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Nonlinear and nonclassical vibration analysis of double walled piezoelectric cylindrical nanoshell

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Abstract. In current paper, nonlocal (NLT), nonlocal strain gradient (NSGT) and Gurtin-Murdoch surface/interface (GMSIT) theories with classical theory (CT) are utilized to investigate vibration and stability analysis of Double Walled Piezoelectric Nanosensor (DWPENS) based on cylindrical nanoshell. DWPENS simultaneously subjected to direct electrostatic voltage DC and harmonic excitations, structural damping, two piezoelectric layers and also nonlinear van der Waals force. For this purpose, Hamilton's principle, Galerkin technique, complex averaging and with arc-length continuation methods are used to analyze nonlinear behavior of DWPENS. For this work, three nonclassical theories compared with classical theory CT to investigate Dimensionless Natural Frequency (DNF), pull-in voltage, nonlinear frequency response and stability analysis of the DWPENS considering the nonlocal, material length scale, surface/interface (S/I) effects, electrostatic and harmonic excitation.

Keywords: double walled piezoelectric nanosensor; Gurtin-Murdoch surface/interface; nonlocal strain gradient theory; nonlinear frequency response; complex averaging method; arc-length continuation

1. Introduction

Nanotechnology is a multidisciplinary branch of science which encompasses numerous diverse fields of science and technology, pharmaceutical, agricultural, environmental, advanced materials, chemical science, physics, electronics, information technology and specially biomedical fields such as imaging agents, drug delivery vehicle, diagnostic tools, etc., to save human life along with other areas by application of engineering skills in surgical diagnosis, monitoring, treatment and therapy etc. (Chatterjee *et al.* 2014, Duan *et al.* 2010, Kosaka *et al.* 2014, Melancon *et al.* 2011, Mousavi *et al.* 2018). Nano structures, especially piezoelectric nano sensors/resonators, due to small size and mass, are widely used in modern technology and have received considerable attention from researchers around the world, due to their unique features and widespread applications (Tzou 2019, Rupitsch 2019, Manbachi and Cobbold 2011). Nowadays for the dynamics analysis and the mathematical modeling of these nano-structures, the size-dependent parameters should be contained in theoretical models. For this purpose, nonclassical theories such as nonlocal (Eringen 1972, 1983, 2002), strain gradient (Lim *et al.* 2015) and Gurtin-Murdoch surface/interface (Gurtin and Murdoch 1975, 1978) theories are presented to investigate nonlinear vibration and dynamic analysis of nanostructures. Corresponding to nonlocal theory, Ebrahimi *et al.* (2019a, b, c) used nonlocal theory to investigate wave propagation analysis of nanotubes and heterogeneous nanobeams and also static stability of nanoplates. For solving electromagnetic waveguide problem, Rabczuk *et al.* (2019) introduced a nonlocal operator method for partial differential equations. Also, Samaniego *et al.* (2020) explored Deep Neural Networks (DNNs) as an option for approximation to solve partial differential equations in computational mechanics. Sensitivity analysis has been studied by Hamdia *et al.* (2018) to identify the key input parameters influencing the Energy Conversion Factor (ECF) of flexoelectric materials by using a NURBS-based IGA formulation exploiting their higher order continuity and hence avoiding a complex mixed formulation and the Latin Hypercube Sampling (LHS) method in the probability space. Also, Sensitivity Analysis (SA) approach is used by Vu-Bac *et al.* (2016) to quantify the effects of correlated input parameters on model outputs. Alizada and Sofiyev (2011) and Alizada *et al.* (2012) investigated the modified Young's moduli of nano-materials taking into account the scale effects and vacancies and the stress analysis of the substrate coated by nanomaterials with vacancies subjected to the uniform extension load. Also, Sofiyev *et al.* (2020a, b) studied the buckling behavior and vibration analysis of functionally graded carbon nanotube reinforced composite conical shells. Zhang *et al.* (2018) employed lattice, finite difference and Eringen's nonlocal models for Modelling vibrating nano-strings. Chen *et al.* (2017) used nonlocal effect to analyze propagation of time-harmonic waves in the three-dimensional, transversely isotropic, magneto-electroelastic and multilayered plates. Arefi (2018) has been shown that increasing of nonlocal parameter leads to increasing of the rotations, in-plane displacements and

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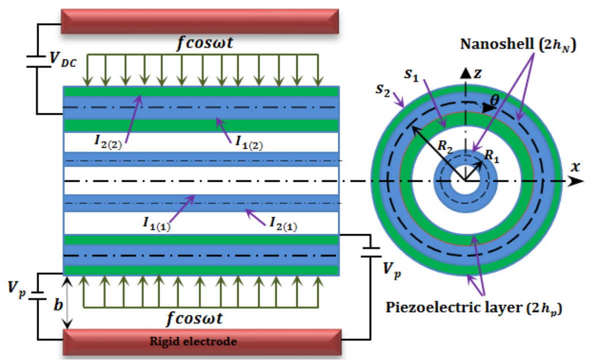


Fig. 1 A DW piezoelectric nanosensor subjected to electrostatic and harmonic excitation

transverse deflection of a piezoelectric nano shell with double curves. Because in the strain gradient theory, stress of small-scale materials appears in both nonlocal stress field (Eringen 1972, 1983, 2002) and pure strain gradient stress field (Aifantis 1992), as a result, this theory has a better prediction to nonlocal one. Buckling and postbuckling of nanobeams were studied by Li and Hu (2015) using the nonlocal stain gradient theory. In another work, Li *et al.* (2016) studied a vibration analysis of rods considering the nonlocal strain gradient theory. Ebrahimi *et al.* (2019d, e) used nonlocal strain gradient theory to investigate vibration analysis of porous rotary nanobeams and dispersion of waves in porous nanoscale plates in thermal environment. Energy harvesting from a Timoshenko beam is studied by Managheb *et al.* (2018) considering the flexoelectric and strain gradient effects. According to Gurtin-Murdoch approach of surface/interface elasticity theory, recently, Hashemi Kachapi *et al.* (2019a, b, c, d, 2020a, b) and Hashemi Kachapi (2020) presented surface/interface effects to investigate linear and nonlinear vibration analysis of multi-walled piezoelectric nanostructures. The active vibration control of a viscoelastic orthotropic piezoelectric doubly-curved nanoshell is studied by Zhu *et al.* (2020) within the framework of surface piezoelectricity theory and Kelvin-Voigt viscoelastic model. In another work by Zhu *et al.* (2017), nonlinear free vibration behavior of orthotropic piezoelectric cylindrical nano-shells is investigated using surface energy effect. In the work done by Sun *et al.* (2018), analytical buckling solutions of piezoelectric cylindrical nanoshells under the combined compressive loads and external voltages are investigated with the coupled nonlocal and surface effects. In the work done by Ghorbani *et al.* (2019), material length scale parameter and nonlocal parameter increases and decreases the natural frequency respectively. Also, Ghorbanpour Arani *et al.* (2014) investigated nonlinear vibration of single and double-walled boron nitride nano sheet with small scale and surface effects using nonlocal and surface piezoelasticity theories. In analysis of nonlinear dynamics of a double-layer piezoelectric and Kelvin-Voigt viscoelastic nanobeams, Rahmanian and Hosseini-Hashemi (2019) concluded that with surface energy effect, the softening behavior of the lower nanobeam is slightly weakened, whereas the softening-type behavior of the upper nanobeam becomes more intense in the presence of the surface effect. In literature there are a few works on vibration and stability analysis of DW piezoelectric nanosensor considering simultaneously nonlocal, nonlocal strain gradient and Gurtin-Murdoch surface/interface effects. In this work, mentioned effects are presented to investigate nonlinear vibration and stability analysis of DW piezoelectric nanosensor compared to classical theory. For this analysis, Hamilton's principle, Galerkin technique, Complex averaging method combined with arc-length continuation is used to compare three nonclassical theories of NLT, NSGT and GMSIT with classical theory to achieve the influences of the nonlocal, material length scale, surface/interface effects, electrostatic and harmonic excitation on dimensionless natural frequency DNF, pull-in voltage, nonlinear frequency response and stability analysis of the DWPENS.

2. Mathematical formulation

A double walled cylindrical piezoelectric nanosensor subjected to the combined electrostatic force with direct electric voltage (V_{DC}) and harmonic excitation with amplitude f and angular frequency ω is shown in Fig. 1.

All of the physical and geometrical properties of the mentioned nanostructures can be seen in reference Hashemi Kachapi *et al.* (2019c, d).

2.1 Nonlocal strain gradient and surface/interface theories

Based on the Nonlocal Strain Gradient Theory (NSGT), the constitutive equation can be written as (Lim *et al.* 2015)

$$(1-\mu\nabla^2)\sigma_{ij}^{NSG} = (1-\eta\nabla^2)\left(C_{ijkl}\varepsilon_{kl} - e_{ijk}\bar{E}_k\right) \quad (1)$$

$$(1-\mu\nabla^2)D_k^{NSG} = (1-\eta\nabla^2)\left(e_{kli}\varepsilon_{kl} - \eta_{ij}\bar{E}_j\right) \quad (2)$$

where σ_{ij}^{SGT} and D_k^{SGT} respectively are the nonlocal strain gradient stress tensor and components of the nonlocal strain gradient electric displacement. The $\mu=(e_0a)^2$ and $\eta=l^2$ respectively denote the influence of nonlocal scale parameter and material length scale parameter that e_0 denotes a constant appropriate to each material, and a is an internal characteristic length of the material. $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}$ is the Laplacian operator. In mentioned equations and in Nonlocal Theory (NLT), η is zero, i.e., $\eta=0$.

The modified nonlocal strain gradient stress and electric displacement considering of the surface/interface effects for the piezoelectric materials may be represented as (Lim *et al.* 2015)

$$(1-\mu\nabla^2)\sigma_{ij}^{NSG(s/I)} = (1-\eta\nabla^2)\left(C_{ijkl}\varepsilon_{kl} - e_{ijk}\bar{E}_k\right) \quad (3)$$

$$(1-\mu\nabla^2)D_k^{NSG(s/I)} = (1-\eta\nabla^2)\left(e_{kli}\varepsilon_{kl} - \eta_{ij}\bar{E}_j\right) \quad (4)$$

The terms $\sigma_{ij}^{NSG(s/I)}$ and $D_k^{NSG(s/I)}$ are the nonlocal surface/interface stress tensor and electric displacement, respectively.

2.2 Nonlocal strain gradient surface/interface and small-scale stress- strain relationships

Based on the Gurtin-Murdoch surface/interface elasticity theory, the normal and shear stresses, $\sigma_{xx(N,p)n}$, $\sigma_{\theta\theta(N,p)n}$ and $\sigma_{x\theta(N,p)n}$ can be written as (Gurtin and Murdoch 1975, 1978, Hashemi Kachapi 2020a, Hashemi Kachapi *et al*. 2019d)

$$\sigma_{xx(N,p)n} = C_{11(N,p)n}\varepsilon_{xxn} + C_{12(N,p)n}\varepsilon_{\theta\theta n} - e_{31p2}\bar{E}_{xp2} + \frac{\upsilon_{(N,p)n}\sigma_{zzn}}{1-\upsilon_{(N,p)n}} \quad (5)$$

$$\sigma_{\theta\theta(N,p)n} = C_{21(N,p)n}\varepsilon_{xxn} + C_{22(N,p)n}\varepsilon_{\theta\theta n} - e_{32p2}\bar{E}_{\theta p2} + \frac{\upsilon_{(N,p)n}\sigma_{zzn}}{1-\upsilon_{(N,p)n}} \quad (6)$$

$$\sigma_{x\theta(N,p)n} = C_{66(N,p)n}\gamma_{x\theta n} \quad (7)$$

where based on nonclassical continuum model, σ_{zzn} is expressed as following

$$\sigma_{zzn} = \frac{z}{h_{Nn}+h_{p2}}\left((\tau_{0n}^{ps}+\tau_{0n}^{NI})(\frac{\partial^2 w_n}{\partial x^2}+\frac{1}{R_n^2}\frac{\partial^2 w_n}{\partial\theta^2}) - (\rho_n^{ps}+\rho_n^{NI})\frac{\partial^2 w_n}{\partial t^2}\right) \quad (8)$$

Also, all coefficients and phrases of Eqs. (5)-(8) such as displacement fields, nonlinear deflection and curvatures, relations of Gurtin-Murdoch surface/interface elasticity theory and etc., can be seen in full detail in references Amabili (2008), Donnell (1976), Hashemi Kachapi (2020a) and Hashemi Kachapi *et al*. (2019d).

