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Eringen's nonlocal theory for non-linear bending analysis of BGF Timoshenko nanobeams

Mojtaba Gorji Azandariani[1a], Mohammad Gholami[*2] and Akbar Nikzad[3b]

[1]*Structural Engineering Division, Faculty of Civil Engineering, Semnan University, Semnan, Iran*
[2]*Department of Civil Engineering, Yasouj University, Yasouj, Iran*
[3]*Department of Civil Engineering, Islamic Azad University of Bushehr, Bushehr, Iran*

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Abstract. In this paper, the non-linear static analysis of Timoshenko nanobeams consisting of bi-directional functionally graded material (BFGM) with immovable ends is investigated. The scratching in the FG nanobeam mid-plane, is the source of nonlinearity of the bending problems. The nonlocal theory is used to investigate the non-linear static deflection of nanobeam. In order to simplify the formulation, the problem formulas is derived according to the physical middle surface. The Hamilton principle is employed to determine governing partial differential equations as well as boundary conditions. Moreover, the differential quadrature method (DQM) and direct iterative method are applied to solve governing equations. Present results for non-linear static deflection were compared with previously published results in order to validate the present formulation. The impacts of the nonlocal factors, beam length and material property gradient on the non-linear static deflection of BFG nanobeams are investigated. It is observed that these parameters are vital in the value of the non-linear static deflection of the BFG nanobeam.

Keywords: Eringen's nonlocal theory; bi-directional functionally graded; nanobeam; non-linear static deflection; Timoshenko theory

1. Introduction

Due to the unique properties included in the functionally graded (FG) nanobeams, including low density, high strength, magneto-electro-viscoelastic, thermal resistance, and toughness, they have attracted much attention among scholars and engineers (Aydogdu and Taskin 2007, Ebrahimi *et al*. 2019, Ebrahimi and Fardshad 2018, Ebrahimi and Haghi 2018, Gao *et al*. 2019, Gholami *et al*. 2021, Gorji Azandariani *et al*. 2021b, e, Rousta *et al*. 2021, Talebizadehsardari *et al*. 2020, Usefvand *et al*. 2021). In the last few years, many research works have been conducted on the different aspects included in FG beams (Amar *et al*. 2017, Berghouti *et al*. 2019, Ebrahimi and Zia 2015, Gorji Azandariani *et al*. 2021f, Li and Batra 2013, Thai and Vo 2012, Vaziri *et al*. 2021). Due to technological developments and achievements, functionally graded materials (FGMs) have begun to find their ways into micro/nanobeams. Given the great potential of micro/nanobeams in engineering applications, gaining a proper understanding of their mechanical behavior is of great importance. The classic continuum theories suffer from inaccurate predictions done for the mechanical behavior of nanostructures (Eringen 1983, Ghanbari-Ghazijahani *et al*. 2020, Gorji Azandariani *et al*. 2021a, Mohammadi *et al*. 2019, 2020). In this regard, a few size-dependent elasticity mechanics have been developed, including the nonlocal theory (Eringen 1983), the theory of modified couple stress (MCS) (Yang *et al*. 2002), Eringen's nonlocal elasticity theory (Nejad *et al*. 2018), and the theory of the nonlocal strain gradient (Lim *et al*. 2015). On the other hand, with the advance of nanotechnology, microbeams, in which their thickness is generally on the order of microns and sub-microns, have been widely used in many applications of microdevices. Micro-electro-mechanical systems are a technology that, in its most general form, can be defined as devices and structures that are made using the techniques of microfabrication in its most general form. The critical physical dimensions of micro-electro-mechanical systems devices can vary from well below one micron on the lower end of the dimensional spectrum, all the way to several millimeters. While the functional elements of micro-electro-mechanical systems are miniaturized structures, sensors, actuators, and microelectronics, the most notable elements are the microsensors and microactuators. However, it is experimentally proved that the size effect also becomes important for the mechanical behavior of microstructures when the dimensions of structures become on the order of microns and sub-microns. Due to this fact, it is inevitable to consider the size effect in the analysis of the mechanical behavior of microstructures. At the same time, since controlled experiments in microscale are both difficult and expensive, the development of appropriate mathematical models for microstructures is an important issue concerning an approximate analysis of microstructures.

Gao *et al*. (2019) investigated non-linear thermal buckling of bi-directional functionally graded beams in the

*Corresponding author, Ph.D.,
E-mail: m.gholami@yu.ac.ir
[a] Ph.D., E-mail: MGorji@semnan.ac.ir
[b] Ph.D. Candidate

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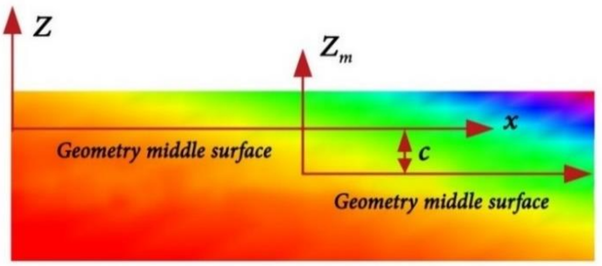
theoretical frameworks of nonlocal strain graded theory. This study was used a higher-order shear deformation theory that contains a physical neutral surface to derive the size-dependent governing equations hybrid with Hamilton's principle and the von Kármán geometric nonlinearity. Also, a parametric study is performed in detail after verifying the analysis, especially for the effects of a nonlocal parameter, a strain gradient length scale parameter, and the ratio of the two on the critical thermal buckling temperature of beams. Ebrahimi and Barati (2018) provide an analytical solution to the buckling governing equations of functionally graded piezoelectric (FGP) nanobeams obtained using a developed third-order shear deformation theory. Employing Hamilton's principle, the nonlocal governing equations of an FG nanobeam made of piezoelectric materials are obtained, and they were solved by applying a Navier-type analytical solution. Ebrahimi and Barati (2018) validated the present model's accuracy by comparing it with nonlocal Timoshenko FG beams. Aydogdu *et al*. (2018) studied the vibration of axially functionally graded nano-rods and beams. The Ritz method with algebraic polynomials and stress gradient elasticity theory with the nonlocal effects formulated the problems.

