finite element analyses were carried out for forced vibration solutions of viscoelastic nanobeams embedded in an elastic medium (Akbas 2016b), functionally graded nanobeams (Akbas 2017), cantilever nanobeam with crack (Akbas 2018c) and cracked functionally graded microbeam (Akbas 2018b). Eltahir *et al.* (2019) calculated Young's modulus, the natural frequency of vibration and the buckling force of single-walled carbon nanotubes by considering 3D beam elements where Carbon bonds were simulated as beams and the atoms as nodes. Civalek formulated vibration problems of embedded nanobeams and nanorods using Hamilton's principle and solved the equations of motions analytically and numerically by the finite element method (Civalek *et al.* 2020).

The analytical methods mentioned above solved fourth-order differential equations with four nonlocal statical and geometric boundary conditions, while other methods solved systems of equations. In contrast, this paper presents a fast and efficient method by applying Castigliano's theorem on a VLB where local statical boundary conditions are required.

Incorporation of the nonlocal effects by adjusting the deflections of a local equivalent is rarely found. However, an iterative procedure of local prediction with nonlocal correction was presented by Polizzotto (2001) by simulating the nonlocal effects with a correction strain in a fictitious local linear elastic continuum. In a different way, the equilibrium of a nonlocal beam segment is assumed to be

maintained under the classical local stress resultants, plus long-range elastic volume forces/moments exchanged by the beam segment with all nonadjacent beam segments for EBBT by Paola *et al.* (2014) and for TBT by Alotta *et al.* (2014). Barretta and de Sciarra (2015) assumed that the nonlocal elastostatic problem of an EB nanobeam was equivalent to a corresponding local nanobeam subjected to a prescribed bending curvature distortion. They simulated the nonlocality effect so that the bending curvature is the sum of an elastic part and an inelastic part.

This paper introduces a VLB made up of a linear elastic material and proposes a procedure to determine static deflections of nonlocal beams. First, the VLB experiences a curvature due to the applied transverse loads, whose equilibrium conditions determine its stress resultants. Then, the nonlocal effects deliver an additional curvature, simulated by an added set of stress-resultants. The resultant stress-resultants mathematically express the final strain energy of the VLB. Closed-form analytical equations of displacements are obtained by Castigliano's theorem, which states that the displacement (of a variable location) is the first derivative of the complementary energy with respect to a virtual point load (applied at that location). As per the authors' best knowledge, the proposed method is novel and applies to all boundary conditions coupled with any transverse load.

2. Methods of solutions

2.1 The EBBT and the TBT in nonlocal elasticity

A beam under transverse loads has displacements (u_1, u_2, u_3) toward three coordinate axes directions as

$$u_1^E = -z \frac{dw^E(x)}{dx}, u_2^E = 0, u_3^E = w^E(x) \quad (1)$$

$$u_1^T = -z \varphi(x), u_2^T = 0, u_3^T = w^T(x) \quad (2)$$

Here w and φ are the transverse displacement and the rotation of the cross-section at a variable location along the length direction x on the mid-plane of the beam ($z = 0$). Superscripts E and T denote quantities belonging to the EBBT and the TBT, respectively, and quantities without superscripts apply to both beam theories. Consequently, the nonzero strains according to the EBBT and the TBT are

$$\varepsilon_{xx}^E = -z \frac{d^2 w^E(x)}{dx^2} \quad (3)$$

$$\varepsilon_{xx}^T = -z \frac{d\varphi^T(x)}{dx}, \gamma_{xz}^T = -\varphi(x) + \frac{dw^T(x)}{dx} \quad (4)$$

Beam equilibrium relations under a transverse load $\hat{q}(x)$ are independent of nonlocality and the choice of beam theory. They are derived either by the equilibrium of a beam segment (Peddieson *et al.* 2003) or by the principle of virtual displacements (Reddy 2007) as

$$\frac{d^2 M_{xx}}{dx^2} = \frac{dV_{xz}}{dx} = \hat{q}(x) \quad (5)$$

If the normal stress and the shear stress on a plane perpendicular to the x -axis are denoted by σ_{xx} and σ_{xz} respectively, the stress resultants on that plane are defined as

$$M_{xx} = - \int z \sigma_{xx} dA, \quad V_{xz} = - \int \sigma_{xz} dA \quad (6)$$

The theory of nonlocal elasticity states that stress at a point of a continuum depends on the strains of all other points of that continuum. The integral constitutive relation of nonlocal elasticity (Eringen and Wegner 2003) was frequently expressed by an equivalent differential form as (Wang and Liew 2007, Reddy 2007)

$$(1 - \mu \nabla^2) \sigma^{nl} = t^l \quad (7)$$

where, σ^{nl} is the nonlocal stress tensor at a certain point, and t^l is its classical local counterpart. The parameter, $\mu = (e_0 a)^2$ is called the nonlocal parameter, comprising a material constant (e_0) that is determined by matching the dispersion curves based on atomistic models (Eringen and Wegner 2003, Karličić *et al.* 2016); and an internal length parameter (a) that depends on internal characteristic length such as the lattice spacing. Eq. (7) is modified for static deflection under transverse loads, where σ_{xx} exists for the EBBT and both σ_{xx} and σ_{xz} exist in the TBT (Wang and Liew 2007, Reddy 2007) as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx} \quad (8)$$

$$\sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G \gamma_{xz} \quad (9)$$

Here E is the Modulus of Elasticity, and G is the shear modulus of the considered material.

Substitutions of Eq. (3) into Eq. (8) and Eq. (4) into Eqs. (8)-(9) lead to the stress displacement relations for the EBBT and the TBT, respectively as

$$\sigma_{xx}^E - \mu \frac{d^2 \sigma_{xx}^E}{dx^2} = -Ez \frac{d^2 w^E}{dx^2} \quad (10)$$

$$\sigma_{xx}^T - \mu \frac{d^2 \sigma_{xx}^T}{dx^2} = -Ez \frac{d\varphi}{dx} \quad (11)$$

$$\sigma_{xz}^T - \mu \frac{d^2 \sigma_{xz}^T}{dx^2} = G \left(-\varphi + \frac{dw^T}{dx} \right) \quad (12)$$

Eqs. (10)-(11) are multiplied with z and integrated over the beam's cross-sectional area to derive Eqs. (13)-(14). The definition of the bending moment stated by Eq. (6) and the definition of the second moment of inertia of the beam cross-section $\int_0^A z^2 dA = I$ were substituted to establish Eqs. (13)-(14). Likewise, Eq. (12) is integrated over the cross-sectional area of the beam and modified by the definition of shear force from Eq. (6) to obtain Eq. (15) as

$$M_{xx}^E - \mu \frac{d^2 M_{xx}^E}{dx^2} = EI \frac{d^2 w}{dx^2} \quad (13)$$

$$M_{xx}^T - \mu \frac{d^2 M_{xx}^T}{dx^2} = EI \frac{d\varphi}{dx} \quad (14)$$

$$V_{xz}^T - \mu \frac{d^2 V_{xz}^T}{dx^2} = KGA \left(\varphi - \frac{dw^T}{dx} \right) \quad (15)$$

The shear correction factor K used in the TBT accounts for the non-uniform distribution of the shear stress over the beam cross-section. Values of K depend on the cross-sectional shape of the beam.