The resulting in-plane loads for piezoelectric nanostructures lead to surface/interface stresses and electric displacement which can be defined using surface/interface effects as follows

$$(1-\mu\nabla^2)\sigma_{xx(N,p)n}^{NSG\left(\frac{s}{I}\right)} = (1-\eta\nabla^2)\times\left(C_{11(N,p)n}\varepsilon_{xxn} + C_{12(N,p)n}\varepsilon_{\theta\theta n} - e_{31p2}\bar{E}_{xp2} + \frac{\upsilon_{(N,p)n}\sigma_{zzn}}{1-\upsilon_{(N,p)n}}\right) \quad (9)$$

$$(1-\mu\nabla^2)\sigma_{\theta\theta(N,p)n}^{NSG\left(\frac{s}{I}\right)} = (1-\eta\nabla^2)\left(C_{21(N,p)n}\varepsilon_{xxn} + C_{22(N,p)n}\varepsilon_{\theta\theta n} - e_{32p2}\bar{E}_{\theta p2} + \frac{\upsilon_{(N,p)n}\sigma_{zzn}}{1-\upsilon_{(N,p)n}}\right) \quad (10)$$

$$(1-\mu^2\nabla^2)\sigma_{x\theta(N,p)n}^{NSG(s/I)} = (1-l^2\nabla^2)\left(C_{66(N,p)n}\gamma_{x\theta n}\right) \quad (11)$$

2.3 Governing equations

In this section, the governing equations of motion and corresponding boundary conditions of the piezoelectric shell are obtained by applying the following Hamilton principle

$$\int_0^t \left(\delta T_n - \delta\pi_n + \delta w_{vdw} + \delta w_{vf} + \delta w_e + \delta W_f\right)dt = 0 \quad (12)$$

where δT, $\delta\pi$, δw_{vdw}, δw_e and δw_f respectively are the first variation of strain energy, kinetic energy, nonlinear van der Waals force, viscoelastic foundation, nonlinear electrostatic force and harmonic excitation.

The first variation of strain energy can be obtained as

$$\begin{aligned}\pi_n &= \int_0^L\int_0^{2\pi}\Big\{\int_{-h_{Nn}}^{h_{Nn}}\left(\sigma_{ijNn}\delta\varepsilon_{ijn}\right)dz + \int_{-h_{N2}-h_{p2}}^{-h_{N2}}\left(\sigma_{ijp2}\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{zp2}\right) \\ &+ \int_{h_{N2}}^{h_{N2}+h_{p2}}\left(\sigma_{ijp2}\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{zp2}\right)dz + \left(\sigma_{ij2}^S\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{i2}^S\right)\left(R_2+h_{N2}+h_{p2}\right) + \left(\sigma_{ij2}^S\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{i2}^S\right) \\ &\left(\sigma_{ij2}^I\delta\varepsilon_{ij2}\right)\left(R_2-h_{N2}\right) + \left(\sigma_{ijn}^I\delta\varepsilon_{ijn}\right)\left(R_n+h_{Nn}\right)\Big\}R_n d\theta dx \\ &= \int_0^L\int_0^{2\pi}\Big\{N_{xxn}\left(\frac{\partial\delta u_n}{\partial x}+\frac{\partial w_n}{\partial x}\frac{\partial\delta w_n}{\partial x}\right) + N_{\theta\theta n}\left(\frac{1}{R_n}\left(\frac{\partial\delta v_n}{\partial\theta}+\delta w_n\right)+\frac{1}{R_n^2}\left(\frac{\partial w_n}{\partial\theta}\frac{\partial\delta w_n}{\partial\theta}\right)\right) + N_{x\theta n}\left(\frac{1}{R_n}\frac{\partial\delta u_n}{\partial\theta}+\frac{\partial\delta v_n}{\partial x}+\right. \\ &\left.\frac{1}{R_n}\frac{\partial\delta w_n}{\partial x}\frac{\partial w_n}{\partial\theta}+\frac{1}{R_n}\frac{\partial w_n}{\partial x}\frac{\partial\delta w_n}{\partial\theta}\right) - M_{xxn}\left(\frac{\partial^2\delta w_n}{\partial x^2}\right) \\ &-M_{\theta\theta n}\left(\frac{1}{R_n^2}\frac{\partial^2\delta w_n}{\partial\theta^2}\right) - M_{x\theta n}\left(\frac{2}{R_n}\frac{\partial^2\delta w_n}{\partial x\partial\theta}\right)\Big\}R_n d\theta dx\end{aligned} \quad (13)$$

In Eq. (13), the forces (N) and moment (M) resultants are defined in reference Hashemi Kachapi *et al*. (2019c, d). Also, the first variation of kinetic energy can be written as

$$\delta T_n = -\iint I_n\left(\left(\frac{\partial^2 u_n}{\partial t^2}\right)\delta u_n + \left(\frac{\partial^2 v_n}{\partial t^2}\right)\delta v_n + \left(\frac{\partial^2 w_n}{\partial t^2}\right)\delta w_n\right)R_n d\theta dx \quad (14)$$

where

$$\begin{aligned}I_n &= \int_{-h_{Nn}}^{h_{Nn}}\rho_{Nn}\,dz + \int_{-h_{Nn}-h_{p2}}^{-h_{Nn}}\rho_{p2}\,dz + \int_{h_{Nn}}^{h_{Nn}+h_{p2}}\rho_{p2}\,dz + \rho_n^{S,I} \\ &= 2\rho_{Nn}h_{Nn} + 2\rho_{p2}h_{p2} + 2\rho_n^{ps} + 2\rho_n^{NI}\end{aligned} \quad (15)$$

and first variation of the work done by the nonlinear van der Waals force, viscoelastic foundation and nonlinear electrostatic force and the external harmonic excitation, respectively, can be written as (Hashemi Kachapi *et al*. 2019c, d, Farokhi *et al*. 2016)

$$\begin{aligned}\delta W_{vdw} &= \int_0^L\int_0^{2\pi}\int_0^{w_1}\left(C_{vdw}^L(w_2-w_1)\right. \\ &\left.C_{vdw}^{NL}(w_2-w_1)^3\right)R_1 d\theta dx \\ &-\int_0^L\int_0^{2\pi}\int_0^{w_2}\left(C_{vdwn}^L(w_2-w_1)\right. \\ &\left.+C_{vdwn}^{NL}(w_2-w_1)^3\right)\delta w_2\,R_2 d\theta dx\end{aligned} \quad (16)$$

$$\delta W_{vf} = -\int_0^L \int_0^{2\pi} \int_0^{w_2} C_w \frac{\partial w_2}{\partial t} \delta w_2 R_2 d\theta dx \quad (17)$$

$$\delta W_e = \int_0^L \int_0^{2\pi} \int_0^{w_2} \frac{\pi \Upsilon V_{DC}^2}{\sqrt{(b_2 - w_2)(2R_2 + b_2 - w_2)}\left[\cosh^{-1}\left(1 + \frac{b_2 - w_2}{R_2}\right)\right]^2} \delta w_2 R_2 d\theta dx \quad (18)$$

$$\delta W_f = \int_0^L \int_0^{2\pi} \int_0^{w_2} (f cos\omega t)\, \delta w_2 R_2 d\theta dx \quad (19)$$

where all coefficients and phrases of Eqs. (16)-(19) can be seen in references Hashemi Kachapi *et al*. (2019c, d).

By substituting Eqs. (13)-(19) into Eq. (12), the governing equations of motion and boundary conditions for PENS respectively are obtained as follows

Equations of motion

$$\delta u_n: \quad \frac{\partial N_{xn}}{\partial x} + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial \theta} = I_n \frac{\partial^2 u_n}{\partial t^2} \quad (20)$$

$$\delta v_n: \quad \frac{\partial N_{x\theta n}}{\partial x} + \frac{1}{R_n}\frac{\partial N_{\theta n}}{\partial \theta} = I_n \frac{\partial^2 v_n}{\partial t^2} \quad (21)$$

$$\begin{aligned} w_n: \quad & \frac{\partial^2 M_{xn}}{\partial x^2} + \frac{2}{R_n}\frac{\partial^2 M_{x\theta n}}{\partial x \partial \theta} + \frac{1}{R_n^2}\frac{\partial^2 M_{\theta n}}{\partial \theta^2} - \frac{N_{\theta n}}{R_n} \\ & + N_{xn}\frac{\partial^2 w_n}{\partial x^2} + \frac{\partial N_{xn}}{\partial x}\frac{\partial w_n}{\partial x} + \frac{N_{\theta n}}{R_n^2}\frac{\partial^2 w_n}{\partial \theta^2} + \frac{1}{R_n^2}\frac{\partial N_{\theta n}}{\partial \theta}\frac{\partial w_n}{\partial \theta} \\ & + \frac{2}{R_n} N_{x\theta n}\frac{\partial^2 w_n}{\partial x \partial \theta} + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial x}\frac{\partial w_n}{\partial \theta} + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial \theta}\frac{\partial w_n}{\partial x} \\ & = I_n \frac{\partial^2 w_n}{\partial t^2} + S_n \\ & - \frac{\pi \Upsilon V_{DC}^2}{\sqrt{(b_2 - w_2)(2R_2 + b_2 - w_2)}\left[\cosh^{-1}\left(1 + \frac{b_2 - w_2}{R_2}\right)\right]^2} \\ & - f cos\omega t \end{aligned} \quad (22)$$

where S_n for the inner and outer layer, respectively, are

$$S_1 = -(C_{vdw}^L (w_2 - w_1) + C_{vdw}^{NL}(w_2 - w_1)^3 \quad (23)$$

$$S_2 = \left(\frac{R_1}{R_2}\right)\left(C_{vdw}^L (w_2 - w_1) + C_{vdw}^{NL}(w_2 - w_1)^3 + C_w \frac{\partial w_2}{\partial t}\right) \quad (24)$$

and boundary conditions

$$\delta u_n = 0 \quad \text{or} \quad N_{xxn} n_x + \frac{1}{R_n} N_{x\theta n} n_\theta = 0 \quad (25)$$

$$\delta v_n = 0 \quad \text{or} \quad N_{x\theta n} n_x + \frac{1}{R_n} N_{\theta n} n_\theta = 0 \quad (26)$$

$$\begin{aligned} & \delta w_n = 0 \quad \text{or} \\ & \left(\frac{\partial M_{xxn}}{\partial x} + \frac{1}{R_n}\frac{\partial M_{x\theta n}}{\partial \theta} + N_{xxn}\frac{\partial w_n}{\partial x} + \frac{N_{x\theta}}{R_n}\frac{\partial w_n}{\partial \theta}\right) n_x \\ & + \left(\frac{1}{R_n}\frac{\partial M_{x\theta n}}{\partial x} + \frac{1}{R_n^2}\frac{\partial M_{\theta n}}{\partial \theta} + \frac{N_{x\theta n}}{R_n}\frac{\partial w_n}{\partial x} + \frac{N_{\theta n}}{R_n^2}\frac{\partial w_n}{\partial \theta}\right) n_\theta = 0 \end{aligned} \quad (27)$$

$$\frac{\partial w_n}{\partial x} = 0 \quad \text{or} \quad M_{xxn} n_x + \frac{1}{R_n} M_{x\theta n} n_\theta = 0 \quad (28)$$

$$\frac{\partial w_n}{\partial \theta} = 0 \quad \text{or} \quad \frac{1}{R_n} M_{x\theta n} n_x + \frac{1}{R_n^2} M_{\theta n} n_\theta = 0 \quad (29)$$