Also, the nonlocal parameter was assumed to change linearly or quadratically along the length of the nanostructure. Frequencies are compared to constant nonlocal parameter cases, and considerable differences are observed between constant and variable nonlocal parameter cases. Mode shapes in various cases are depicted to explain the effects of axial grading. Luat *et al*. (2021) investigated the bending, free vibration, and buckling analysis of a novel bi-functionally graded sandwich nanobeam via a nonlocal refined simple shear deformation theory. The novel sandwich beam included one ceramic core and two different functionally graded face sheets. Eringen's nonlocal elastic theory was utilized in cooperation with a refined simple shear deformation theory and Hamilton's principle to derive the equations of motion. A closed-form solution based on Navier's technique is used to solve the equations of motion of supported nanobeams. Also, in this study, the results were compared with existing solutions to evaluate the accuracy of the proposed theory. Zidi *et al*. (2017) proposed a novel simple higher-order shear deformation theory for bending and free vibration analysis of functionally graded (FG) beams. In this method, equations of motion were obtained via Hamilton's principle. Zidi *et al*. analytical equations for the bending and free vibration analysis were presented for simply supported beams. In the final, the numerical results compared with those of other higher-order shear deformation beam theories were obtained and validated.

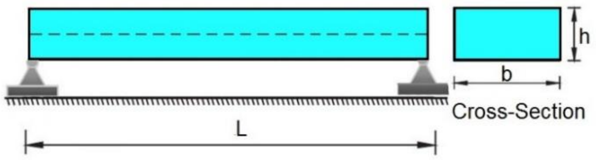
The buckling behavior and the vibration behavior of FG micro/nanobeams have been investigated in many studies. For example, in a study, the FG Timoshenko microbeams' free vibration was investigated according to the MCS theory (Asghari *et al*. 2011). In another study (Ansari *et al*. 2011), the free vibrations of FG microbeams were analyzed using the theory of the strain gradient. A group of researchers has studied the nonlinearity of the free vibration in the FG microbeams using the theory of MCS (Ke *et al*. 2012). By applying the nonlocal theory, a finite element formulation was proposed to analyze the free vibrations in the FG nanobeams (Nejad *et al*. 2017, Sanjay Anandrao *et al*. 2012, Setoodeh and Rezaei 2017). In these works, the static buckling behavior in the FG nanobeams depended on size was also studied using the nonlocal continuum model. In another work, the bending/buckling behaviors of FG nanobeam have been examined through an analytical approach concerning nonlocal theory based on Timoshenko and Euler Bernoulli beam (Simsek and Yurtcu 2013). In addition, the vibrations were also examined in the simply supported (SS) Timoshenko FG nanobeams applying the MCS theory (Rahmani and Pedram 2014). Through an analytical study, the size-dependent bending/ buckling behaviors were examined in the FG nanobeams considering the effects induced by thickness stretching (Chaht *et al*. 2015). Moreover, in many other types of research, the vibration behavior and the buckling behavior of the FG micro/nanobeams have been investigated (Chaht *et al*. 2015, Gholhaki *et al*. 2021, Gorji Azandariani *et al*. 2021h, e, Rahmani *et al*. 2017, Tagrara *et al*. 2015).

One of the critical issues in structural engineering is the non-linear bending of the beam exposed to very large displacements. The beam bending behavior is in a non-linear regime when the ends of a beam subjected to large transverse loads are axially immovable because of axial tension induced by a large deflection. Lately, a great number of researchers have been tried to investigate the non-linear behavior of FG beams. Through a study conducted on the non-linear behavior in the FG structures, the size dependency of non-linear vibrations has been investigated in the FG microbeams using the Casimir force, combined electrostatic force, and temperature changes (Jia *et al*. 2015). In another study, the non-linear bending behavior was investigated in the tapered FG beam subjected to the thermal/mechanical loads (Niknam *et al*. 2014, Zenkour and Abouelregal 2015). In a different work, an exact solution was proposed by some scholars to solve the non-linear forced vibrations in the FG nanobeams by taking into account the surface effect (Akbaş 2018, Ansari *et al*. 2015). The non-linear vibrations of the size-dependent beam were investigated according to Eringen's nonlocal theory and the non-linear geometric theory (Ahouel *et al*. 2016, Gorji Azandariani *et al*. 2020a, 2021c, g, Şimşek 2014). In addition to the done research, some other investigations were also conducted to study the non-linear vibration of the FG micro/nanobeams (Gorji Azandariani *et al*. 2020b, 2021d, Setoodeh and Rezaei 2017, Şimşek 2016).

The material properties of the FG beam gradually change along their thickness direction, while the material properties of bi-directional functionally graded (BFG) nanobeams change along both length and thickness of the nanobeam (Fig. 1(a)). It worths mentioning that the studies, as mentioned earlier, are related to FG micro/nanobeams. The reported results regarding the BFG micro/nanobeams are not remarkable. Eringen's nonlocal theory was developed based on the differential quadrature method (DQM) using the Euler in a few studies on the BFG nanobeams considering the buckling, bending, and vibration



(a)



(b)

Fig. 1 Bi-directional functionally graded (BFG) nanobeam; (a) variation in mechanical properties and (b) physical neutral surface and geometric mid-surface

behaviors–Bernoulli theory. The buckling and vibration behaviors were also analyzed in the BFG porous tapered micro/nanobeams and imperfect BFG porous micro/ nanobeams applying theories of Euler–Bernoulli and Timoshenko beams.