The equilibrium relations stated in Eq. (5) are utilized to establish the following equations

$$M_{xx}^E(x) - \mu \hat{q}(x) = EI \frac{d^2 w(x)}{dx^2} = EI \kappa^E(x) \quad (16)$$

$$M_{xx}^T(x) - \mu \hat{q}(x) = EI \frac{d\varphi(x)}{dx} = EI \kappa^T(x) \quad (17)$$

$$V_{xz}^T(x) - \mu \frac{d\hat{q}(x)}{dx} = KGA \left[\varphi(x) - \frac{dw^T(x)}{dx} \right] \quad (18)$$

2.2 Nonlocal strain gradient theory

This section brings in a Virtual Local Beam (VLB) made up of linear elastic material. The material parameters (E , G) are identical to those of the considered nonlocal beam. The VLB is acted by transverse loads only. The nonzero displacements and strains of the VLB remain the same as stated in Eqs. (1)-(4), respectively. The equilibrium relations of Eq. (5) and the definitions of stress-resultants of Eq. (6) are also valid. The strain energy \bar{U} of the VLB under static bending due to transverse load is given by

$$\bar{U} = \iiint \frac{1}{2} (\bar{\sigma}_{xx} \bar{\varepsilon}_{xx} + \bar{\sigma}_{xz} \bar{\gamma}_{xz}) dV \quad (19)$$

The quantities with 'bar' are relevant to the VLB. The terms of Eq. (19) within the volume integral retain in the TBT, while the second term (shear term) vanishes in the EBBT. Since the VLB is made up of linear elastic material, Hooke's Law is applicable. Thus $\bar{\varepsilon}_{xx} = \bar{\sigma}_{xx}/E$ and $\bar{\gamma}_{xz} = \bar{\sigma}_{xz}/G$ are substituted into Eq. (19) to express strain energy as

$$\bar{U} = \iiint \frac{1}{2} \left(\frac{\bar{\sigma}_{xx}^2}{E} + \frac{\bar{\sigma}_{xz}^2}{G} \right) dV \quad (20)$$

If the bending moment and the shear force of the VLB are \bar{M}_{xx} and \bar{V}_{xz} respectively, and the formulas for the bending stress $\bar{\sigma}_{xx} = \frac{\bar{M}_{xx} z}{I}$ and the shear stress $\bar{\sigma}_{xz} = \frac{V_{xz} Q}{Ib}$ are substituted into Eq. (20), the strain energy is expressed as (Hibbeller 2016)

$$\bar{U} = \frac{1}{2} \int_0^L \frac{[\bar{M}_{xx}(x)]^2}{EI} dx + \frac{1}{2} \int_0^L \frac{[\bar{V}_{xz}(x)]^2}{KGA} dx \quad (21)$$

The stress-resultants of the VLB, $\bar{M}_{xx}(x)$ and $\bar{V}_{xz}(x)$, are amended by the nonlocal effects before Castigliano's theorem is applied to strain energy expression of Eq. (21).

2.3 Nonlocal effects in the VLB

The moment-curvature relation of the nonlocal EBBT provided by Eq. (16) is implemented in the VLB in two steps (Fig. 1). First, the external load $\hat{q}(x)$ is applied to the VLB to produce a curvature $\bar{\kappa}_1^E(x)$ with relevant

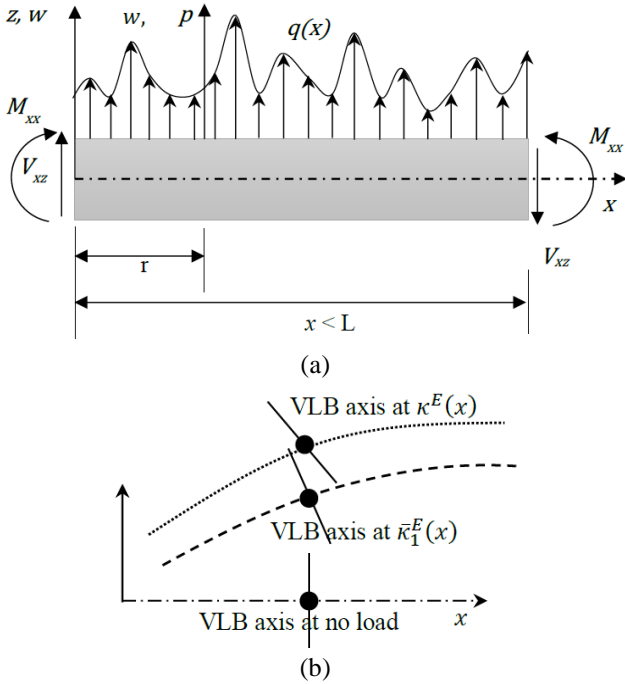


Fig. 1 (a) A beam segment of the VLB with positive directions of all parameters; (b) superposition of nonlocal effects on local VLB curvature

displacements $\bar{w}_1^E(x)$. The magnitudes of $\bar{\kappa}_1^E(x)$ and $\bar{w}_1^E(x)$ are not equal to those of nonlocal beam curvatures $\kappa^E(x)$ and displacements $w^E(x)$ respectively, because of the absence of the long range stresses due to strains of surrounding points. Thus, the bending moment of the VLB without any nonlocal effect by the EBBT is given by

$$\bar{M}_{xx}^{E,1}(x) = EI \frac{d^2 \bar{w}_1^E(x)}{dx^2} = EI \bar{\kappa}_1^E(x) \quad (22)$$

To simulate the nonlocal effects, an additional curvature $\bar{\kappa}_2^E(x)$ is superposed on $\bar{\kappa}_1^E(x)$ in the VLB to obtain a resultant desired curvature $\kappa^E(x)$. The superposed curvature makes the displacements of the VLB equal to those of the nonlocal beam. The bending moment relevant to the additional curvature $\bar{\kappa}_2^E(x)$ is

$$\bar{M}_{xx}^{E,2}(x) = -\mu \hat{q}(x) = EI \bar{\kappa}_2^E(x) \quad (23)$$

Eq. (23) indicates that the superposed bending moment $\bar{M}_{xx}^{E,2}$ due to the nonlocal effects is negative with a negative curvature for a positive (upward) applied load. Equivalently, a positive $\bar{M}_{xx}^{E,2}$ with a positive curvature $\bar{\kappa}_2^E(x)$ is obtained by negative (downward) applied force. A positive bending moment means tensile stresses are below the neutral axis.

The above postulate inevitably ensues superposition of rotation of the normals to the neutral surface of the VLB. At first, the normals tilt with the application of the external load with a curvature $\bar{\kappa}_1^E(x)$, and displacements $\bar{w}_1^E(x)$ occur. A second tilting of the normals are superposed by the curvature $\bar{\kappa}_2^E(x)$ and another round of displacements $\bar{w}_2^E(x)$ take place. The second set simulates the nonlocal effects in such a way that the resultant curvature of the VLB is equal to that of the nonlocal beam $\kappa^E(x)$ and the

resultant displacements of the VLB are identical to the nonlocal beam displacements $w^E(x)$.