Corresponding to reference Hashemi Kachapi *et al*. (2019d) and considering of nonlocal and material length scale effects and so nonlocal strain gradient surface/ interface effects, we have

$$(1 - \mu\nabla^2) N_{xxn} = (1 - \eta\nabla^2) \begin{pmatrix} A_{11n}\varepsilon_{xxn}^0 + A_{12n}\varepsilon_{\theta\theta}^0 \kappa_{\theta\theta n} - \left(\tau_{0n}^{ps} + \tau_{0n}^{NI}\right)\left(\frac{\partial w_n}{\partial x}\right)^2 \\ +2\left(\tau_{0n}^{ps} + \tau_{0n}^{NI}\right) - N_{xpn} \end{pmatrix} \quad (30)$$

$$(1 - \mu\nabla^2) N_{\theta\theta n} = (1 - \eta\nabla^2) \begin{pmatrix} A_{21n}\varepsilon_{xxn}^0 + A_{22n}\varepsilon_{\theta\theta n}^0 + 2\left(\tau_{0n}^{ps} + \tau_{0n}^{NI}\right) - N_{\theta pn} \\ -\left(\tau_{0n}^{ps} + \tau_{0n}^{NI}\right)\left(\frac{2w_n}{R_n} + \frac{1}{R_n^2}\left(\frac{\partial w_n}{\partial \theta}\right)^2\right) \end{pmatrix} \quad (31)$$

$$(1 - \mu\nabla^2) N_{x\theta n} = (1 - \eta\nabla^2)(A_{66n}\gamma_{x\theta n}^0) \quad (32)$$

$$(1 - \mu\nabla^2) M_{xxn} = (1 - \eta\nabla^2) \begin{pmatrix} D_{11n}\kappa_{xxn} + D_{12n}\kappa_{\theta\theta n} - M_{xpn} \\ +E_{11n}^*\left(\frac{\partial^2 w_n}{\partial x^2} + \frac{1}{R_n^2}\frac{\partial^2 w_n}{\partial \theta^2}\right) - G_{11n}^*\frac{\partial^2 w_n}{\partial t^2} \end{pmatrix} \quad (33)$$

$$(1 - \mu\nabla^2) M_{\theta\theta n} = (1 - \eta\nabla^2) \begin{pmatrix} D_{21n}\kappa_{xxn} + D_{22n}\kappa_{\theta\theta n} - M_{\theta pn} \\ +E_{11n}^*\left(\frac{\partial^2 w_n}{\partial x^2} + \frac{1}{R_n^2}\frac{\partial^2 w_n}{\partial \theta^2}\right) - G_{11n}^*\frac{\partial^2 w_n}{\partial t^2} \end{pmatrix}, \quad (34)$$

$$(1 - \mu\nabla^2) M_{x\theta n} = (1 - \eta\nabla^2)(D_{66n}\kappa_{x\theta n}) \quad (35)$$

By substituting Eqs. (30)-(35) into the Eq. (20)-(22) and (25)-(29) and using dimensionless parameters Appendix 1, the dimensionless governing equations can be expressed as

$$\begin{aligned} & (1 - \bar{\eta}\,\bar{\nabla}^2) \begin{pmatrix} \alpha_{1un}\frac{\partial^2 \bar{u}_n}{\partial \xi^2} + \alpha_{2un}\frac{\partial^2 \bar{u}_n}{\partial \theta^2} + \alpha_{3un}\frac{\partial^2 \bar{v}_n}{\partial \xi \partial \theta} \\ +\alpha_{4un}\frac{\partial \bar{w}_n}{\partial \xi} + \alpha_{5un}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \xi^2} \\ +\alpha_{6un}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \theta^2} + \alpha_{7un}\frac{\partial^2 \bar{w}_n}{\partial \xi \partial \theta}\frac{\partial \bar{w}_n}{\partial \theta} \end{pmatrix} \\ & = (1 - \bar{\mu}\,\bar{\nabla}^2)\frac{\partial^2 \bar{u}_n}{\partial \tau^2} \end{aligned} \quad (36)$$

$$\begin{aligned} & (1 - \bar{\eta}\,\bar{\nabla}^2) \times \\ & \begin{pmatrix} \alpha_{1vn}\frac{\partial^2 \bar{u}_n}{\partial \xi \partial \theta} + \alpha_{2vn}\frac{\partial^2 \bar{v}_n}{\partial \xi^2} + \alpha_{3vn}\frac{\partial^2 \bar{v}_n}{\partial \theta^2} + \alpha_{4vn}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \xi \partial \theta} \\ +\alpha_{5vn}\frac{\partial^2 \bar{w}_n}{\partial \xi^2}\frac{\partial \bar{w}_n}{\partial \theta} + \alpha_{6vn}\frac{\partial \bar{w}_n}{\partial \theta} + \alpha_{7vn}\frac{\partial \bar{w}_n}{\partial \theta}\frac{\partial^2 \bar{w}_n}{\partial \theta^2} \end{pmatrix} \\ & = (1 - \bar{\mu}\,\bar{\nabla}^2)\frac{\partial^2 \bar{v}_n}{\partial \tau^2} \end{aligned} \quad (37)$$

$$(1-\eta\,\bar{\nabla}^2)\left(\begin{array}{c}\alpha_{1wn}\frac{\partial\bar{u}_n}{\partial\xi}+\alpha_{2wn}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{3wn}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{4wn}\frac{\partial^2\bar{u}_n}{\partial\xi^2}\frac{\partial\bar{w}_n}{\partial\xi}\\+\alpha_{5wn}\frac{\partial^2\bar{u}_n}{\partial\xi\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{6wn}\frac{\partial^2\bar{u}_n}{\partial\xi\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{7wn}\frac{\partial\bar{u}_n}{\partial\theta}\frac{\partial^2\bar{w}}{\partial\xi\partial\theta}\\+\alpha_{8wn}\frac{\partial^2\bar{u}_n}{\partial\theta^2}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{9wn}\frac{\partial\bar{v}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}+\alpha_{10wn}\frac{\partial^2\bar{v}_n}{\partial\xi^2}\frac{\partial\bar{w}_n}{\partial\theta}\\+\alpha_{11wn}\frac{\partial^2\bar{v}_n}{\partial\xi\partial\theta}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{12wn}\frac{\partial\bar{v}_n}{\partial\theta}+\alpha_{13wn}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial^2\bar{w}_n}{\partial\xi^2}\\+\alpha_{14wn}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{15wn}\frac{\partial^2\bar{v}_n}{\partial\theta^2}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{16wn}\bar{w}_n+\alpha_{17wn}\bar{w}_n\frac{\partial^2\bar{w}_n}{\partial\xi^2}\\+\alpha_{18wn}\bar{w}_n\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{19wn}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{20wn}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}\\+\alpha_{21wn}\frac{\partial^2\bar{w}_n}{\partial\xi^2}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2+\alpha_{22wn}\frac{\partial^2\bar{w}_n}{\partial\xi^2}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{23wn}\frac{\partial^4\bar{w}_n}{\partial\xi^4}\\+\alpha_{24wn}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2+\alpha_{26wn}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{27wn}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{28wn}\frac{\partial^4\bar{w}_n}{\partial\theta^4}\\+\alpha_{29wn}\frac{\partial^2\bar{w}_n}{\partial\theta^2}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{30wn}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{31wn}\frac{\partial^4\bar{w}_n}{\partial\xi^2\partial\tau^2}\\+\alpha_{32wn}\frac{\partial^4\bar{w}_n}{\partial\theta^2\partial\tau^2}+\alpha_{33wn}\end{array}\right)$$
$$=(1-\bar{\mu}\,\bar{\nabla}^2)\times\left(\begin{array}{c}\frac{\partial^2\bar{w}_n}{\partial\tau^2}+\bar{S}_n-\bar{F}cos\bar{\Omega}\tau\\-\frac{\bar{F}_{e2}\bar{V}_{DC2}^2}{\sqrt{(m_{12}\bar{b}_2-\bar{w}_2)(2m_{42}\bar{R}_2+m_{12}\bar{b}_2-\bar{w}_2)}\left[cosh^{-1}\left(1+\frac{m_{12}\bar{b}_2-\bar{w}_2}{m_{42}\bar{R}_2}\right)\right]^2}\end{array}\right) \tag{38}$$

where $\bar{\nabla}^2=\frac{\partial^2}{\partial\xi^2}+m_0^2\frac{\partial^2}{\partial\theta^2}$ and $\bar{S}_n$ for the inner and outer layer, respectively, are

$$\bar{S}_1=-(\bar{C}_{vdw}^{L}(\bar{w}_2-\bar{w}_1)+\bar{C}_{vdw}^{NL}(\bar{w}_2-\bar{w}_1)^3) \tag{39}$$

$$\bar{S}_2=\left(\frac{\bar{R}_1}{\bar{R}_2}\right)\left(\bar{C}_{vdw}^{L}(\bar{w}_2-\bar{w}_1)+\bar{C}_{vdw}^{NL}(\bar{w}_2-\bar{w}_1)^3+\bar{C}_w\frac{\partial\bar{w}_2}{\partial t}\right) \tag{40}$$