A review of the history of the above research clearly shows that no attention has been paid to the mechanical behavior of nanobeams with geometric nonlinearity and to the author's knowledge, no published work on eringen's nonlocal theory for non-linear bending analysis of bi-directional functionally graded Timoshenko nanobeams there is no pair. Accordingly, in the present study, the impact of the nonlinearity related to geometry is thoroughly examined on the static deflection of BFG nanobeams applying Eringen's nonlocal theory. In order to simplify the formulation, the problem formulas are derived according to the physical middle surface. The Hamilton principle is employed to determine governing partial differential equations as well as boundary conditions. Moreover, the differential quadrature method (DQM) and direct iterative method are applied to solve governing equations. Present results for non-linear static deflection were compared with previously published results in order to validate the present formulation. The impacts of the nonlocal factors, beam length and material property gradient on the non-linear static deflection of BFG nanobeams are investigated. It is observed that these parameters are vital in the value of the non-linear static deflection of the BFG nanobeam.

2. Preliminaries

2.1 Functionally graded materials

Assum a BFG nanobeam with a length defined as L and a rectangular cross-section defined as $b \times h$ (b = width and h = height), and with immovable simple supports at both ends (Fig. 1(b)). Here, to simplify the calculations, the assumption is that Poisson's ratio ν is constant. Moreover, mass density ρ, as well as elasticity modulus E, is defined as follows (Nejad and Hadi 2016):

$$E(x, z_{ms}) = e^{N_x \frac{x}{L}}\left[(E_c - E_m)\left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^{N_z} + E_m\right] \quad (1)$$

$$\rho(x, z_{ms}) = e^{N_x \frac{x}{L}}\left[(\rho_c - \rho_m)\left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^{N_z} + \rho_m\right] \quad (2)$$

where m is the metallic constituent, c is the ceramic constituent, N_x and N_z indicate the gradient parameters dictating the alteration profile of the material along the thickness direction and length direction of the nanobeam, respectively.

It is worth mentioning that in Eq. (1), the geometric mid-surface z_{ms} are considered a reference for the properties of the material. As is known, the geometric mid-surface may not coincide with the neutral surface in the BFG nanobeams when the material characteristics of the BFG nanobeam are asymmetric regarding its geometric mid-surface, resulting in stretching and bending coupling to develop. However, the stretching and bending coupling will be eliminated if the physical neutral surface is identified as the coordinate system's origin (Fig. 1(b)).

The relationship between the geometric mid-surface and physical neutral surface is expressed as follows (Şimşek 2016):

$$z_{ms} = z_c + e, \qquad e = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z_{ms})\, dz_{ms}}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z_{ms})\, dz_{ms}} \quad (3)$$

In case the physical middle surface is considered as a reference, the effective characteristics of the material become:

$$E(x, z_c) = e^{N_x \frac{x}{L}}\left[(E_c - E_m)\left(\frac{z_c + e}{h} + \frac{1}{2}\right)^{N_z} + E_m\right] \quad (4)$$

$$\rho(x, z_c) = e^{N_x \frac{x}{L}}\left[(\rho_c - \rho_m)\left(\frac{z_c + e}{h} + \frac{1}{2}\right)^{N_z} + \rho_m\right] \quad (5)$$

2.2 Eringen's nonlocal theory

Eringen's nonlocal theory (Eringen 1983) assumes that the stress tensor matrix at any point of x in the material domain is dependent on two factors: **(i)** the strain tensor matrix corresponding to point x, **(ii)** strain tensor matrix at all of the rest of points in the domain. It is consistent with experimental data reported in the literature based on the lattice dynamics and the atomic theory. Based on the mentioned theory, components of the tensor matrix of the nonlocal stress σ_{ij} (x) at an arbitrary point of x in the homogeneous elastic body could be defined by the following equation:

$$\sigma_{ij}(x) = \int_{\Omega}^{l} \alpha(|x' - x|, \tau) t_{ij}(x') d\Omega(x') \tag{6}$$

where $t_{ij}(x')$ is the component for the classic local stress-tensor at point x, a relationship can be defined between classic local stress-tensor and linear tensor matrix of the strain ε_{kl} using the fundamental equations for a Hookean material:

$$t_{ij} = c_{ilkl}\varepsilon_{kl} \tag{7}$$

In Eq. (6), α is the kernel function known as nonlocal modulus, which depends on two variables $|x'-x|$ defined as the distance in Euclidean form and τ that is determined as follows:

$$\tau = \frac{e_0 a}{l} \tag{8}$$

where τ represents the ratio of a characteristic internal length (e.g., granular distance, lattice parameter) to a characteristic external length l (e.g., wavelength, crack length) related to each material using an adjusting constantly, e_0, the e_0 magnitude is estimated through two ways: **(i)** by fitting the dispersion curves of plane waves to the dispersion curves of atomic lattice dynamics or, **(ii)** experimentally.

According to a study conducted by Eringen, the nonlocal integral fundamental equation (Eq. (6)) could be expressed in differential form in the following way if the appropriate kernels function is chosen:

$$(1 - (e_0 a)^2 \nabla^2)\sigma_{kl} = t_{kl} \tag{9}$$

where ∇^2 indicates the Laplacian operator. For one dimension, it is possible to simplify the nonlocal constitutive relation (Eq. (9)) for an elastic material as follows:

$$\sigma_{xx} - (e_0)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \tag{10a}$$

$$\sigma_{xz} - (e_0)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\gamma_{xz} \tag{10b}$$

where E, G, σ_{xx}, σ_{xz}, ε_{xx}, and γ_{xz} are the elasticity modulus, shear modulus, normal axial stress, shear stress, axial and shear strain, respectively.