The net bending moment of the VLB for the EBBT that simulates the identical deflection of its nonlocal counterpart is

$$\bar{M}_{xx}^E(x) = \bar{M}_{xx}^{E,1}(x) + \bar{M}_{xx}^{E,2}(x) \quad (24)$$

The postulate of curvature superposition applies to the TBT because it also considers that the normals to the beam neutral-surface remain straight after deformation. The net stress resultants (bending moment \bar{M}_{xx}^T and transverse shear force \bar{V}_{xz}^T) of the Timoshenko VLB that simulate the identical curvature of its nonlocal counterpart are calculated by summations of two sets of bending moments and shear forces. The first set of Eqs. (25) and (26) rotates the normals to the neutral surface so that bending strain $\bar{\kappa}_1^T(x)$ and shear strain $\bar{\gamma}_1^T(x)$ are achieved. Then, the second set given by Eqs. (27) and (28) is superposed to allow additional rotations of the normals that produce additional bending strain $\bar{\kappa}_2^T(x)$ and additional shear strain $\bar{\gamma}_2^T(x)$.

$$\bar{M}_{xx}^{T,1}(x) = EI \frac{d\bar{\varphi}(x)}{dx} = EI \bar{\kappa}_1^T(x) \quad (25)$$

$$\bar{V}_{xz}^{T,1}(x) = KAG \left[\bar{\varphi}(x) - \frac{d\bar{w}^T(x)}{dx} \right] = KAG \bar{\gamma}_1^T(x) \quad (26)$$

$$\bar{M}_{xx}^{T,2}(x) = -\mu \hat{q}(x) = EI \bar{\kappa}_2^T(x) \quad (27)$$

$$\bar{V}_{xz}^{T,2}(x) = -\mu \frac{d\hat{q}(x)}{dx} = KAG \bar{\gamma}_2^T(x) \quad (28)$$

The final stress-resultants of the Timoshenko VLB that simulate the identical deflection of its nonlocal counterpart are

$$\bar{M}_{xx}^T(x) = \bar{M}_{xx}^{T,1}(x) + \bar{M}_{xx}^{T,2}(x) \quad (29)$$

$$\bar{V}_{xz}^T(x) = \bar{V}_{xz}^{T,1}(x) + \bar{V}_{xz}^{T,2}(x) \quad (30)$$

2.4 Castigliano's theorem

Castigliano's theorem applies to materials with linear-elastic behavior where the strain energy is equal to the complementary energy. It is stated as

$$\delta \bar{w} = \frac{\partial \bar{U}}{\partial p} \quad (31)$$

p is a dummy point load applied at a variable location $0 < r < L$ where the displacement \bar{w} is sought. The dummy point load is simulated by the Dirac Delta function as

$$\hat{p}(x) = p \delta(x - r) \quad (32)$$

Thus the load function $\hat{q}(x)$ in all preceding equations are updated as

$$\hat{q}(x) = q(x) + p \delta(x - r) \quad (33)$$

where $q(x)$ is the externally applied transverse load that can have any random distribution. This paper considers

three distributions of applied loads; (i) a uniformly distributed load (UDL) Q applied on the entire beam length, (ii) a point load F applied at a variable location b along the beam length, and (iii) a sinusoidal load with an amplitude B and a wavelength $2L$. All loads are contained within a single equation, Eq. (34) with a view to revealing one by setting others zero.

$$q(x) = Q + B \sin \frac{\pi x}{L} + F \delta(x - b) \quad (34)$$

2.5 Calculation of stress resultants

The stress resultants do not depend on the choice of a beam theory. They are the same in the EBBT and the TBT but are specific to the applied loads and boundary conditions. The expressions of the stress resultants $\bar{M}_{xx}^{E,1}(x)$, $\bar{M}_{xx}^{T,1}(x)$, $\bar{V}_{xz}^{T,1}(x)$ of a statically determinate beam are determined straightforwardly by the conditions of the statical equilibrium of a beam segment (Fig. 1), while those of the indeterminate beam are derived either by following a displacement-based method (e.g., Stiffness method) or by the Force method.

In a more general way, stress resultants $\bar{M}_{xx}^{E,1}(x)$, $\bar{M}_{xx}^{T,1}(x)$, $\bar{V}_{xz}^{T,1}(x)$ are derived from the expression of displacements $\bar{w}_1^E(x)$, $\bar{\varphi}(x)$, $\bar{w}^T(x)$ as indicated by Eqs. (22)-(26). For this reason, the expressions of displacements are required from the solutions of governing differential equations of the VLB without nonlocal amendments. The governing equation of a local EB beam under transverse loads $q(x)$ along with an additional fictitious point load p at r is the following fourth order differential equation

$$EI \frac{d^4 \bar{w}_1^E(x)}{dx^4} = q(x) + p \delta(x - r) \quad (35)$$

The solution of Eq. (35) depends on the distribution of external loads $q(x)$. If the load given in Eq. (34) are applied, the solution of Eq. (35)

$$\begin{aligned} \bar{w}_1^E(x) &= \frac{Qx^4}{24EI} + \frac{BL^4}{\pi^4 EI} \sin \frac{\pi x}{L} \\ &+ \frac{F}{6EI} (x - b)^3 [1 + H(x - b)] \\ &+ \frac{p}{6EI} (x - r)^3 [1 + H(x - r)] \\ &+ C_1^E + xC_2^E + x^2C_3^E + x^3C_4^E \end{aligned} \quad (36)$$

The constants of Eq. (36) are estimated from the local boundary conditions of the VLB. In the same manner, the governing equations of the Timoshenko VLB subjected to a transverse load $q(x)$ are

$$EI \frac{d^3 \bar{\varphi}}{dx^3} = q(x) + p \delta(x - r) \quad (37)$$

$$\frac{d\bar{w}^T}{dx} = \bar{\varphi} - \frac{EI}{KAG} \frac{d^2 \bar{\varphi}}{dx^2} \quad (38)$$

Simultaneous differential equations, Eqs. (37) and (38) are solved by applying loads of Eq. (34) on the VLB to work out the following expressions of $\bar{\varphi}(x)$ and $\bar{w}^T(x)$

$$\begin{aligned} \bar{\varphi}(x) &= \frac{Qx^3}{6EI} + \frac{BL^3}{\pi^3 EI} \cos \frac{\pi x}{L} + \frac{F}{2EI} (x - b)^2 H(x - b) \\ &+ \frac{p}{2EI} (x - r)^2 H(x - r) + C_1^T + xC_2^T + x^2C_3^T \end{aligned} \quad (39)$$

$$\begin{aligned} \bar{w}^T(x) &= Q \left[\frac{x^4}{24EI} - \frac{x^2}{2\Omega} \right] + B \left(\frac{L^4}{\pi^4 EI} + \frac{L^2}{\pi^2 \Omega} \right) \sin \frac{\pi x}{L} \\ &+ F \left[\frac{1}{6EI} (x - b)^3 + \frac{1}{\Omega} (x - b) \{1 + H(x - b)\} - \right. \\ &\left. \frac{(x - b)^2}{2\Omega} \delta(x - b) \right] + p \left[\frac{1}{6EI} (x - r)^3 + \frac{1}{\Omega} (x - r) \{1 + \right. \\ &\left. H(x - r)\} - \frac{(x - r)^2}{2\Omega} \delta(x - r) \right] + xC_1^T + \frac{x^2}{2} C_2^T + \\ &\left(\frac{x^3}{3} - \frac{2xEI}{\Omega} \right) C_3^T + x^3 C_4^T \end{aligned} \quad (40)$$

With the expressions of displacements presented in Eqs. (36), (39) and (40), the stress resultants without nonlocal amendments, $\bar{M}_{xx}^{E,1}(x)$, $\bar{M}_{xx}^{T,1}(x)$ and $\bar{V}_{xz}^{T,1}(x)$ are determined by differentiation shown in Eqs. (22), (25) and (26). The constants are to be determined for the specific boundary conditions at the beam ends.