Also, all coefficients of $\alpha_{iu}(i=1..7),\alpha_{jv}(j=1..7)$ and $\alpha_{kw}(k=1,\ldots,33)$ are introduced in Appendix 2. Also V_0 is a reference voltage, set to 1 volt throughout this study (Farokhi *et al*. 2016). The associated boundary conditions can be obtained in the dimensionless form as

$$\delta\bar{u}_n=0:\ \left(\begin{array}{c}\alpha_{1un}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}+\alpha_{2un}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}+\alpha_{3un}^{bc}\bar{w}_n+\alpha_{4un}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2\\+\alpha_{5un}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{6un}^{bc}\end{array}\right)\delta\bar{u}_{\xi n}|+\left(\alpha_{7un}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}+\alpha_{8un}^{bc}\frac{\partial\bar{v}_n}{\partial\xi}+\alpha_{9un}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}\right)\delta\bar{u}_{\theta n}|\ =0 \tag{41}$$

$$\delta\bar{v}_n=0:\ \left(\alpha_{1vn}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}+\alpha_{2vn}^{bc}\frac{\partial\bar{v}_n}{\partial\xi}+\alpha_{3vn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}\right)\delta\bar{v}_{\xi n}|+\left(\begin{array}{c}\alpha_{4vn}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}+\alpha_{5vn}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}+\alpha_{6vn}^{bc}\bar{w}_n+\alpha_{7vn}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2\\+\alpha_{8vn}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{9vn}^{bc}\end{array}\right)\delta\bar{v}_{\theta n}|=0 \tag{42}$$

$$\begin{aligned}\delta\bar{w}_n=0:\ &\Big(+\alpha_{1wn}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{2wn}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}+\\&+\alpha_{3wn}^{bc}\frac{\partial\bar{v}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{4wn}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{5wn}^{bc}\bar{w}_n\frac{\partial\bar{w}_n}{\partial\xi}\\&+\alpha_{6wn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{7wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi^3}+\alpha_{8wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi\partial\theta^2}+\alpha_{9wn}^{bc}\\&\frac{\partial\bar{w}_n}{\partial\xi}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2+\alpha_{10wn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{11wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi\partial\tau^2}\Big)\\&\delta\bar{w}_{\xi n}|\ \Big(+\alpha_{12wn}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{13wn}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{14wn}^{bc}\\&\frac{\partial\bar{v}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{15wn}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{16wn}^{bc}\bar{w}_n\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{17wn}^{bc}\\&\frac{\partial^3\bar{w}_n}{\partial\xi^2\partial\theta}+\alpha_{18wn}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{19wn}^{bc}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{20wn}^{bc}\\&\frac{\partial^3\bar{w}_n}{\partial\theta^3}+\alpha_{21wn}^{bc}\frac{\partial\bar{w}_n}{\partial\theta}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{22wn}^{bc}\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial\bar{w}_n}{\partial\xi}=0:\ &\Big(\alpha_{23wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{24wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{25wn}^{bc}\\&+\alpha_{26wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{27wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{28wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\tau^2}\Big)\delta\left(\frac{\partial\bar{w}_{\xi n}}{\partial\xi}\right)|\\&+\left(\alpha_{29wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\right)\delta(\frac{\partial\bar{w}_{\theta n}}{\partial\xi})|\ =0\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial\bar{w}_n}{\partial\theta}=0:\ &\left(\alpha_{30wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\right)\delta\left(\frac{\partial\bar{w}_{\xi n}}{\partial\theta}\right)|\\&+\Big(\alpha_{31wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{32wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{33wn}^{bc}+\alpha_{34wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}\end{aligned} \tag{45}$$

$$+\alpha_{35wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{36wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\tau^2}\Bigg)\,\delta\left(\frac{\partial\bar{w}_{\theta n}}{\partial\theta}\right)|\ =0$$

Coefficients of $\alpha_{lu}^{bc}(l=1,\ \dots\ ,9)$, $\alpha_{mv}^{bc}(m=1,\ \dots\ ,9)$ and $\alpha_{nw}^{bc}(n=1,\ \dots,36)$ are introduced in Appendix 3. In current study with expressing the electrostatic force Eq. (18) as a polynomial form that as nonlinear curve-fitting problem is solved by lsqcurvefit function in Matlab toolbox using least-squares, the dimensionless work done by electrostatic force can be express in following equation

$$\delta W_e=\int_0^L\int_0^{2\pi}\int_0^{\bar{w}_2}\bar{F}_{e2}(\bar{V}_{DC2}+\bar{V}_{AC2}\cos(\varpi\tau))^2\begin{pmatrix}\bar{C}_1+\bar{C}_2\bar{w}_2+\bar{C}_3\bar{w}_2^2\\+\cdots+\bar{C}_n\bar{w}_2^{n-1}\end{pmatrix}\delta\bar{w}_2\bar{R}_2\delta\theta\delta\xi \tag{46}$$

which $\bar{C}_1-\bar{C}_\mathrm{n}$ are constant.

2.4 Solution procedure

In order to solve the equations of motions, the Galerkin method is employed to convert the governing equations to ordinary differential equations, so displacements are written in terms of generalized coordinate and mode function as follows (Amabili 2008)

$$\begin{bmatrix}u_n(x,\theta,t)\\v_n(x,\theta,t)\\w_n(x,\theta,t)\end{bmatrix}=\sum_{m=1}^{M_1}\sum_{j=1}^{N}\begin{bmatrix}[u_{m,j,c}(\tau)\cos(j\theta)+u_{m,j,s}(\tau)\sin(j\theta)]\chi_{mj}(\xi)\\ [v_{m,j,c}(\tau)\sin(j\theta)+v_{m,j,s}(\tau)\cos(j\theta)]\phi_{mj}(\xi)\\ [w_{m,j,c}(\tau)\cos(j\theta)+w_{m,j,s}(\tau)\sin(j\theta)]\beta_{mj}(\xi)\end{bmatrix}+\sum_{m=1}^{M_2}\begin{bmatrix}u_{m,0}(\tau)\chi_{m0}(\xi)\\v_{m,0}(\tau)\phi_{m0}(\xi)\\w_{m,0}(\tau)\beta_{m0}(\xi)\end{bmatrix}=\sum_{(i,r,s)=1}^{M_2+M_1\times N}\begin{bmatrix}u_{ni}(\tau)\chi_{ni}(\xi)\vartheta_{ni}(\theta)\\v_{nr}(\tau)\phi_{nr}(\xi)\alpha_{nr}(\theta)\\w_{ns}(\tau)\beta_{ns}(\xi)\psi_{ns}(\theta)\end{bmatrix} \tag{47}$$

In the Galerkin method, the functions $\chi_i(\xi)$, $\phi_r(\xi)$ and $\beta_s(\xi)$ must satisfy all the geometric and natural boundary conditions. By substituting Eq. (47) into Eqs. (36)-(38) and (41)-(45) and applying the Galerkin technique, the reduced-order equation of motion is written to the following form

$$[(K)_u^u+(K_{bc})_u^u]_n\{\bar{u}_n\}+[(K)_u^v+(K_{bc})_u^v]_n\{\bar{v}_n\}+[(K)_u^w+(K_{bc})_u^w]_n\{\bar{w}_n\}+[(NL)_u^w+(NL_{bc})_u^w]_n\{\bar{w}_n^2\}=[(M)_u^u]_n\{\ddot{\bar{u}}_n\}+\bar{F}_{upn}^{bc} \tag{48}$$

$$[(K)_v^u+(K_{bc})_v^u]_n\{\bar{u}_n\}+[(K)_v^v+(K_{bc})_v^v]_n\{\bar{v}_n\}+[(K)_v^w+(K_{bc})_v^w]_n\{\bar{w}_n\}+[(NL)_v^w+(NL_{bc})_v^w]_n\{\bar{w}_n^2\}=[(M)_v^v]_n\{\ddot{\bar{v}}_n\}+\bar{F}_{vpn}^{bc} \tag{49}$$

$$\begin{aligned}&[(K)_w^u]_n\{\bar{u}_n\}+[(K)_w^v]_n\{\bar{v}_n\}+[(K)_w^w+(K_{bc})_w^w-(K_{vp})_w^w-(K_{e2})_w^w]_n\{\bar{w}_n\}+[(NL)_w^u+(NL_{bc})_w^u]_n\{\bar{w}_n\bar{u}_n\}+[(NL)_w^v\\&+(NL_{bc})_w^v]_n\{\bar{w}_n\bar{v}_n\}+[(NL)_{w2}^w+(NL_{bc})_{w2}^w-(NL_{2e})_{w2}^w]_n\{\bar{w}_n^2\}+[(NL)_{w3}^w+(NL_{bc})_{w3}^w-(NL_{3e})_{w3}^w]_n\{\bar{w}_n^3\}\\&=([(M)_w^w+(M_{bc})_w^w]_n)\{\ddot{\bar{w}}_n\}+([(C)_w^w]+[(C_{bc})_w^w]_n)\{\dot{\bar{w}}_n\}\\&+(-1)^p\left(\frac{\bar{R}_1}{\bar{R}_2}\right)^q\bar{C}_{vdw}^L([(K)_{w2n}^{vdw}]\{\bar{w}_2\}-[(K)_{w1n}^{vdw}]\{\bar{w}_1\})\\&+\bar{F}_{wpn}+\bar{F}_{wpn}^{bc}-\bar{F}_{we2}-[\bar{F}cos\bar{\Omega}\tau]+(-1)^p\left(\frac{\bar{R}_1}{\bar{R}_2}\right)^q\bar{C}_{vdw}^{NL}\\&([(NL)_{w1}^{vdw}]\bar{w}_2^3-3[(NL)_{w2}^{vdw}]\bar{w}_2^2\bar{w}_1+3[(NL)_{w3}^{vdw}]\bar{w}_2\bar{w}_1^2-[(NL)_{w4}^{vdw}]\bar{w}_1^3)-\bar{F}_{e2}\{((\bar{V}_{AC2}cos\varpi\tau)^2+2\bar{V}_{AC2}\bar{V}_{DC2}cos\varpi\tau)\\&(\bar{C}_4(NL_e)_{w3}^w+\bar{C}_3(NL_e)_{w2}^w+\bar{C}_2(K_e)_w^w+\bar{C}_1\bar{F}_1)\}\end{aligned} \tag{50}$$

where all coefficients and phrases of Eqs. (48)-(50) are defined in references Hashemi Kachapi *et al*. (2019d) and Hashemi Kachapi (2020a) and are presented in Appendix 4 of current work. Also, $[(K)_w^{vdw}]_n$ is stiffness matrix for van der Walls effect, for $p=1{:}\,q=0$ and for $p=2{:}\,q=1$. In order to solve nonlinear equations of a nonlinear system Eqs. (48)-(50), the complex averaging method combined with the arc-length continuation method are used (Hashemi Kachapi *et al*. 2019c, d, Manevitch and Manevitch 2005).

3. Results and discussions

In this section, at first, the time response of DW piezoelectric nanosensor with nonlocal, nonlocal strain gradient and surface/interface effects at the steady state is investigated for verification and accuracy the complex-averaging and the arc-length continuation methods compared with numerical Runge-kutta approach. Then the effects of different material and geometrical parameters with and without nonlocal, nonlocal strain gradient and surface/interface effects on dimensionless natural frequency, frequency response and stability analysis using numerical method based on arc-length continuation are presented. For this purpose, different boundary condition such as clamped edge (CC), simply supported edge (SS), clamped-simply supported edge (CS) and, clamped-free edge (CF) are presented. The surface and bulk material properties of Aluminum (Al) nanoshell and PZT piezoelectric layer are shown in Tables 1 and 2, respectively (Hashemi Kachapi *et al*. 2019c, d).