3. The governing equation and BCs

The Timoshenko beam theory is applied in the present analysis. Displacements of u_1, u_2, and u_3 along the X, Y, and Z directions of a point on the cross-section can be expressed as follows (Rahmani and Pedram 2014):

$$\begin{aligned} u_1(x,z,t) &= u(x,t) - z_c\phi(x,t) \\ u_2(x,z,t) &= 0 \\ u_3(x,z,t) &= w(x,t) \end{aligned} \tag{11}$$

where $u(x,t)$ and $w(x,t)$ are the axial and transverse displacements of a point on the beam mid-surface, respectively, $\phi(x,t)$ signifies the cross-section flexural rotations at any point on the neutral beam axis. t stands for time. The strains are expressed by following Eq. (12) (Li and Hu 2016):

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z_c \frac{\partial \phi}{\partial x} \\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} - \phi\right) \end{aligned} \tag{12}$$

Note the shear strain $\gamma = 2\varepsilon_{xz}$.

Using the Eq. (12), the potential energy U of the nano FGM is given by (Li and Hu 2016):

$$\begin{aligned} U &= \frac{1}{2}\int_0^L \int_A^d (\sigma_{xx}\varepsilon_{xx} + 2\sigma_{xz}\varepsilon_{xz}) dA\,dy \\ &= \frac{1}{2}\int_0^L h\left[N_{xx}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right. \\ &\left. - M_{xx}\frac{\partial \theta}{dx} + Q_{xz}\left(\frac{\partial w}{\partial x} - \phi\right)\right] dx \end{aligned} \tag{13}$$

where the resultant normal force N_{xx}, shearing force Q_x, and flexural moment M_{xx} are expressed in the form of as follows:

$$\begin{aligned} N_{xx} &= \int_A^s \sigma_{xx}\, dA \\ M_{xx} &= \int_A^j z_c \sigma_{xx}\, dA \\ Q_{xz} &= \int_A^d \sigma_{xz}\, dA \end{aligned} \tag{14}$$

The virtual work carried on due to the applied external forces could be expressed as:

$$W = \int_0^L q w\, dx \tag{15}$$

The Hamilton principle is implemented as follows:

$$\delta \int_{t_1}^{t_2} (U + W)\, dt = 0 \tag{16}$$

The Substitution of the terms for U and W, respectively, from Eqs. (13) and (15) into Eq. (16) and integrating by parts according to t and also x to relieve the virtual alterations δu and δw of any differentiation, and then by using the fundamental lemma of calculus can result in obtaining some governing equations in the form of as follows:

$$\delta_u \Rightarrow \frac{\partial N_{xx}}{\partial x} = 0 \tag{17a}$$

$$\delta_\phi \Rightarrow Q_{xz} - \frac{\partial M_{xx}}{\partial x} = 0 \tag{17b}$$

$$\delta_w \Rightarrow \frac{\partial Q_{xx}}{\partial x} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) - q = 0 \tag{17c}$$

And boundary conditions at the point of $x = 0, L$ are:

$$\delta_u \Rightarrow N_{xx} = 0 \quad or \quad u = 0 \tag{18a}$$

$$\delta_w \Rightarrow Q_{xx} + N_{xx}\frac{\partial w}{\partial x} \qquad or \qquad w = 0 \tag{18b}$$

$$\delta_\phi \Rightarrow M_{xx} = 0 \qquad or \qquad \phi = 0 \tag{18c}$$

With the help of Eq. (17a) and Eq. (17c) can be rewritten Eq. (17c) as fallow:

$$\frac{\partial Q_{xx}}{\partial x} + N_{xx}\frac{\partial^2 w}{\partial x^2} = I_0 e^{B\frac{x}{L}}\,\frac{\partial^2 w}{\partial t^2} \tag{19}$$

By using Eqs. (10a), (10b), (12), and (14), relations of the load-displacement and the moment- displacement based on the nonlocal theory could be expressed in the form of as fallow:

$$N_{xx} - \mu\frac{\partial^2 N_{xx}}{\partial x^2} = \int_A^g E(x,z_c)\begin{bmatrix}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ -z_c\frac{\partial^2 w}{\partial x^2}\end{bmatrix} dA = A_0\, e^{B\frac{x}{L}}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{20a}$$

$$Q_{xz} - \mu\frac{\partial^2 Q_{xz}}{\partial x^2} = \int_A^g G(x,z_c)\left(\frac{\partial w}{\partial x} - \phi\right) dA = A_3\, e^{B\frac{x}{L}}\left(\frac{\partial w}{\partial x} - \phi\right) \tag{20b}$$

$$M_{xx} - \mu\frac{\partial^2 M_{xx}}{\partial x^2} = \int_A^g E(x,z_c)\left[z_c\frac{\partial u}{\partial x} + \frac{1}{2}z_c\left(\frac{\partial w}{\partial x}\right)^2 - Z_c^2\frac{\partial \phi}{\partial x}\right] = -A_2 e^{B\frac{x}{L}}\,\frac{\partial \phi}{\partial x} \tag{20c}$$

In the above relations, the stiffness coefficients are expressed as follow:

$$(A_0\,, A_2\,) = b\int_{\frac{h}{2}-e}^{\frac{h}{2}+e}(1, z_c^2)\left[\frac{(E_c - E_m)}{\left(\frac{z_c + e}{h} + \frac{1}{2}\right)^{N_z}} + E_m\right] dz \tag{21a}$$

$$A_3 = \frac{bK_s}{2(1+v)}\int_{\frac{h}{2}-e}^{\frac{h}{2}+e}\left[\frac{(E_c - E_m)}{\left(\frac{z_c + e}{h} + \frac{1}{2}\right)^{N_z}} + E_m\right] dz \tag{21b}$$

$$A_1 = b\int_{\frac{h}{2}-e}^{\frac{h}{2}+e} z_c\left[\frac{(E_c - E_m)}{\left(\frac{z_c + e}{h} + \frac{1}{2}\right)^{N_z}} + E_m\right] dz = 0 \tag{21c}$$

It worths mentioning that the stiffness coefficient causing the coupling of the stretching and bending becomes zero, in other words, is obtained from Eq. (21c) (Şimşek 2016).