When loads of Eq. (34) are applied, the amendments to the bending moment required by the Eqs. (23) and (27), and the amendments to the shear force required by Eq. (28) respectively are finalized as

$$\begin{aligned} \bar{M}_{xx}^{E,2} = \bar{M}_{xx}^{T,2} &= -\mu \left[Q + F \delta(x - b) + B \sin \frac{\pi x}{L} + \right. \\ &\left. p \delta(x - r) \right] \end{aligned} \quad (41)$$

$$\bar{V}_{xz}^{T,2} = -\frac{\mu \pi B}{L} \cos \frac{\pi x}{L} \quad (42)$$

3. Solution

3.1 General solutions

The stress resultants are amended according to Eq. (24) for the EBBT and by Eqs. (29) and (30) for the TBT. The amended stress-resultants are substituted into Eq. (21) for general expressions of the strain energy in terms of the stress-resultants by the EBBT and the TBT, respectively as Similarly, the application of Eq. (31) along with the strain energy expression given by Eq. (43) yields the following expression of displacement for nonlocal beam by the TBT

$$\bar{U}^E = \frac{1}{2} \int_0^L \frac{1}{EI} [\bar{M}_{xx}^{E,1}(x) + \bar{M}_{xx}^{E,2}(x)]^2 dx \quad (43)$$

$$\begin{aligned} \bar{U}^T &= \frac{1}{2} \int_0^L \frac{1}{EI} [\bar{M}_{xx}^{T,1}(x) + \bar{M}_{xx}^{T,2}(x)]^2 dx \\ &+ \frac{1}{2} \int_0^L \frac{1}{KGA} [\bar{V}_{xz}^{T,1}(x) + \bar{V}_{xz}^{T,2}(x)]^2 dx \end{aligned} \quad (44)$$

Consequently, Eq. (31) is implemented with the strain energy expression of Eq. (43) to evaluate the displacement of the nonlocal beam by the EBBT as

$$w^E(x) = \frac{1}{EI} \int_0^L \left[\begin{array}{c} \bar{M}_{xx}^{E,1}(x) \\ + \bar{M}_{xx}^{E,2}(x) \end{array} \right] \frac{\partial}{\partial P} \left[\begin{array}{c} \bar{M}_{xx}^{E,1}(x) \\ + \bar{M}_{xx}^{E,2}(x) \end{array} \right] dx \quad (45)$$

Table 1 Equations of displacements of S-S nonlocal beam

Load distribution	EBBT
Q	$\frac{Qx}{24EI} \{(L-x)(L^2 + Lx - x^2 + 12\mu)\}$
$B \sin \frac{\pi x}{L}$	$\frac{B}{\pi^4 EI} \{L^2(L^2 + \pi^2\mu)\} \sin \frac{\pi x}{L}$
$F \delta(x-b)$	$\frac{F}{6EIL} \left[\begin{array}{l} bx(2L^2 - 3Lx + x^2) - b^3(L-x) \\ -L(x-b)^3 H(b-x) \\ +6\mu\{x(L-b) + L(x-b)H(x-b)\} \end{array} \right]$
TBT	
Q	$\frac{Qx}{24EI} (L-x)[L^2 + Lx - x^2 + 12\Omega + 12\mu]$
$B \sin \frac{\pi x}{L}$	$\frac{B}{\pi^4 EI} [(L^2 + \pi^2\Omega)(L^2 + \pi^2\mu)] \sin \frac{\pi x}{L}$
$F \delta(x-b)$	$\frac{F}{6EIL} \left[\begin{array}{l} x(L-b)\{2bL - b^2 - x^2 + 6\Omega\} \\ +L(x-b)\{(b-x)^2 - 6\Omega\}H(x-b) \\ +6\mu\{b(L-x) + L(x-b)H(b-x)\} \end{array} \right]$

Table 2 Equations of displacements of the C-C nonlocal beam

Load distribution	EBBT
Q	$\frac{Qx^2}{24EI} [(6L^2 - 4Lx + x^2) - 12\mu]$
$B \sin \frac{\pi x}{L}$	$\frac{BL}{6\pi^4 EI} \left[\begin{array}{l} 6L(L^2 + \pi^2\mu) \sin \frac{\pi x}{L} \\ -\pi x\{6L^2 - 3\pi^2 Lx + \pi^2 x^2\} \\ -6\pi^3 \mu x \end{array} \right]$
$F \delta(x-b)$	$\frac{F}{6EI} \left[\begin{array}{l} b^2(3x-b) - (x-b)^3 H(b-x) \\ -6\mu(x-b) \end{array} \right] H(x-b)$
TBT	
Q	$\frac{Qx^2}{24EI} \{[(6L^2 - 4Lx + x^2) - 12\Omega(1 - 2L/x)] - 12\mu\}$
$B \sin \frac{\pi x}{L}$	$\frac{B}{\pi^4 EI} (L^2 + \pi^2\mu) \left\{ (L^2 + \pi^2\Omega) \sin \frac{\pi x}{L} - \pi x(L-x) \right\}$
$F \delta(x-b)$	$\frac{F}{6EIL(12\Omega + L^2)} \left[\begin{array}{l} x(b-L) \left\{ \begin{array}{l} (x^2 - 6\Omega)(L^2 + 12\Omega) \\ +b^2(3Lx - 2x^2 + 12\Omega) \\ +b(Lx(x-3L) - 6\Omega(L+3x)\Omega) \end{array} \right\} \\ +L(x-b)\{(b-x)^2 - 6\Omega\}(L^2 + 12\Omega)H(x-b) \\ +6\mu \left\{ \begin{array}{l} b(6L^3 - 3Lx^2 + 2x^3 + 12(L-x)\Omega) \\ -x(L-x)(L^2 - Lx + 6\Omega) \end{array} \right\} - 6\mu L(b-x)(L^2 + 12\Omega)H(b-x) \end{array} \right]$

$$w^T(x) = \frac{1}{EI} \int_0^L \left[\begin{array}{l} \bar{M}_{xx}^{T,1}(x) \\ +\bar{M}_{xx}^{T,2}(x) \end{array} \right] \frac{\partial}{\partial P} \left[\begin{array}{l} \bar{M}_{xx}^{T,1}(x) \\ +\bar{M}_{xx}^{T,2}(x) \end{array} \right] dx + \frac{1}{KAG} \int_0^L \left[\begin{array}{l} \bar{V}_{xz}^{T,1}(x) \\ +\bar{V}_{xz}^{T,2}(x) \end{array} \right] \frac{\partial}{\partial p} \left[\begin{array}{l} \bar{V}_{xz}^{T,1}(x) \\ +\bar{V}_{xz}^{T,2}(x) \end{array} \right] dx \quad (46)$$

The first step of the proposed procedure is the determination of the constants $\{C_1^E, C_2^E, C_3^E, C_4^E\}$ and $\{C_1^T, C_2^T, C_3^T, C_4^T\}$ of the expressions of displacements $\bar{w}_1^E(x), \bar{\varphi}(x), \bar{w}^T(x)$ given by Eqs. (36), (39) and (40) using the boundary conditions of the VLB without any

nonlocal amendment. The second step is to determine the stress resultants $\bar{M}_{xx}^{E,1}(x), \bar{M}_{xx}^{T,1}(x)$ and $\bar{V}_{xz}^{T,1}(x)$ relevant to the VLB without nonlocal amendment. The third step is to amend stress resultants by superposing $\bar{M}_{xx}^{E,2}(x), \bar{M}_{xx}^{T,2}(x)$ and $\bar{V}_{xz}^{T,2}(x)$ from Eqs. (41) and (42). Then, the displacements of the nonlocal beam are obtained by using Eqs. (45) and (46) by the EBBT and the TBT, respectively. The final step includes setting the virtual load $p = 0$.