The others geometrical parameters for bulk and surface of DWPENS in all following results are shown in Table 3

Table 1 Surface and bulk properties of Al

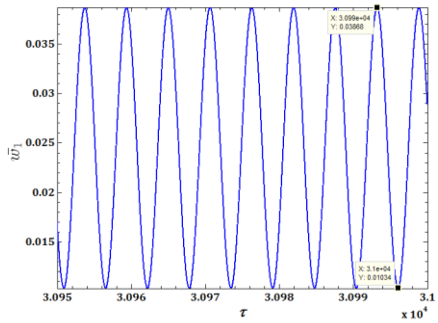
E_{Nn} (GPa)	υ_{Nn}	ρ_{Nn} (kg/m^3)	λ_n^I (N/m)	μ_n^I (N/m)	τ_{0n}^I (N/m)	ρ_n^I (kg/m^2)
70	0.33	2700	3.786	1.95	0.9108	5.46×10^{-8}

Table 2 Surface and bulk properties of PZT-4

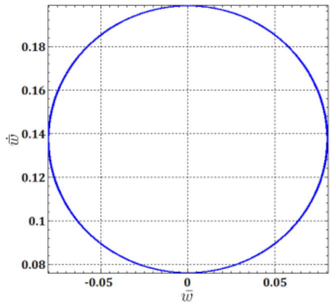
C_{11p2} (GPa)	C_{22p2} (GPa)	C_{12p2} (GPa)	C_{21p2} (GPa)	C_{66p2} (GPa)	E_{p2} (GPa)
139	139	77.8	77.8	30.5	95
υ_{p2}	ρ_{p2} (kgm^{-3})	η_{33p2} (10^{-8} F/m)	λ_n^S (N/m)	μ_n^S (N/m)	τ_{0n}^S (N/m)
0.3	7500	8.91	4.488	2.774	0.6048
e_{31p2} (C/m^2)	e_{32p2} (C/m^2)	e_{31p2}^S (C/m)	e_{32p2}^S (C/m)	ρ_n^S (kg/m^2)	
-5.2	-5.2	-3×10^{-8}	-3×10^{-8}	5.61×10^{-7}	

Table 3 The material and geometrical parameters

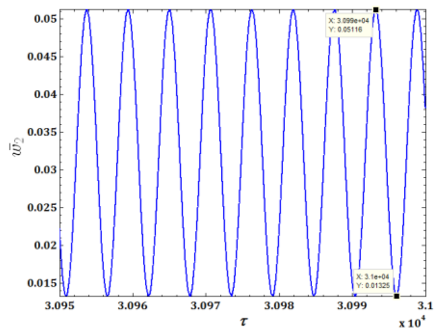
$R_1(m)$	$R_2(m)$	L/R_1	$h_{N_{1,2}}/R_1$	h_{p_2}/R_1
1×10^{-9}	1.5×10^{-9}	10	0.01	0.005
b_2/R_2	$V_{p_2}(V)$	V_{02}	$F(N)$	$V_{DC_2}(V)$
0.1	1×10^{-5}	1	50	1.5
ζ_w	$C^{L}_{vdw_2}$ (N/m^3)	$C^{NL}_{vdw_2}$ (N/m^3)	$\bar{\mu}$	$\eta(m^2)$
1×10^{-7}	9.91866693×10^{19}	2.201667×10^{31}	0.05	$(0.01\times10^{-9})^2$



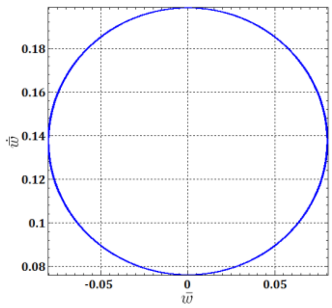
(a) Displacement in $\bar{w}_1$ direction



(b) Phase space in $\bar{w}_1$ direction

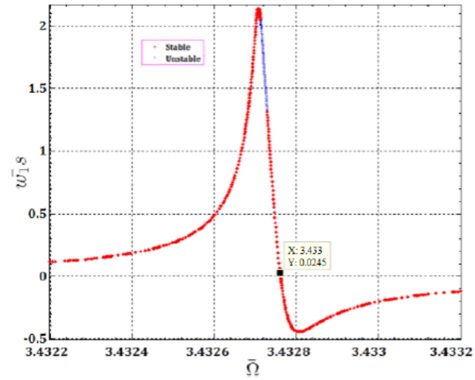


(c) Displacement in $\bar{w}_2$ direction

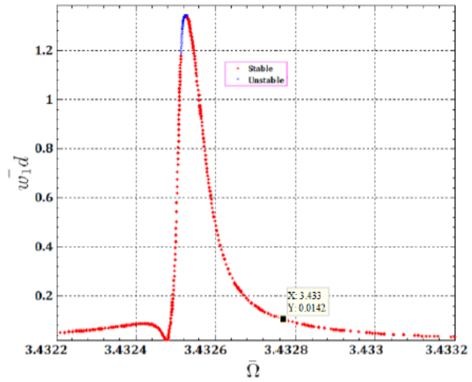


(d) Phase space in $\bar{w}_2$ direction

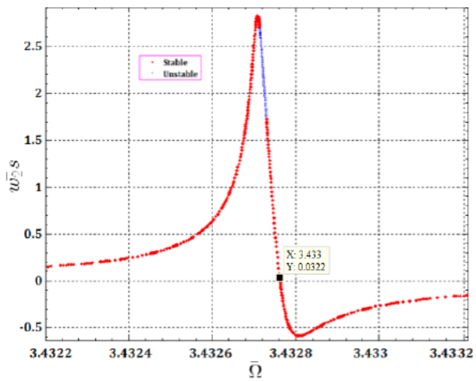
Fig. 2 The nonlinear dynamic response of the SS DWPENS



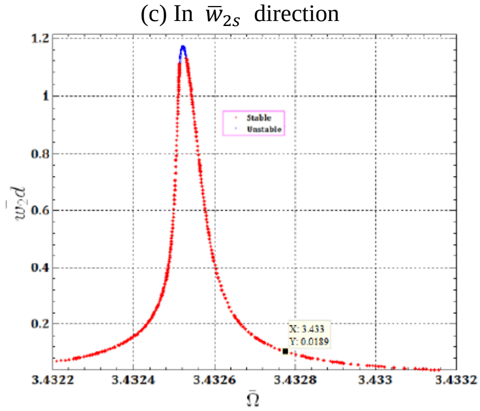
(a) In $\bar{w}_{1s}$ direction



(b) In $\bar{w}_{1d}$ direction



(c) In $\bar{w}_{2s}$ direction



(d) In $\bar{w}_{2d}$ direction

Fig. 3 The static (a, c) and dynamic (b, d) frequency responses of the SS DWPENS

(Hashemi Kachapi *et al*. 2019c, d, Ghorbanpour Arani *et al*. 2014).

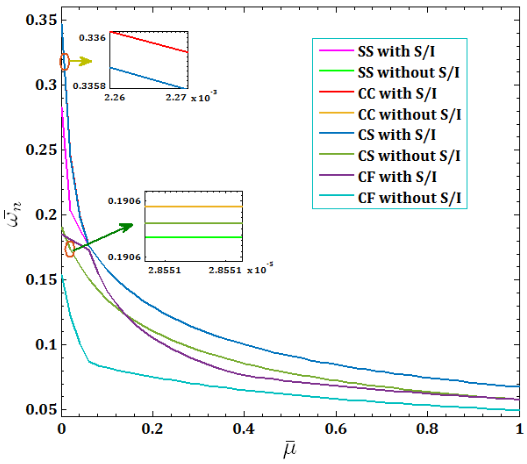


Fig. 4 The surface/ interface effects on natural frequency versus dimensionless nonlocal scale parameter $\bar{\mu}$ with $\bar{\eta} = 0.01$ for different boundary conditions

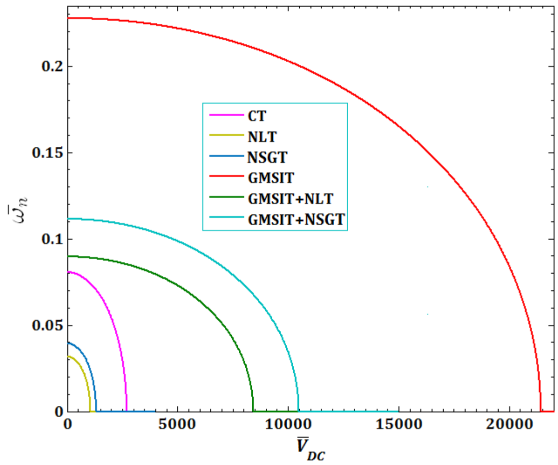


Fig. 6 Comparison of nonclassical theories with classical theory on dimensionless natural frequency versus different direct pull in voltage DC $(\bar{V}_{DC})$ of SS DWPENS

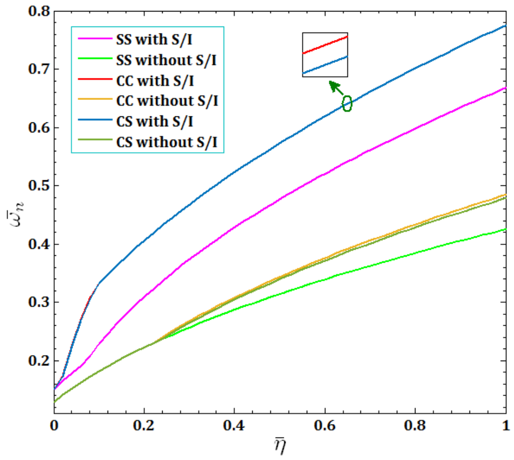


Fig. 5 The surface/ interface effects on natural frequency versus dimensionless material length scale parameter $\bar{\eta}$ with $\bar{\mu} = 0.1$ for different boundary conditions

3.1 Verification and comparison

In this subsection, considering all nonlocal and S/I effects of GMSIT + NSGT, the verification and accuracy of the solution by the complex-averaging and the arc-length continuation methods is compared with Runge-kutta method. Figs. 2(a)-(d) represent the nonlinear steady state response and phase space of the DWPENS for fixed values of the system parameters with $m = 3$ and $n = 1$ number of modes.

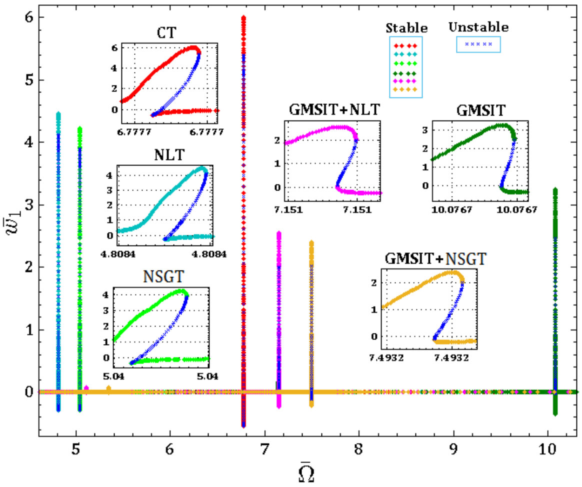
By comparing the static $(\bar{w}_{1s} = 0.0245,\ \bar{w}_{2s} = 0.0322)$ and dynamic $(\bar{w}_{1d} = 0.0142,\ \bar{w}_{2d} = 0.0189)$ displacements in the nonlinear dynamic response of Figs. 2(a) and (c) and the static and dynamic frequency response of Figs. 3 (a)-(d) at the resonance frequency $\bar{\Omega} = 3.433$, it can be concluded that the result of this simulation suggests a complete agreement between the numerical solutions and the averaging method.

3.2 Nonlinear frequency response and stability analysis

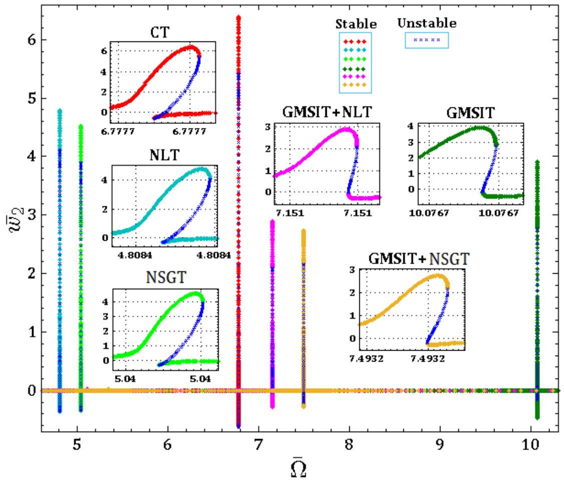
The main purpose of this section is to compare three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT. For this purpose, the effect of different material and geometrical parameters with and without nonlocal, nonlocal strain gradient and surface/interface effects will be discussed on Dimensionless Natural Frequency (DNF), nonlinear frequency response and stability analysis using numerical method based on arc-length continuation of the DW piezoelectric nanosensor with specifications mentioned to Tables 1-3. It is noted that in all following results, in NLT only consider $\bar{\mu}$ effect; in NSGT consider both $\bar{\mu}$ and $\bar{\eta}$ effects; in GMSIT only consider all S/I effects and CT not consider $\bar{\mu}$, $\bar{\eta}$ and all S/I effects.