Using the size-dependent equilibrium Eqs. (17a), (17b) and (20), the axial force, shearing force, and the moment could be as expressed:

$$N_{xx} = A_0\, e^{B\frac{x}{L}}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{22a}$$

$$Q_{xz} = A_3\, e^{B\frac{x}{L}}\left(\frac{\partial w}{\partial x} - \phi\right) + \mu\left[\frac{\partial q}{\partial x} - \left(A_0\, e^{B\frac{x}{L}}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right)\frac{\partial^3 w}{\partial x^3}\right] \tag{22b}$$

$$M_{xx} = -A_2 e^{B\frac{x}{L}}\,\frac{\partial \emptyset}{\partial x} + \mu\left[q - \left(A_0\, e^{B\frac{x}{L}}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right)\frac{\partial^2 w}{\partial x^2}\right] \tag{22c}$$

With the help of Eqs. (22a), (22b) and (22c), the non-linear equations of motion for an FG Timoshenko nanobeam in terms of the displacements can be obtained as:

$$\frac{B}{L}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{23a}$$

$$A_2\frac{\partial^2 \phi}{\partial x^2} + \frac{B}{L}A_2\,\frac{\partial \phi}{\partial x} + A_3\left(\frac{\partial w}{\partial x} - \phi\right) = 0 \tag{23b}$$

$$A_3\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) + \frac{B}{L}A_3\left(\frac{\partial w}{\partial x} - \phi\right) + \mu\frac{\partial^2 q}{\partial x^2} - q + A_0\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left(1 - \mu\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 w}{\partial x^2} = 0 \tag{23c}$$

The orders of non-linear equations of motion in the present BFG Timoshenko nanobeam model are greater than the orders of the model of the classical Timoshenko beam (Su and Banerjee 2015).

4. Solution procedure

In this section, the differential quadrature method (DQM) is used to solve Eqs. (23a)-(23c). First of all, in this section, a review is provided on the method. The main application of this approach is estimating the derivatives of a dependent variable using a weighted linear combination of variable values at the nodes (predefined points in the domain). Consider a continuous function on $x(f(x))$ over the $1D$ domain ($0 \le X \le L$), which can be expressed using Lagrange interpolation with predefined N points.

$$f(x) = \sum_{j=1}^{N} L_j(x) f_j \text{ for j} = 1,2,\ \dots,\text{N} \quad \text{with } f_j = f(x_j) \text{ and } x_j \in [0, L] \tag{24}$$

where $L_j(x)$ indicates the Lagrange interpolating function and f_j is the functional value corresponding to the j-th node. Thus, the approximation of the n-th derivative of the function is given by:

$$f_i^{(n)} = \sum_{j=1}^{N} c_{ij}^{(n)}(x) f_i \text{ for i} = 1,2,\dots,\text{N} \quad \text{with } f_i^{(n)} = \left.\frac{d^n f(x)}{dx^n}\right|_{x=x_i} \tag{25}$$

where $c_{ij}^{(n)}$ indicates the weight coefficient, representing the contribution of the functional value at the j-th node to the n-th derivative value at the i-th node, the following equations indicate the weight coefficient:

For n = 1:

$$c_{ij}^{(1)} = \frac{M(x_i)}{(x_i - x_j)M(x_j)} : i, j = 1,2,\dots\text{n, and i} \ne j \tag{26}$$

$$c_{ij}^{(1)} = -\sum_{\substack{j=1\\ j\neq i}}^{n_x} c_{ij}^{(n)}: \qquad i=j$$

where

$$M(X_i) = \prod_{j=1, j\neq i}^{n} (x_i - x_j) \tag{27}$$

For $2 \leq n \leq (N\text{-}1)$:

$$\begin{aligned} &c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)}c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{x_i - x_j}\right) \\ &i,j = 1,2,\ldots,N \qquad j\neq i \\ &c_{ii}^{(n)} = -\sum_{\substack{j=1\\ j\neq i}}^{n_x} c_{ij}^{(n)}, \qquad \begin{cases} i = 1,\ldots,N \\ n = 1,2,\ldots,N-1 \end{cases} \end{aligned} \tag{28}$$

In this study, the interpolation points' locations are being taken as Gauss-Lobatto-Chebyshev type as they are considered to be resulting in higher accuracy:

$$X_i = \frac{l}{2}\left(1 - cos\left(\frac{(i-1)}{(n-1)}\pi\right)\right) \quad i = 1,2,3, \ \ldots n \tag{29}$$