The equations of displacement for three distributions of external loads are sorted row-wise in Tables 1-4 in the same order; a UDL Q , a sinusoidal load with amplitude B , and a point load of magnitude F at a location $x = b$. In all equations, $H(\cdot)$ means the Heaviside function and substitutions of the shear parameter $\Omega = \frac{EI}{AGK}$ have been made. Superscripts E and T are omitted from the boundary conditions and the stress resultants because they have the same expressions in both the EBBT and the TBT.

3.2 Simply supported beams

The VLB simply supported at both the left and the right ends has the following conditions.

$$\bar{w} = \bar{M}_{xx} = 0 \quad \text{at } x = 0, L \quad (47)$$

The stress resultants of the VLB without any nonlocal amendment are calculated as

$$M_{xx}^1 = \frac{Qx}{2}(-L+x) - \frac{BL^2}{\pi^2} \sin \frac{\pi x}{L} + Fx \left(-1 + \frac{b}{L}\right) + F(x-b)H(x-b) + px \left(-1 + \frac{r}{L}\right) + p(x-r)H(x-r) \quad (48)$$

$$V_{xz}^1 = \frac{Q}{2}(-L+2x) - \frac{BL}{\pi} \cos \frac{\pi x}{L} + F \left(-1 + \frac{b}{L}\right) + FH(x-b) + p \left(-1 + \frac{r}{L}\right) + pH(x-r) \quad (49)$$

By substituting the above stress resultants along with the nonlocal amendments from Eqs. (41) and (42) into the Eqs. (45) and (46), the equations of displacements for S-S nonlocal beams for various distributions of loads are calculated and presented in Table 1.

3.3 Clamped-clamped beams

The boundary conditions for a VLB clamped at both ends are

$$\bar{w}^E = \bar{w}^T = \frac{d\bar{w}^E}{dx} = \varphi = 0 \quad \text{at } x = 0, L \quad (50)$$

The boundary conditions of Eq. (50) are used to determine the stress resultants of the VLB without nonlocal amendments as

$$M_{xx}^1 = \frac{Q}{12}(L^2 - 6Lx + 6x^2) - \frac{BL^2}{\pi^2} \sin \frac{\pi x}{L} + \frac{2BL^2}{\pi^3} + \frac{F}{L^3} [L(L-b)^2(L-3x) - (L-b)^3(L-2x) + L^3(x-b)H(x-b)] + \frac{p}{L^3} [L(L-r)^2(L-3x) - (L-r)^3(L-2x) + L^3(x-r)H(x-r)] \quad (51)$$

$$V_{xz}^1 = \frac{Q}{2}(2x - L) - \frac{BL}{\pi} \cos \frac{\pi x}{L} + \frac{F}{L^3} [2(L - b)^3 - 3L(L - b)^2 + L^3 H(x - b)] + \frac{p}{L^3} [2(L - r)^3 - 3L(L - r)^2 + L^3 H(x - r)] \quad (52)$$

These expressions of the stress-resultants plus the nonlocal amendments from Eqs. (41) and (42) are replaced in the Eqs. (45) and (46) to develop the equations of displacements for C-C nonlocal beams and presented in Table 2.

3.4 Cantilevers

The VLB clamped at the left end and free at the right end has the following conditions

$$\bar{w}^E = \bar{w}^T = \frac{d\bar{w}^E}{dx} = \bar{\varphi} = 0 \text{ at } x = 0 \quad (53)$$

$$\bar{M}_{xx}^E = \bar{M}_{xx}^T = \bar{V}_{xz}^E = \bar{V}_{xz}^T = 0 \text{ at } x = L \quad (54)$$

The boundary conditions of the cantilever provided by Eqs. (53) and (54) are used to determine the stress resultants of the VLB without nonlocal amendments as

$$M_{xx}^1 = \frac{Q}{2}(L - x)^2 + \frac{BL^2}{\pi} - \frac{BLx}{\pi} - \frac{BL^2}{\pi^2} \sin \frac{\pi x}{L} + F(b - x)H(b - x) + p(r - x)H(r - x) \quad (55)$$

$$V_{xz}^1 = Q(x - L) - \frac{2BL}{\pi} \cos^2 \frac{\pi x}{2L} + FH(b - x) + pH(r - x) \quad (56)$$

The expressions of the stress-resultants from Eqs. (55) and (56) plus the nonlocal amendments from Eqs. (41) and (42) are replaced in the Eqs. (45) and (46) to obtain the equations of displacements for nonlocal cantilevers. They are presented in Table 3.

3.5 Propped cantilevers

The VLB clamped at the left end and simply supported at the right end has the following conditions

$$\bar{w}^E = \bar{w}^T = \frac{d\bar{w}^E}{dx} = \bar{\varphi} = 0 \text{ at } x = 0 \quad (57)$$

$$\bar{w}^E = \bar{w}^T = \bar{M}_{xx}^E = M_{xx}^T = 0 \text{ at } x = L \quad (58)$$

The above boundary conditions are utilized to calculate the stress resultants of the VLB without nonlocal amendments as

$$M_{xx}^1 = \frac{Q}{8}(L - 4x)(L - x) - \frac{BL}{\pi^3} \left[3(-L + x) + \pi L \sin \frac{\pi x}{L} \right] + \frac{F}{2L^3} (L - b) \{ 2L^2(b - x) + b^2x - Lb(b + 2x) \} + F(x - b)H(x - b) + \frac{p}{2L^3} (L - r) \{ 2L^2(r - x) + r^2x \} + p(x - r)H(x - r) \quad (59)$$

The expressions of the stress-resultants along with the nonlocal amendments from Eqs. (41) and (42) are replaced in the Eqs. (45) and (46) to obtain the equations of

Table 3 Equations of displacements of the nonlocal cantilever

Load distribution	EBBT
Q	$\frac{Qx^2}{24EI} [(6L^2 - 4Lx + x^2) - 12\mu]$
$B \sin \frac{\pi x}{L}$	$\frac{BL}{6\pi^4 EI} \left[\begin{array}{l} 6L(L^2 + \pi^2\mu) \sin \frac{\pi x}{L} \\ -\pi x \{ 6L^2 - 3\pi^2 Lx + \pi^2 x^2 \} - 6\pi^3 \mu x \end{array} \right]$
$F \delta(x - b)$	$\frac{F}{6EI} [b^2(3x - b) - (x - b)^3 H(b - x) - 6\mu(x - b)H(x - b)]$
TBT	
Q	$\frac{Qx^2}{24EI} \{ [(6L^2 - 4Lx + x^2) - 12\Omega(1 - 2L/x)] - 12\mu \}$
$B \sin \frac{\pi x}{L}$	$\frac{BL}{6\pi^4 EI} \left[\begin{array}{l} \left\{ 6L(L^2 + \pi^2\Omega + \pi^2\mu) - \frac{6\pi^4\mu\Omega}{L} \right\} \sin \frac{\pi x}{L} \\ -\pi x \{ 6L^2 - 3\pi^2 Lx + \pi^2 x^2 \} + 6\pi^3\Omega x - 6\pi^3\mu \end{array} \right]$
$F \delta(x - b)$	$\frac{F}{6EI} \left[\begin{array}{l} x^2(3b - x) + 6x\Omega \\ + (x - b) \{ (x - b)^2 - 6\Omega \} H(x - b) \\ - 6\mu(x - b)H(x - b) \end{array} \right]$