In following, first dimensionless natural frequencies of DW piezoelectric nanosensor versus dimensionless of nonlocal scale parameter $\bar{\mu}$ and dimensionless material length scale parameter $\bar{\eta}$ with and without surface/ interface respectively are presented in Figs. 4 and 5. It is clear from the Fig. 4 that in all boundary conditions, considering surface/interface effects leads to increasing of DWPENS stiffness, as a result cause to increasing of the dimensionless natural frequency compared to case of without S/I effects. Also, in all cases, due to decreasing of DWPENS stiffness, the DNF decreases with increasing nonlocal scale parameter $\bar{\mu}$ with $\bar{\eta} = 0.01$.

Also, from Fig. 5, in addition to mentioned results for effect of surface/interface densities, it is clear that in all boundary conditions, due to increasing of DWPENS stiffness, the DNF increases with increasing dimensionless material length scale parameter $\bar{\eta}$ with $\bar{\mu} = 0.1$. Results of Figs. 4 and 5 indicate that nonlocal scale parameter $\bar{\mu}$ and material length scale parameter $\bar{\eta}$ respectively lead to increasing and decreasing of DWPENS stiffness and results of lead to increasing and decreasing the DNF of DWPENS. Furthermore, in both cases of nonlocal and material length



(a)



(b)

Fig. 7 Comparison of nonclassical theories with classical theory on the nonlinear vibration and stability analysis of SS DWPENS (a) $\bar{w}_1$ direction; (b) $\bar{w}_2$ direction

scale parameters, the dimensionless natural frequency related to CC boundary condition is higher than that related to other boundary conditions. This is due to the fact that the CC boundary condition is stiffer than other boundary conditions.

The compare three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT on the natural frequency versus direct pull in voltage DC of SS nanoshell are presented in Fig. 6. As it is known, considering the surface/interface effects causes rigidity of the system and leads to more DC voltage to reach the pull in voltages. The three theories of NLT, NSGT and CT, will soon reach the pull in voltage. Also, for zero natural frequency, SS DWPENS becomes unstable and this physically implies that first the DWPENS losses its stability due to the divergence via a pitchfork bifurcation.

The comparison of three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT on nonlinear frequency response of SS DWPENS with $\bar{V}_{DC} = 1.5$ and $\bar{F} = 5 \times 10^{-7}$ respectively are presented in Fig. 7. It can be seen from Figure that considering the results of the frequency analysis and the softness and rigidity of SS DWPENS with regard to the type of theory, the consideration of the S/I effects in the GMSIT + NSGT theory leads to higher hardening behavior of the DWPENS, and it reduces the resonance amplitude of DWPENS. And also, in CT lower hardening type, maximum oscillation amplitude and instability with saddle-node bifurcations and nonlinear hardening behavior occurs in the system to all theories.

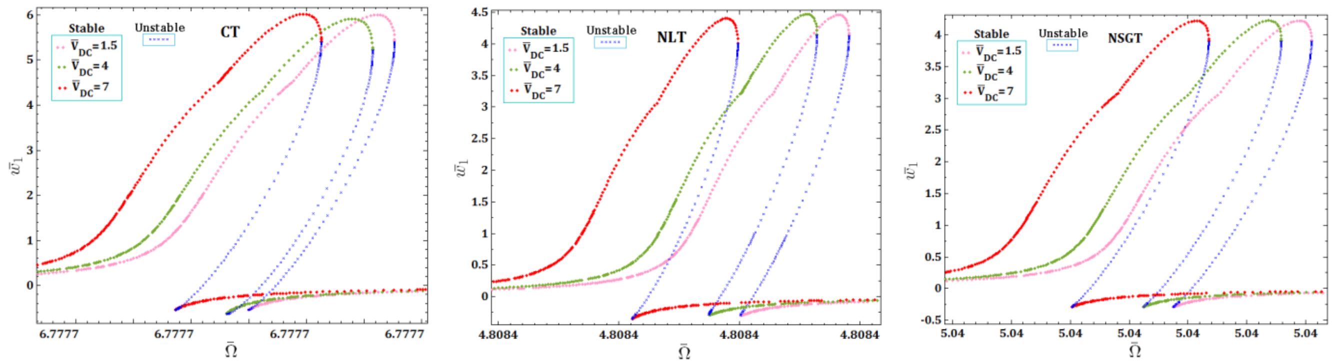
Figs. 8 and 9 present frequency response and stability analysis SS DWPENS for different values of direct electrostatic voltage ($\bar{V}_{DC}$) with harmonic excitation $\bar{F} = 5 \times 10^{-7}$ respectively in w_1 and w_2 directions based on different theories of classical theory CT, nonlocal theory NLT ($\bar{\mu} = 0.1$), nonlocal strain gradient theory NSGT ($\bar{\mu} = 0.1$, $\bar{\eta} = 0.01$), Gurtin-Murdoch surface/interface theory GMSIT, GMSIT combined with NLT and NSGT. As can be concluded from the Figures, the lowest resonance frequency for the unstable amplitude is related to the NLT theory and the most value is related to the GMSIT, which has the highest resonance frequency for the unstable amplitude.

According to Figs. 8 and 9, and since the nonlinear electrostatic force is non-harmonic and applied only statically, the resonance amplitude and instability does not have much effect and in all cases, with increasing DC voltage, the resonance amplitude is almost constant and the range of system instability with a slight slop decrease. When the S/I effects are not considered, i.e., for CT, NLT and NSGT, with increasing DC voltage and due to reduced system stiffness, the resonance frequency decreases and when S/I is considered, i.e., in the case of GMSIT and its combination with NLT and NSGT, the resonance amplitude is increased and the system instability range is greater than the first three cases (Figs. 8 and 9(a)-(c)), but there is no significant change in the resonant frequency.

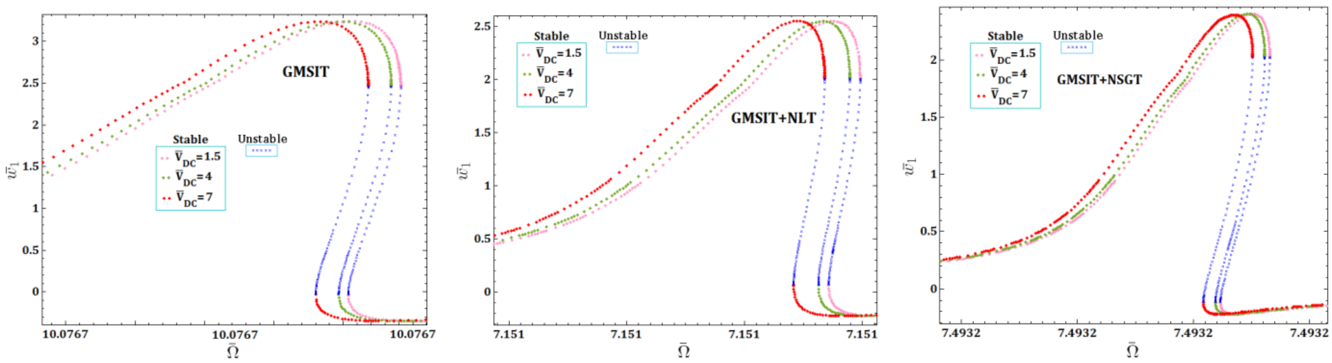
Furthermore, the results show that in all cases, the system displays hardening-type nonlinear behaviour with two saddle-node bifurcations. Also, in most frequencies, the amplitude of motion is almost constant, which indicates that the nanoshell has a static deformation, and in these regions dynamic behavior or amplitude of the dynamic response is not significant. This is due to the high rigidity of the shell and because of it has very small dimensions. Dynamic effects are only noticeable in the resonance region, where the amplitude of the system in these areas is significantly jumped, which is consistent with the main concept of the sensor.

Figs. 10 and 11 present frequency response and stability analysis SS DWPENS for different values of harmonic excitation ($\bar{F}$) with direct electrostatic voltage $\bar{V}_{DC} = 1.5$ respectively in w_1 and w_2 directions based on different theories of classical theory CT, nonlocal theory NLT ($\bar{\mu} = 0.1$), nonlocal strain gradient theory NSGT ($\bar{\mu} = 0.1, \bar{\eta} = 0.01$), Gurtin-Murdoch surface/interface theory GMSIT, GMSIT combined with NLT and NSGT.

The results of the resonant frequencies analysis are similar to those of the previous Figs. 8 and 9 as you have seen. In Figs. 10 and 11, as can be seen, the changes in the

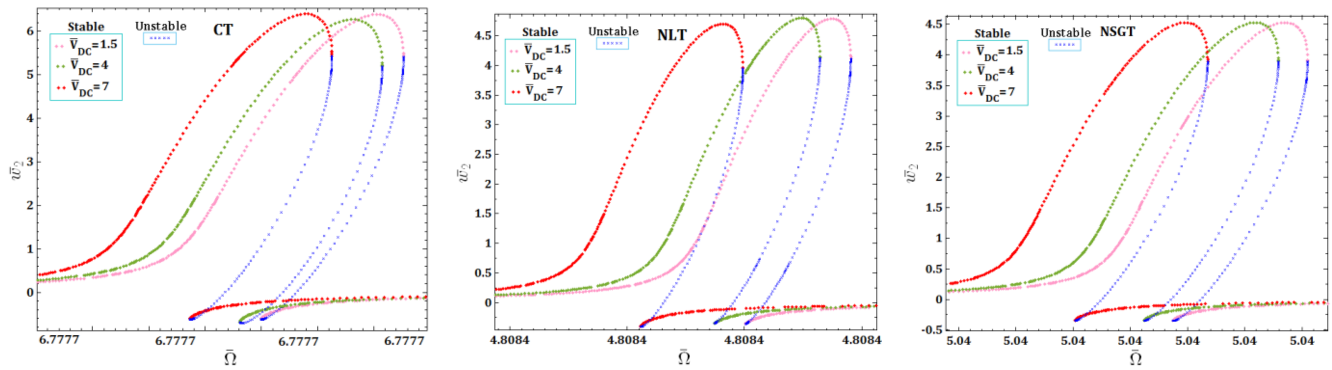


(a) CT (b) NLT (c) NSGT

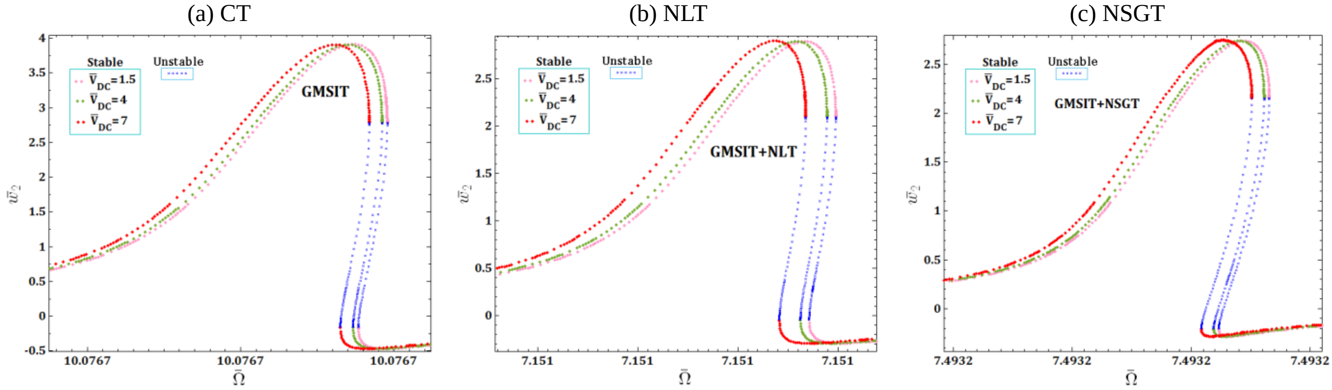


(d) GMSIT (e) GMSIT + NLT (f) GMSIT + NSGT

Fig. 8 The effect of direct electrostatic voltage ($\bar{V}_{DC}$) with harmonic excitation $\bar{F} = 5 \times 10^{-7}$ on frequency response and stability analysis of SS DWPENS in w_1 direction based on (a) CT; (b) NLT; (c) NSGT; (d) GMSIT; (e) GMSIT + NLT; (f) GMSIT+NSGT