Substituting Eqs. (24) and (25) into Eqs. (23a)-(23c) will lead to a set of differential equations:

$$\begin{aligned} &\frac{B}{L}\left[\sum_{m=1}^{n} C_{1m}^{(1)}u_m + \frac{1}{2}\left(\sum_{m=1}^{n} C_{1m}^{(1)}w_m\right)^2\right] \\ &+\left(\sum_{m=1}^{n} C_{1m}^{(2)}u_m + \sum_{m=1}^{n} C_{1m}^{(1)}w_m \sum_{m=1}^{n} C_{1m}^{(2)}w_m\right) = 0 \end{aligned} \tag{30a}$$

$$\begin{aligned} &A_2\sum_{m=1}^{N} c_{im}^{(2)}\phi_m + \frac{B}{L}A_2\sum_{m=1}^{N} c_{im}^{(1)}\phi_m \\ &+A_3\left(\sum_{m=1}^{N} c_{im}^{(1)}w_m - \phi_i\right) = 0 \end{aligned} \tag{30b}$$

$$\begin{aligned} &A_3\left(\sum_{m=1}^{N} c_{im}^{(2)}w_m - \sum_{m=1}^{N} c_{im}^{(1)}\phi_m\right) \\ &+\frac{B}{L}A_3\left(\sum_{m=1}^{N} c_{im}^{(1)}w_m - \phi_i\right) + \mu\sum_{m=1}^{N} c_{im}^{(2)}q_m - q_i \\ &+\left[A_0\left[\sum_{m=1}^{N} c_{im}^{(1)}u_m + \frac{1}{2}\left(\sum_{m=1}^{N} c_{im}^{(1)}w_m\right)^2\right]\right]\times \\ &\left(\sum_{m=1}^{N} c_{im}^{(2)}w_m - \mu\sum_{m=1}^{N} c_{im}^{(4)}w_m\right) = 0 \end{aligned} \tag{30c}$$

where i = 1, 2, ... , N.

It is possible to estimate the relevant boundary conditions in a similar way. That is, for a clamped - clamped (CC) and Simple – simple (SS) BFG microbeam:

$$\begin{aligned} u = w = \phi = 0 \quad at \ x = 0 \\ u = w = \phi = 0 \ \ at \ \ x = L \quad \text{for (CC)} \end{aligned} \tag{31a}$$

$$\begin{aligned} u = w = 0 \quad at \ x = 0 \\ u = w = 0 \ \ at \ \ x = L \quad \text{for (SS)} \end{aligned} \tag{31b}$$

By placing the boundary conditions in Eqs. (30), are expressed as follows:

$$\begin{aligned} &u_1 = w_1 = 0 \\ &\mu A_0\left[\begin{matrix}\sum_{m=1}^{N} c_{1m}^{(1)}u_m \\ +\frac{1}{2}\left(\sum_{m=1}^{N} c_{1m}^{(1)}w_m\right)^2\end{matrix}\right]\sum_{m=1}^{N} c_{1m}^{(2)}w_m \\ &-A_2\sum_{m=1}^{N} c_{1m}^{(1)}\phi_m = 0 \quad at \quad x = 0 \end{aligned} \tag{32a}$$

$$\begin{aligned} &u_N = w_N = 0 \\ &\mu A_0\left[\begin{matrix}\sum_{m=1}^{N} c_{Nm}^{(1)}u_m \\ +\frac{1}{2}\left(\sum_{m=1}^{N} c_{Nm}^{(1)}w_m\right)^2\end{matrix}\right]\sum_{m=1}^{N} c_{Nm}^{(2)}w_m \\ &-A_2\sum_{m=1}^{N} c_{Nm}^{(1)}\phi_m = 0 \quad at \quad x = L \end{aligned} \tag{32b}$$

Denoting the mode shape vector of a non-linear vibration is expressed as follows:

$$d = \{\{u_i\}^T, \{w_i\}^T, \{\emptyset_i\}^T\}^T \tag{33}$$

Eqs. (30a)-(30c) and corresponding boundary conditions (31) and (32) can be expressed in the form of the matrix as:

$$F = d\left(K_L + \frac{1}{2}K_{NL1} + \frac{1}{3}K_{NL2}\right) \tag{34}$$

where F is the matrix of force; K_L is the linear stiffness matrix; K_{NL1} and K_{NL2} are non-linear stiffness matrices that are linear and second-order functions in d, respectively. K_L, K_{NL1}, and K_{NL2} are $3N \times 3N$ symmetric matrices, and F is $3N \times 1$ matric. These equations can be resolved by applying a direct iterative algorithm.

5. Numerical results

This section contains two separate subsections. In the first subsection, the verification of the proposed nonlocal model is presented based on comparisons with previously published models. In the second subsection, the significance of several parameters, including the nonlocal parameter, the material distribution, as well as the boundary conditions on the non-linear static deflections of the BFG nanobeams, will be presented.

5.1 Model validation

According to Table 2, to approve the reliability of the proposed formulation, the linear deflection ratios, i.e., nonlocal linear deflection to classical linear deflection ratios, for BFG nanobeams with clamped-clamped (CC)

Table 1 Types of the considered boundary conditions

Boundary condition type	Conditions	
	$x = 0$	$x = L$
Simple – simple	$u = w = 0$	$u = w = 0$
Clamped – clamped	$u = w = \phi = 0$	$u = w = \phi = 0$