Table 4 Equations of displacements of propped cantilever nonlocal beam

Load distribution	EBBT
Q	$\frac{Qx^2}{48EI L} [(L - x)(3L^2 - 2Lx + 12\mu)]$
$B \sin \frac{\pi x}{L}$	$\frac{B}{\pi^4 EI} (L^2 + \pi^2\mu) \left\{ L^2 \sin \frac{\pi x}{L} - \frac{\pi x}{2L} (2L^2 - 3Lx + x^2) \right\}$
$F \delta(x - b)$	$\frac{F}{12EI L^3} [b^2(L - x) \{ 2Lx(3L - b) - x^2(3L - b) - 2bL^2 \} - 6\mu x^2(b - L)(3L - x) + 2L^3(b - x) \{ (b - x)^2 H(b - x) + 6\mu H(x - b) \}]$
TBT	
Q	$\frac{Qx(L - x)}{48EI(L^2 + 3\Omega)} [3L^3x - 2L^2(x^2 - 15\Omega) - 6\Omega x^2 + 6Lx(\Omega + 2\mu) + 72\Omega(\Omega + \mu)]^a$
$B \sin \frac{\pi x}{L}$	$\frac{B}{2\pi^4 EI} \frac{(L^2 + \pi^2\mu)}{(L^2 + 3\Omega)} \left[2(L^2 + 3\Omega)(L^2 + \pi^2\Omega) \sin \frac{\pi x}{L} - \pi x L(2L^2 - 3Lx + x^2) \right]$
$F \delta(x - b)$	$\frac{F}{12EI(L^2 + 3\Omega)} \left[\begin{array}{l} 3b^2Lx(x^2 - 3Lx - 6\Omega) + b^3x(3Lx - x^2 + 6\Omega) \\ + 6b \left\{ L^3(x^2 + 2\mu) - 3L\mu(x^2 - 2\Omega) \right\} \\ + x(x^2 - 6\Omega)(\mu + \Omega) \right\} \\ - 2Lx \left\{ -9Lx\mu + L^2(x^2 + 6\mu - 6\Omega) \right\} \\ + 3(-6\Omega^2 + x^2(\mu + \Omega)) \right\} \\ - 2L(b - x)(L^2 + 3\Omega) \\ + \{ (b - x)^2 - 6\Omega \} H(x - b) + 6\mu H(b - x) \end{array} \right]$

$$V_{xz}^1 = Q \left(-\frac{5L}{8} + x \right) - \frac{BL}{\pi^3} \left[3 + \pi^2 \cos \frac{\pi x}{L} \right] - \frac{F}{2L^3} (2L^3 - 3Lb^2 + b^3) + FH(x - b) - \frac{p}{2L^3} (2L^3 - 3Lr^2 + r^3) + pH(x - r) \quad (60)$$

displacements for nonlocal propped cantilever as given in Table 4.

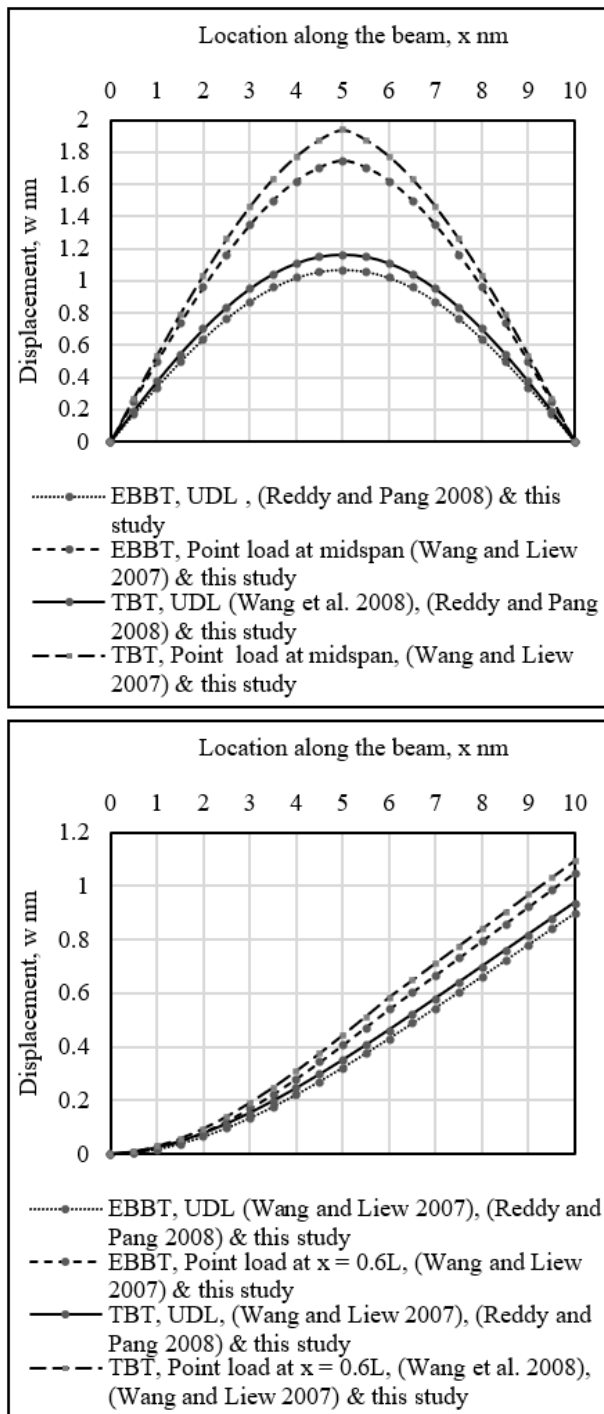


Fig. 2 Graphical validation of equations of displacements

4. Results and discussions

4.1 General comments

The presented equations of displacements of Tables 1-4 separated the nonlocal terms from others with a view to obtaining the regular local beam displacements by a vanishing nonlocal parameter. The variable location of the point load ($x = b$) is generalized by exploiting the Heaviside function; $H(x - b)$ vanishes for $x < b$ and $H(b - x)$ vanishes for $b < x$, thus making the equations

useful for any location of the point load along the length of the beam. The equations reveal that the nonlocal effects increase the static displacements for all boundary conditions except for cantilevers, as remarked by a few other researchers (Peddieson *et al.* 2003, Reddy and Pang 2008). This property is demonstrated by Figs. 3 and 4, where the presented equations of displacements are plotted in the context of a nanobeam that has a rectangular cross-section 1 nm wide and 2 nm deep in a 10 nm span. The material properties are relevant to steel with a modulus of elasticity 200 GPa and a shear modulus 79.3 GPa.