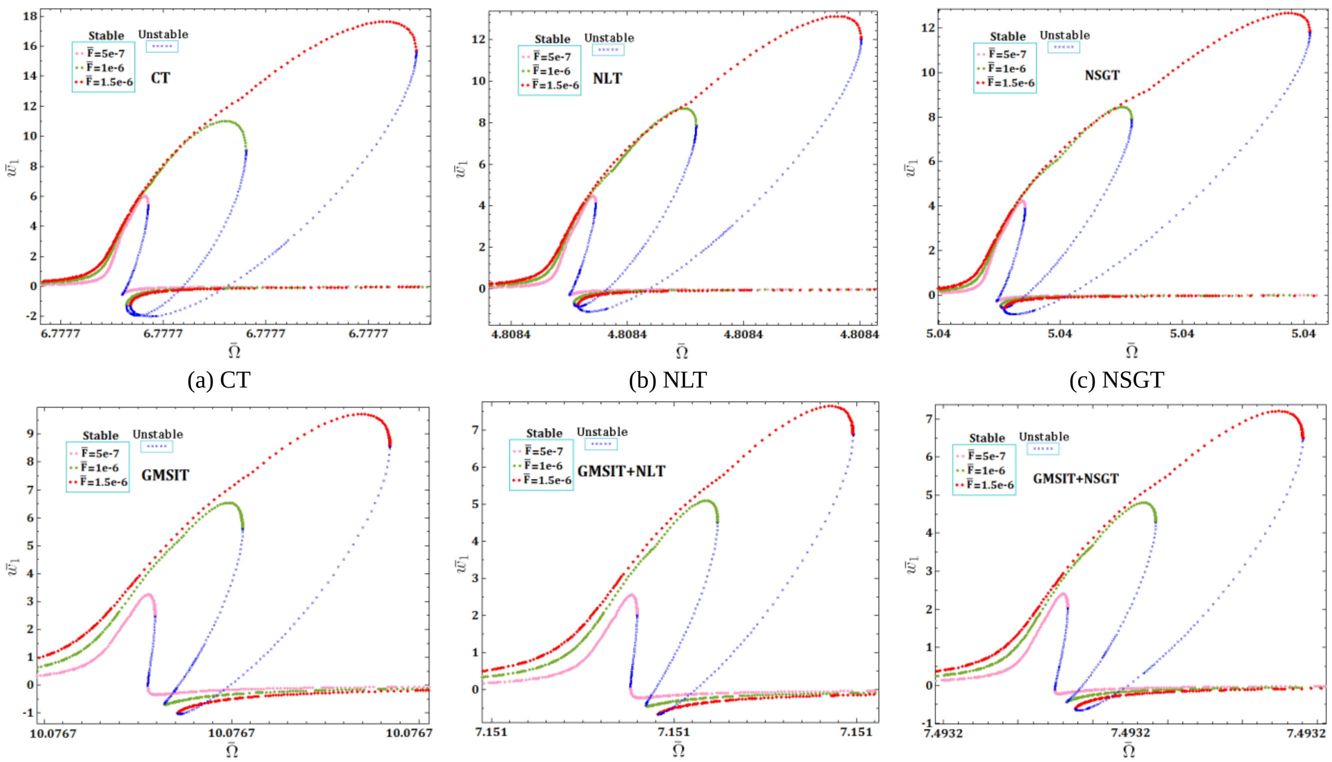


(a) CT (b) NLT (c) NSGT



(d) GMSIT (e) GMSIT + NLT (f) GMSIT + NSGT

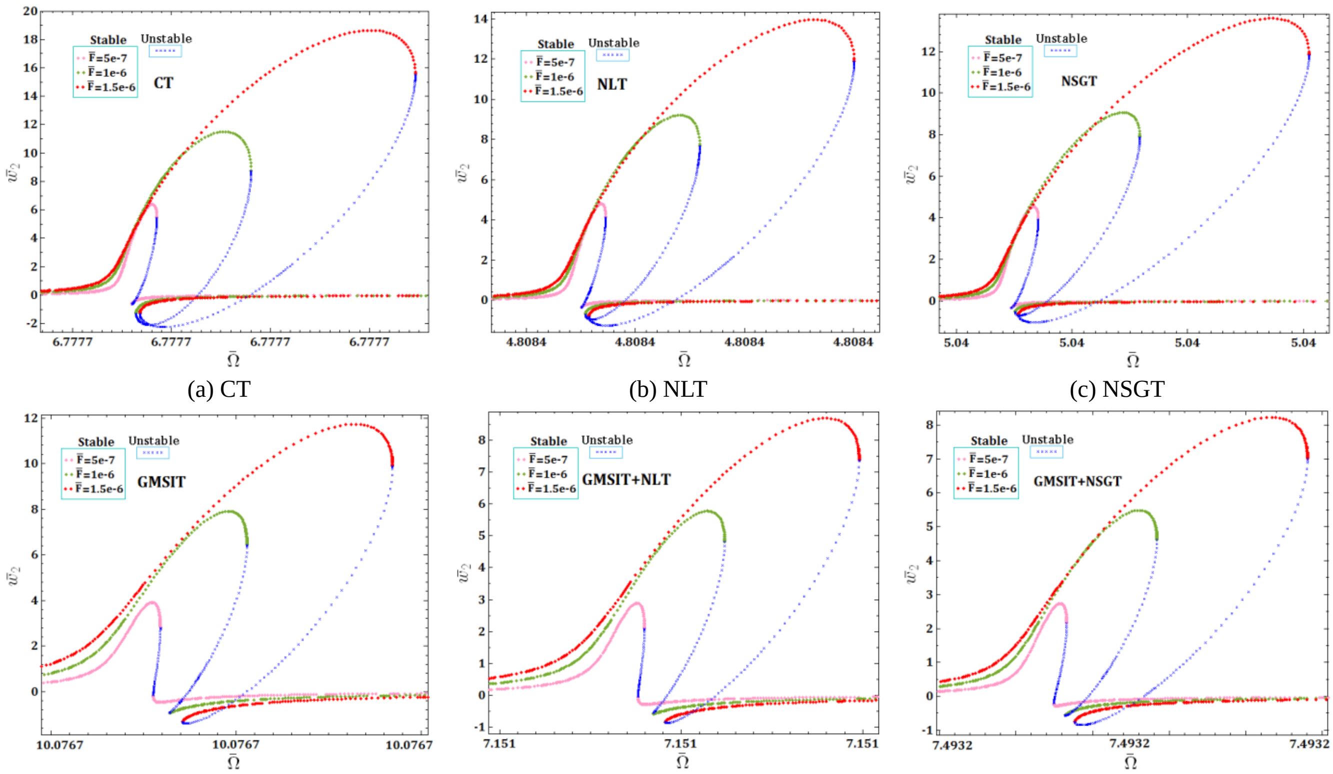
Fig. 9 The effect of direct electrostatic voltage ($\bar{V}_{DC}$) with harmonic excitation $\bar{F} = 5 \times 10^{-7}$ on frequency response and stability analysis of SS DWPENS in w_2 direction based on (a) CT; (b) NLT; (c) NSGT; (d) GMSIT; (e) GMSIT + NLT; (f) GMSIT + NSGT



(a) CT (b) NLT (c) NSGT

(d) GMSIT (e) GMSIT + NLT (f) GMSIT + NSGT

Fig. 10 The effect of different values of harmonic excitation ($\bar{F}$) with direct electrostatic voltage $\bar{V}_{DC} = 1.5$ on frequency response and stability analysis of SS DWPENS in w_1 direction based on (a) CT; (b) NLT; (c) NSGT; (d) GMSIT; (e) GMSIT+NLT; (f) GMSIT+NSGT



(a) CT (b) NLT (c) NSGT

(d) GMSIT (e) GMSIT + NLT (f) GMSIT + NSGT

Fig. 11 The effect of different values of harmonic excitation ($\bar{F}$) with direct electrostatic voltage $\bar{V}_{DC} = 1.5$ on frequency response and stability analysis of SS DWPENS in w_2 direction based on (a) CT; (b) NLT; (c) NSGT; (d) GMSIT; (e) GMSIT+NLT; (f) GMSIT+NSGT harmonic excitation amplitude, $\bar{F}$, will not affect the resonance frequency. As the $\bar{F}$ increases, in all the cases shown in Figs. 10 and 11(a)-(f), the resonance amplitude and system instability increase. Considering the S/I effects, the resonance amplitude and range of instability of the SS DWPENS are decreased compared to CT, NLT and NSGT.

4. Conclusions

In current study, surface/interface, nonlocal and nonlocal strain gradient effects are presented to investigate nonlinear vibration and stability analysis of DW piezoelectric nanosensor subjected to electrostatic and harmonic excitation. For this analysis, Hamilton's principle, Galerkin technique and also complex averaging method combined with arc-length continuation are used. These results shown that in all boundary conditions, considering surface/interface effects leads to increasing of DWPENS stiffness, as a result cause to increasing of the dimensionless natural frequency compared to case of without S/I effects and nonlocal scale parameter $\bar{\mu}$ and material length scale parameter $\bar{\eta}$ respectively lead to increasing and decreasing of DWPENS stiffness and results of lead to increasing and decreasing the DNF of DWPENS. Also considering the surface/interface effects leads to more DC voltage to reach the pull in voltages. The three theories of NLT, NSGT and CT, will soon reach the pull in voltage. In addition, consideration of the S/I effects in the GMSIT + NSGT theory, due to higher hardening behavior of the DWPENS, reduces the resonance amplitude of DWPENS and in CT, due to lower hardening behavior of the DWPENS, maximum oscillation amplitude and instability with saddle-node bifurcations and nonlinear hardening behavior occurs in the system to all theories. The lowest resonance frequency for the unstable amplitude is related to the NLT theory and the most value is related to the GMSIT. Also, since the nonlinear electrostatic force is non-harmonic and applied only statically, the resonance amplitude and instability does not have much effect. In all cases, with increasing DC voltage, the resonance amplitude is almost constant and the range of system instability with a slight slop decrease. When the S/I is considered, the resonance amplitude is increased and the system instability range is greater than the first three cases, but there is no significant change in the resonant frequency and as the $\bar{F}$ increases, in all the cases, the resonance amplitude and system instability increase.