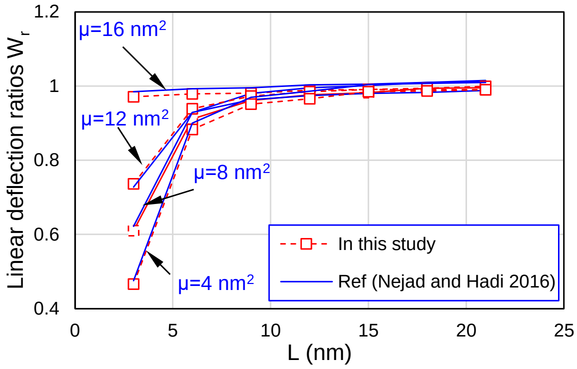
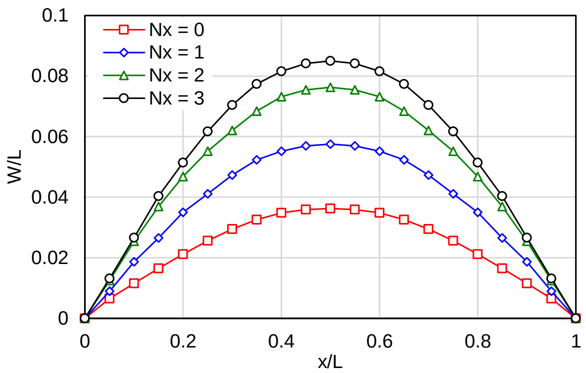
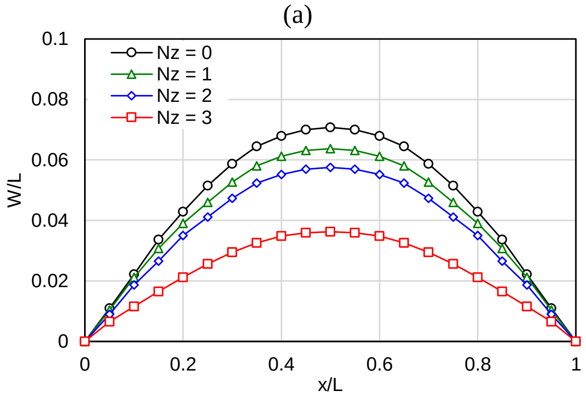


Fig. 2 Comparison of linear deflection ratios W_r



(a)



(b)

Fig. 3 Variation of the linear and non-linear dimensionless deflection W/L versus distance along the length of a SS nanobeam: (a) for different gradient index along an axis, N_z, when $N_x = 1$ and (b) for different gradient index along the length, N_x, when $N_z = 2$

boundary conditions have been compared with ones of Nejad and Hadi (2016) for various nonlocal parameters μ and nanobeam length *L*. The features related to geometry and material properties of applied nanobeam are derived from (Nejad and Hadi 2016). It should be noted that the linear deflection ratios of the BFG nanobeams can be obtained by neglecting the non-linear terms. The authors of (Nejad and Hadi 2016) used the nonlocal theory based on the theory of Euler–Bernoulli beam to derive results. Table 1 indicates that the present results follow the ones of Nejad *et al*. (2017), and they have little difference.

Also, to confirm the formulation, the nonlocal linear deflection to classical linear deflection ratios, for BFG nanobeams with clamped-clamped (CC) boundary obtained by Nejad and Hadi (2016) were examined. In Fig. 2, the linear deflection ratios-length curves of CC nanobeam of the formulation and data obtained by Nejad and Hadi (2016) results are provided. According to Tables 2 and Fig. 2, the mean error of the formulation and data obtained by Nejad and Hadi (2016) is 1.3%. In the formulation was observed that have appropriately evaluated the linear deflection ratios-length of CC isotropic nanobeam.

5.2 Parametric results

This section presents some numerical examples to investigate the impacts of the nonlocal factor, material distribution, boundary conditions on the non-linear static deflection of BFG nanobeam with a thickness (h) equal to 100 nm and width (b) equals to 10 nm. The BFG nanobeam material is similar to those applied in Reference (Nejad and Hadi 2016). To capture the different parameters' effects on the non-linear deflection in a better way, the results are presented in terms of the dimensionless deflection W/L and the dimensionless load parameter defined as following:

$$P = \frac{qL^4}{E_c h^4} \tag{35}$$

Fig. 3 demonstrates the linear and non-linear dimensionless deflection W/L versus distance along the length of a SS nanobeam for different gradient parameters N_z and N_x, when $\mu = 4$ nm^2, $L/h = 5$, and $P = 0.0006$.

It worths noting that the behavior of FG beams with CC boundary conditions was the same as the behavior of the beam with the SS boundary conditions. Hence, to avoid repetition, the figure for CC boundary conditions is not depicted here. The increase of gradient index N_z or decrease of gradient index N_x, reduces the W/L; the reason is that increasing N_z as well as decreasing N_x, increases the effective stiffness of BFG beams. Moreover, impacts of N_z and N_x are more obvious for $2 < N_z < 3$ and $0 < N_x < 2$.

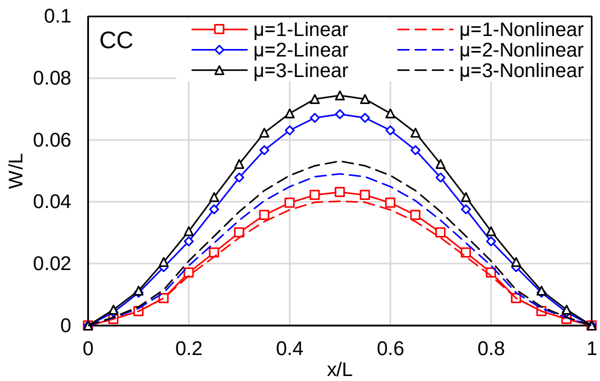
In Fig. 4 the dimensionless linear and nonlinear deflection along the non-dimensional length of nanobeam are plotted for CC and SS boundary conditions for various values of nonlocal factor of μ, when $N_z = 2$, $N_x = 1$, $L/h = 5$ and $P = 0.0006$. According to the curve, there is a significant impact on the dimensionless deflection concerning the change of μ, such as the dimensionless deflection increases as the nonlocal factor increased. But the impact of the nonlocal factor is clearer when increased from 2 to 3 than that of 1 to 2. Also, Fig. 4 shows that for all values of μ, the non-linear deflections are less than their linear counterparts. This indicates that considering the geometric nonlinearity causes to increase in the stiffness of the nanobeam. Furthermore, it is clear from Fig. 4 that the

Table 2 Comparison of linear deflection ratios W_r of clamped-clamped BFG nanobeam