A few equations of displacements of Tables 1-4 matched exactly with previously presented equations by Reddy and Pang (2008), Wang *et al.* (2008) and Wang and Liew in relevant cases as depicted in Fig. 2. The mentioned researchers neither adopted a virtual local beam nor applied Castigliano's theorem. Reddy and Pang (2008) followed the direct integration method to solve the fourth-order differential equation relevant to nonlocal beams and presented equations of displacements for UDL cases for the EBBT and the TBT; their equations matched precisely with all pertinent equations of Tables 1-4. Wang *et al.* (2008) also followed the direct integration method on TBT governing differential equation and presented equations of displacements for UDL on S-S and C-C conditions, sinusoidal load on S-S and a point load at cantilever's free edge. Exactly the same equations have been derived in the present study (Tables 1-3) by applying Castigliano's theorem on the VLB. Peddieson *et al.* (2003) and Wang and Liew (2007) solved the differential equation of the nonlocal beam with specific loading conditions on S-S, C-C and Cantilever and presented equations of displacements that are reproduced in Tables 1-3.

The above comparisons validate the postulate proposed in this paper that the bending deflection of a nonlocal beam under transverse loads might be computed by superposing the deflections due to nonlocal effects on the deflections by the local effects on a VLB. Thus, the application of Castigliano's theorem on the VLB with an amendment of stress resultants for nonlocal effects is authenticated for EBBT and TBT.

4.2 Comments on S-S and C-C beams

The equations of displacements presented in Tables 1 and 2 are plotted in Fig. 3 for two load cases, each on S-S and C-C boundary conditions by varying the nonlocal parameter. Loads with various distributions were applied so that the total load remained the same for all cases. Small size effects due to nonlocality are demonstrated by choosing three values of the nonlocal parameter, while the shear parameter remained $\Omega = 0.01$ for all cases. The effects of nonlocality increase the displacements for all load cases of these two support conditions, except for the C-C beam under a uniformly distributed load. This situation has no nonlocal effect for both the EBBT and the TBT, although shear deformation effects are present. The nonlocal effects are further demonstrated in Table 5, where maximum displacements for S-S and C-C for three load cases are presented with non-dimensionalized nonlocal parameter $\bar{\mu} = \frac{\mu}{l^2}$ and non-dimensionalized shear parameter $\bar{\Omega} = \frac{\Omega}{l^2}$.

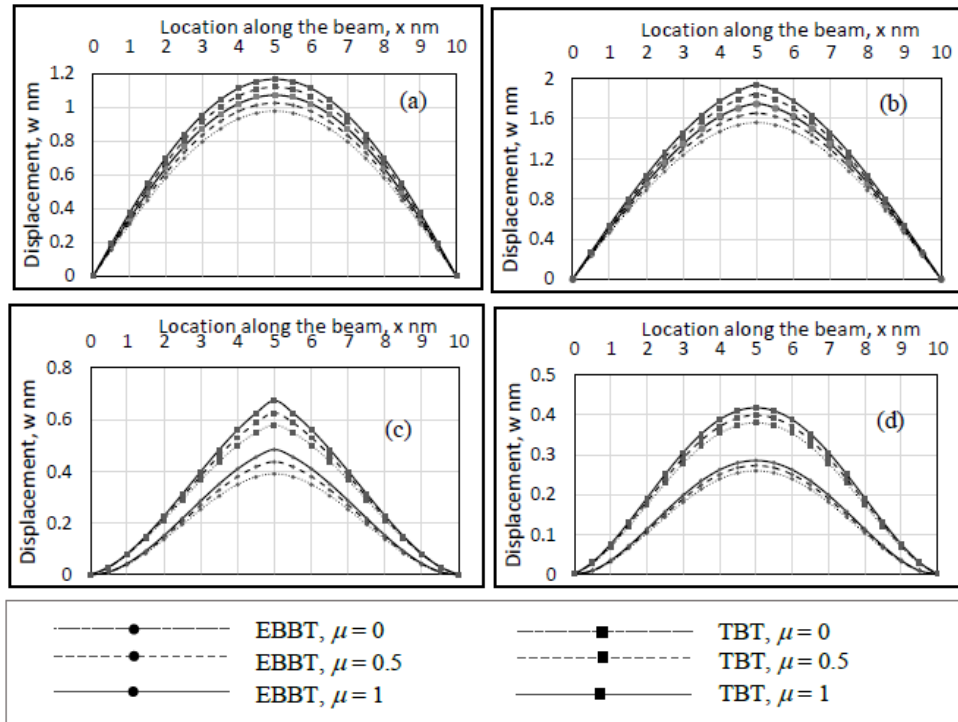


Fig. 3 Displacement profiles of (a) (b) S-S and (c) (d) C-C nonlocal beams

Table 5 Maximum displacements of S-S and C-C nonlocal beams

		S-S	C-C
UDL	EBBT	$(5 + 48\bar{\mu}) \frac{QL^4}{384EI}$	$\frac{QL^4}{384EI}$
	TBT	$(5 + 48\bar{\Omega} + 48\bar{\mu}) \frac{QL^4}{384EI}$	$(1 + 48\bar{\Omega}) \frac{QL^4}{384EI}$
Sinusoidal load	EBBT	$(1 + \pi^2\bar{\mu}) \frac{BL^4}{\pi^4EI}$	$[(1 + \pi^2\bar{\mu})(4 - \pi)] \frac{BL^4}{4\pi^4EI}$
	TBT	$[(1 + \pi^2\bar{\mu})(1 + \pi^2\bar{\Omega})] \frac{BL^4}{\pi^4EI}$	$[(1 + \pi^2\bar{\mu})(4 - \pi + 4\pi^2\bar{\Omega})] \frac{BL^4}{4\pi^4EI}$
Point Load at mid-span	EBBT	$(1 + 12\bar{\mu}) \frac{FL^3}{48EI}$	$(1 + 24\bar{\mu}) \frac{FL^3}{192EI}$
	TBT	$(1 + 12\bar{\Omega} + 12\bar{\mu}) \frac{FL^3}{48EI}$	$(\frac{1}{4} + 12\bar{\Omega} + 6\bar{\mu}) \frac{FL^3}{48EI}$

Table 6 Accuracy of the maximum displacements of S-S beams compared to other presented values

μ	UDL					A point load at the mid-span				
	Navier Method, (Reddy 2007)	Navier Method, (Aydogdu, 2009)	Analytical, (Wang and Liew 2007)	Finite element, (Nguyen <i>et al.</i> , 2015)	This study	Navier Method, (Reddy 2007)	Navier Method, (Aydogdu, 2009)	Analytical, (Wang and Liew 2007)	Finite element, (Nguyen <i>et al.</i> 2015)	This study
0	1.3130	1.3130	1.3021	1.3021	1.30208	1.9444	2.2222	2.0833	2.0833	2.0833
1	1.4487	1.4487	1.4271	1.4271	1.42708	2.1111	2.5535	2.3333	2.3333	2.3333
2	1.5844	1.5844	1.5521	1.5521	1.55208	2.2778	2.8848	2.5833	2.5833	2.5833
3	1.7201	1.7201	1.6771	1.6771	1.67708	2.4444	3.2162	2.8333	2.8333	2.8333
4	1.8558	1.8558	1.8021	1.8021	1.80208	2.6111	3.5475	3.0833	3.0833	3.08333

The variation of the maximum displacement magnitude with respect to the nonlocal parameter is linear, while the locations of the maximum displacement for the cases presented in Table 5 do not depend on the nonlocal

parameter. The accuracy of the maximum displacements presented in Table 5 is established by comparisons with the values reported by other researchers in Tables 6 and 7 for the EBBT and the TBT, respectively.