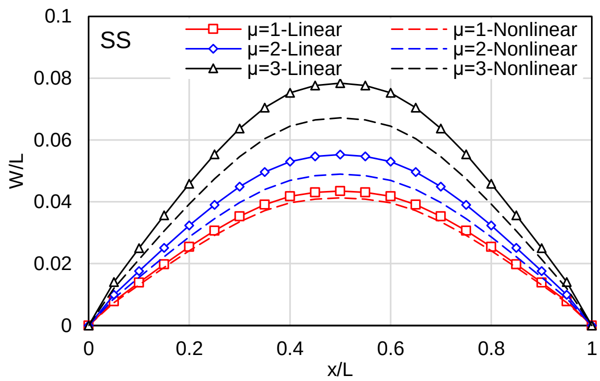
L (nm)	Linear deflection ratios W_r							
	μ = 4 (nm²)		μ = 8 (nm²)		μ = 12 (nm²)		μ = 16 (nm²)	
	In this study	Ref (Nejad and Hadi 2016)	In this study	Ref (Nejad and Hadi 2016)	In this study	Ref (Nejad and Hadi 2016)	In this study	Ref (Nejad and Hadi 2016)
3	0.971	0.985	0.736	0.728	0.610	0.622	0.4659	0.475
6	0.979	0.993	0.939	0.929	0.911	0.929	0.883	0.900
9	0.981	0.995	0.973	0.963	0.961	0.980	0.951	0.970
12	0.989	1.003	0.985	0.974	0.976	0.995	0.966	0.986
15	0.991	1.005	0.991	0.980	0.982	1.001	0.985	1.004
18	0.099	0.100	0.994	0.983	0.994	1.010	0.987	1.007
21	0.099	0.101	0.999	0.988	0.995	1.015	0.990	1.010

Table 3 Non-dimensional central displacement of a nanobeam obtained by Eltaher *et al*. (2013), Nazmul and Devnath (2020) and present study

N_z	μ (nm²)	Eltaher *et al.* (2013)	Nazmul and Devnath (2020)	Present study	N_z	μ (nm²)	Eltaher *et al.* (2013)	Nazmul and Devnath (2020)	Present study
0.1	0	1.376	1.432	1.418	1.0	0	1.703	1.860	1.799
	1	1.442	1.500	1.486		1	1.784	1.949	1.886
	2	1.508	1.569	1.554		2	1.866	2.039	1.972
	3	1.574	1.638	1.622		3	1.947	2.128	2.058
0.2	0	1.439	1.530	1.500	2.0	0	1.814	1.972	1.912
	1	1.508	1.604	1.572		1	1.900	2.066	2.003
	2	1.577	1.677	1.643		2	1.987	2.161	2.095
	3	1.646	1.751	1.715		3	2.073	2.256	2.186
0.5	0	1.577	1.717	1.663	5.0	0	1.951	2.099	2.046
	1	1.652	1.799	1.743		1	2.044	2.200	2.143
	2	1.728	1.881	1.822		2	2.138	2.301	2.241
	3	1.803	1.964	1.902		3	2.231	2.402	2.339



(a)



(b)

Fig. 4 Variation of the linear and non-linear dimensionless deflection W/L along the length of nanobeam is plotted for different values of nonlocal parameter μ for: (a) CC nanobeam, and (b) SS nanobeam

difference between linear and non-linear deflection in CC nanobeam is negligible compared to SS nanobeam.

5.3 Comparison of the present study with the studies of Eltaher et al. (2013) and Nazmul and Devnath (2020)

The objective of this section is to present displacements of bi-directional functionally graded nanobeams by varying material parameters along with both directions. First, thepresent study obtained the central displacement of a functionally graded nanobeam considered by Eltaher *et al*. (2013) and Nazmul and Devnath (2020). The considered nanobeam had a width of 1000 nm and a depth of 100 nm with a length of 10,000 nm. The material at the bottom was steel having Young's modulus 210 GPa, and at the top, it was Al_2O_3 with Young's modulus 390 GPa. The material proper- ties varied along the thickness direction only with various values of the material parameter N_z with $N_x = 0$. Table 3 compares the present study's central displacement results with those of Eltaher *et al*. (2013) and Nazmul and Devnath (2020). The current analytical results are slightly higher than those of Eltaher *et al*. (2013), primarily because the displacement field had been approximated by the differential quadrature method (DQM) and direct iterative method by Eltaher *et al*. (2013) while applying the Galerkin method to the variational statement. As can be noted, the obtained results are in good agreement with those of Nazmul and Devnath (2020) and even more conservative than those presented by Eltaher *et al*. (2013). The nonlocal parameter increases as the non-dimensional central displacement of the nanobeam increases. This emphasizes the significance of the nonlocal effect on the central displacement response of nanobeams.

6. Conclusions

The present study proposed a Timoshenko beam model for non-linear static analysis of the BFG nanobeam within the framework of the nonlocal theory. The being non-linear due to the stretching impact of the FG beam's mid-plane is the main origin of nonlinearity of assumed free vibration issues. The corresponding boundary conditions and the governing equations of motion are captured using the principle of Hamilton; the governing equations are solved by the method of differential quadrature and direct iterative method. The developed method includes material property variations by an exponential law along the axial direction and a power law along the thickness direction. Analytical expressions are derived as benchmarks for application in the design of nanobeams and are valuable sources for validating other approximate methods. Accuracy verification of the proposed formulation is done by comparing the captured non-linear static deflections with the literature. The effect of nonlocal parameters, material property gradient parameters, and boundary conditions on the non-linear static deflection of the BFG nanobeams is explained in detail. The results show that the increase of gradient index N_z and the decrease of gradient index N_x reduces the dimensionless deflection. On the other hand, there is a significant effect on the dimensionless deflection concerning the change of μ, such as the dimensionless deflection with increasing non-local parameters. Also, for all μ values, the nonlinear deviations are less than their linear counterparts.

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