Table 7 Comparisons of maximum displacements

L/h	μ	Navier Method, (Reddy 2007)	Nonlocal Shear Deformation, (Thai 2012)	Analytical, (Reddy and Pang 2008)	This study
	0	1.3483	1.3346	1.33458	1.33458
	1	1.4937	1.4622	1.45958	1.45958
10	2	1.6391	1.5898	1.58458	1.58458
	3	1.7845	1.7173	1.70958	1.70958
	4	1.9299	1.8449	1.83458	1.83458

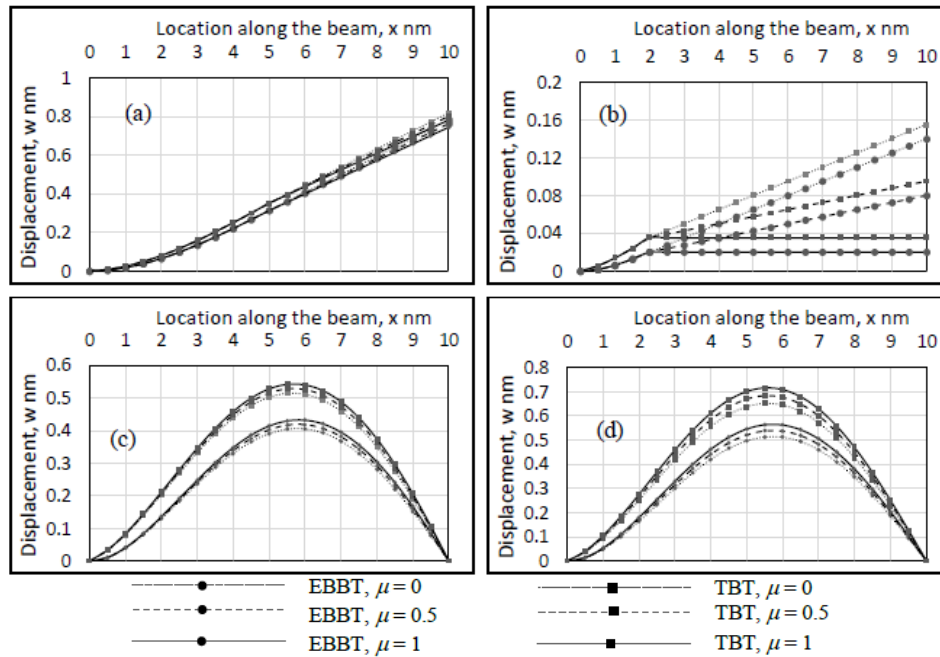


Fig. 4 Displacement profiles of the nonlocal cantilever and propped cantilevers

Table 8 Displacements of the free end of the nonlocal cantilever

UDL	Sinusoidal load	Point Load	
		At the free end	At mid-span
$(1 - 4\bar{\mu}) \frac{QL^4}{8EI}$	$(\pi^2 - 3 - 3\pi^2\bar{\mu}) \frac{BL^4}{3\pi^3EI}$	$\frac{FL^3}{3EI}$	$(5 - 24\bar{\mu}) \frac{FL^3}{48EI}$
$(1 + 4\bar{\Omega} - 4\bar{\mu}) \frac{QL^4}{8EI}$	$(\pi^2 - 3 + 3\pi^2\bar{\Omega} - 3\pi^2\bar{\mu}) \frac{BL^4}{3\pi^3EI}$	$(1 + 3\bar{\Omega}) \frac{FL^3}{3EI}$	$(5 + 24\bar{\Omega} - 24\bar{\mu}) \frac{FL^3}{48EI}$
$(1 - 4\bar{\mu}) \frac{QL^4}{8EI}$	$(\pi^2 - 3 - 3\pi^2\bar{\mu}) \frac{BL^4}{3\pi^3EI}$	$\frac{FL^3}{3EI}$	$(5 - 24\bar{\mu}) \frac{FL^3}{48EI}$
1.9299	1.8449	1.83458	1.83458

4.3 Cantilever and propped cantilevers

Fig. 4 presents the small size effects on the displacement profiles of cantilevers and propped cantilevers by choosing three values of the nonlocal parameter. The displacements decrease for a cantilever with the increase of the nonlocal parameter, contrary to the other three boundary conditions. Furthermore, the nonlocal effects on displacements under a point load exist at locations where $x > b$, with a nonzero value of the Heaviside function $H(x - b)$. No nonlocal effect remains for $b = L$, if the point load is applied at the free end. Table 8 presents the non-dimensional

displacements of the free end of the nonlocal cantilever under the various distribution of loads. The accuracy of the presented equations of displacements of the nonlocal cantilever (Table 3) and its maximum displacements (Table 8) is demonstrated by some numerical comparisons of displacements in Table 9.

The location of the maximum displacement of the propped cantilever depends on the nonlocal parameter, even for symmetric loading. This is evidenced by solving the first derivative of the equation of displacement under UDL for x . The location of the maximum deflection in terms of non-dimensionalized location $\bar{x} = x/L$ of a propped cantilever

Table 9 Displacements of the free end of the cantilever under point load at various locations

L/h	μ	point load at $x/L = 0.2$		point load at $x/L = 0.6$	
		(Wang and Liew 2007)	This study	(Wang and Liew 2007)	This study
10	0	1.866667	1.866667	14.40	14.40
	1	1.066667	1.066667	14.00	14.00
	2	0.266667	0.266667	13.60	13.60
	3	-0.533333	-0.533333	13.20	13.20
	4	-1.333333	-1.333333	12.80	12.80

Table 10 Maximum displacement and its location of the propped cantilever

$\bar{\mu}$	UDL				Sinusoidal load			
	EBBT		TBT ($\bar{\Omega} = 0.01$)		EBBT		TBT ($\bar{\Omega} = 0.01$)	
	\bar{x}	$w * 48 \frac{EI}{QL^4}$	\bar{x}	$w * 48 \frac{EI}{QL^4}$	\bar{x}	$w * \frac{\pi^4 EI}{BL^4}$	\bar{x}	$w * \frac{\pi^4 EI}{BL^4}$
0	0.578	0.260	0.565	0.329		0.425		0.533
0.5	0.647	1.139	0.634	1.258	0.573	2.525	0.589	3.162
1	0.656	2.026	0.644	2.197		4.624		5.792

under UDL by the EBBT is given by

$$\bar{x} = \frac{1}{16} (15 + 36\bar{\mu} - \sqrt{3} \sqrt{11 + 104\bar{\mu} + 432\bar{\mu}^2}) \quad (61)$$

Table 10 presents a few numerical values of maximum displacements of the propped cantilever under UDL and sinusoidal loads, along with the location of maximum displacements.

5. Conclusions

This paper developed analytical expressions for static bending deflections of nonlocal beams for Euler-Bernoulli and Timoshenko beams by applying Castigliano's theorem in a virtual local beam. The postulate of superimposed curvature on VLB and the presented method are validated by matching the derived analytical expressions and maximum displacements with those reported by other researchers. The superimposed curvature due to nonlocal effects on a VLB helps to explicitly understand the size effects of nanobeams on its deflections and stress-resultants. The proposed method provides quick and easy bending solutions of nonlocal beams by using familiar analysis techniques of regular local beams with local boundary conditions under any transverse load. The method has the potential for further extensions to other mechanics problems due to its generality and simplicity.

The presented analytical expressions are helpful to engineers who are designing micro- and nanoelectron-mechanical devices and carbon nanotubes. Moreover, the exact solutions are benchmarks for checking other mathematical models' validity, convergence, and accuracy.

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