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Appendix 1

Dimensionless parameters:

$$\bar{u}_n=\frac{u_n}{h_{Nn}},\quad \bar{v}_n=\frac{v_n}{h_{Nn}},\quad \bar{w}_n=\frac{w_n}{h_{Nn}},\quad \xi_n=\frac{x_n}{L},\quad \bar{b}_n=\frac{b_n}{L},\quad \bar{A}_{ijNn}=\frac{A_{ijNn}}{A_{11Nn}},\quad \bar{B}_{ijNn}=\frac{B_{ijNn}}{A_{11Nn}h_{Nn}}$$

$$\bar{D}_{ijNn}=\frac{D_{ijNn}}{A_{11Nn}h_{Nn}^2},\quad \bar{A}_{ijpn}=\frac{A_{ijpn}}{A_{11Nn}},\quad \bar{A}_{ijn}^*=\frac{A_{ijn}^*}{A_{11Nn}},\quad \bar{B}_{ijpn}=\frac{B_{ijpn}}{A_{11Nn}h_{Nn}},\quad \bar{B}_{ijn}^*=\frac{B_{ijn}^*}{A_{11Nn}h_{Nn}},\quad \bar{D}_{ijpn}=\frac{D_{ijpn}}{A_{11Nn}h_{Nn}^2}$$

$$\bar{D}_{ijn}^*=\frac{D_{ijn}^*}{A_{11Nn}h_{Nn}^2},\quad \bar{F}_{11Nn}^*=\frac{F_{11Nn}^*}{A_{11Nn}h_{Nn}},\quad \bar{F}_{11pn}^*=\frac{F_{11pn}^*}{A_{11Nn}h_{Nn}},\quad \bar{E}_{11Nn}^*=\frac{E_{11Nn}^*}{A_{11Nn}h_{Nn}^2},\quad \bar{E}_{11pn}^*=\frac{E_{11pn}^*}{A_{11Nn}h_{Nn}^2}$$

$$\bar{J}_{11Nn}^*=\frac{J_{11Nn}^*}{\rho_{Nn}h_{Nn}^2},\quad J_{11pn}^*=\frac{J_{11pn}^*}{\rho_{Nn}h_{Nn}^2},\quad \bar{G}_{11Nn}^*=\frac{G_{11Nn}^*}{\rho_{Nn}h_{Nn}^3},\quad G_{11pn}^*=\frac{G_{11pn}^*}{\rho_{Nn}h_{Nn}^3},\quad \bar{N}_{xpn}^*=\frac{N_{xpn}^*V_0}{A_{11Nn}},\quad \bar{N}_{\theta pn}^*=\frac{N_{\theta pn}^*V_0}{A_{11Nn}}$$

$$\bar{M}_{xpn}^*=\frac{M_{xpn}^*V_0}{A_{11Nn}h_{Nn}},\quad \bar{M}_{\theta pn}^*=\frac{M_{\theta pn}^*V_0}{A_{11Nn}h_{Nn}},\quad \bar{\tau}_0^{sn}=\frac{\tau_0^{sn}}{A_{11Nn}},\quad m_{0n}=\frac{L}{R_n}=\frac{1}{\bar{R}_n},\quad m_{1n}=\frac{L}{h_{Nn}}$$

$$m_{2n}=\frac{h_{Nn}}{R_n}=\bar{h}_{Nn},\quad \bar{h}_{pn}=\frac{h_{pn}}{R_n},\quad m_{3n}=\frac{I_n}{2\rho_{Nn}h_{Nn}},\quad m_{4n}=\frac{h_{pn}}{h_{Nn}},\quad \Omega=\sqrt{\frac{A_{11Nn}}{2\rho_{Nn}h_{Nn}L^2}},\quad \tau=\Omega t$$

$$\bar{\Omega}=\frac{\omega}{\Omega},\quad \bar{K}_w=\frac{K_wL^2}{m_3A_{11N}},\quad \bar{K}_p=\frac{K_p}{m_3A_{11N}},\quad \bar{C}_{w2}=\frac{C_w\Omega L^2}{m_{3n}A_{11Nn}},\quad \bar{C}_{vdwn}^{\,L}=\frac{C_{vdwn}^{\,L}L^2}{m_{3n}A_{11Nn}},\quad \bar{\mu}=\left(\frac{e_0a}{L}\right)^2$$

$$\bar{\eta}=\left(\frac{l}{L}\right)^2,\quad \bar{C}_{vdwn}^{\,NL}=\frac{C_{vdwn}^{\,NL}L^2h_{Nn}^2}{m_{3n}A_{11Nn}},\quad \bar{V}_{DC2}=\frac{V_{DC2}}{V_{02}},\quad \bar{V}_{p2}=\frac{V_{p2}}{V_{02}},\quad \bar{F}_{e2}=\frac{\pi m_{12}^2V_{02}^2\Upsilon}{m_{32}A_{11N2}},\quad \bar{F}=\frac{fL^2}{A_{11N}m_3{h_N}^2}$$

Appendix 2

$$\alpha_{1un}=-\frac{1}{m_{3n}}\bar{A}_{11n},\quad \alpha_{2un}=-\frac{m_{0n}^2}{m_{3n}}\bar{A}_{66n},\quad \alpha_{3un}=-\frac{m_{0n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n}),\quad \alpha_{4un}=-\frac{m_{0n}}{m_{3n}}\bar{A}_{12n}$$

$$\alpha_{5un}=-\frac{1}{m_{1n}m_{3n}}\left(\bar{A}_{11n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)\right),\quad \alpha_{6un}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{7un}=-\frac{m_{0n}m_{2n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n})$$

$$\alpha_{1vn}=-\frac{m_{0n}}{m_{3n}}(\bar{A}_{21n}+\bar{A}_{66n}),\quad \alpha_{2vn}=-\frac{1}{m_{3n}}\bar{A}_{66n},\quad \alpha_{3vn}=-\frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n},\quad \alpha_{4vn}=-\frac{m_{2n}}{m_{3n}}(\bar{A}_{21n}+\bar{A}_{66n})$$

$$\alpha_{5vn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{6vn}=-\frac{m_{0n}^2}{m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)\right),\quad \alpha_{7vn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)\right)$$

$$\alpha_{1wn}=\frac{m_{0n}}{m_{3n}}\bar{A}_{21n},\quad \alpha_{2wn}=-\frac{1}{m_{1n}m_{3n}}\bar{A}_{11n},\quad \alpha_{3wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},\quad \alpha_{4wn}=-\frac{1}{m_{1n}m_{3n}}\bar{A}_{11n}$$

$$\alpha_{5wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{6wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},\quad \alpha_{7wn}=-\frac{2m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{8wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n}$$

$$\alpha_{9wn}=-\frac{2m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{10wn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{11wn}=-\frac{m_{2n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n}),\quad \alpha_{12wn}=\frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n}$$

$$\alpha_{13wn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{12n},\quad \alpha_{14wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\bar{A}_{22n},\quad \alpha_{15wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\bar{A}_{22n}$$

$$\alpha_{16wn}=-\frac{m_{0n}^2}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)-\bar{A}_{22n}\right),\quad \alpha_{17wn}=-\frac{m_{2n}}{m_{n3}}\bar{A}_{12n},\quad \alpha_{18wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)\right)$$

$$\alpha_{19wn}=-\frac{1}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)-\bar{N}_{xp2}\right),\quad \alpha_{20wn}=-\frac{m_{2n}^2}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{21n}+4\bar{A}_{66n})$$

$$\alpha_{21wn}=-\frac{3}{2m_{1n}^2m_{3n}}\left(\bar{A}_{11n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I)\right),\quad \alpha_{22wn}=-\frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{12n}+2\bar{A}_{66n})$$

$$\alpha_{23wn}=-\frac{1}{m_{1n}^2m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{11n}),\quad \alpha_{24wn}=\frac{m_{2n}}{2m_{3n}}\bar{A}_{21n},\quad \alpha_{25wn}=-\frac{m_{2n}^2}{m_{3n}}(2\bar{E}_{11n}^*-\bar{D}_{12n}-4\bar{D}_{66n})$$

$$\alpha_{26wn}=-\frac{m_2^2n}{2m_{3n}}(\bar{A}_{21n}+2\bar{A}_{66n}),\quad \alpha_{27wn}=-\frac{m_{0n}^2}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)-\bar{N}_{\theta p2}\right),\quad \alpha_{28wn}=-\frac{m_{0n}^2m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{22n})$$

$$\alpha_{29wn}=-\frac{3m_{0n}^2m_{2n}^2}{2m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right),\quad \alpha_{30wn}=-\frac{m_{0n}^2m_{2n}}{2m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right),\quad \alpha_{31wn}=\frac{1}{2m_{1n}^2m_{3n}}\bar{G}_{11n}^*,$$

$$\alpha_{32wn}=\frac{m_{2n}^2}{2m_{3n}}\bar{G}_{11n}^*,\quad \alpha_{33wn}=-\frac{m_{0n}m_{1n}}{m_{3n}}\left(\bar{N}_{\theta p2}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right)$$

Appendix 3

$$\alpha_{1un}^{bc}=\frac{1}{m_{3n}}\bar{A}_{11n},\quad \alpha_{2un}^{bc}=\frac{m_{0n}}{m_{3n}}\bar{A}_{12n},\quad \alpha_{3un}^{bc}=\frac{m_{0n}}{m_{3n}}\bar{A}_{12n},\quad \alpha_{4un}^{bcn}=\frac{1}{2m_{1n}m_{3n}}\left(\bar{A}_{11n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right)$$

$$\alpha_{5un}^{bc}=\frac{m_{0n}m_{2n}}{2m_{3n}}\bar{A}_{12n},\quad \alpha_{6un}^{bc}=\frac{m_{1n}}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)-\bar{N}_{xp2}\right),\quad \alpha_{7un}^{bc}=\frac{m_0^2}{m_{3n}}\bar{A}_{66n},\quad \alpha_{8un}^{bc}=\frac{m_{0n}}{m_{3n}}\bar{A}_{66n}$$

$$\alpha_{9un}^{bc}=\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{1vn}^{bc}=\frac{m_0}{m_{3n}}\bar{A}_{66},\quad \alpha_{2vn}^{bc}=\frac{1}{m_{3n}}\bar{A}_{66n},\quad \alpha_{3vn}^{bc}=\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{4vn}^{bc}=\frac{m_{0n}}{m_{3n}}\bar{A}_{21n}$$

$$\alpha_{5vn}^{bc}=\frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n},\quad \alpha_{6vn}^{bc}=\frac{m_{0n}^2}{m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right),\quad \alpha_{7vn}^{bc}=\frac{m_{2n}}{2m_{3n}}\bar{A}_{21n}$$

$$\alpha_{8vn}^{bc}=\frac{m_{0n}^2m_{2n}}{2m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right),\quad \alpha_{9vn}^{bc}=\frac{m_{0n}m_{1n}}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)-\bar{N}_{\theta p2}\right),\quad \alpha_{1wn}^{bc}=\frac{1}{m_{1n}\,m_{3n}}\bar{A}_{11n}$$

$$\alpha_{2wn}^{bc}=\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{3wn}^{bc}=\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{4wn}^{bc}=\frac{m_{2n}}{m_{3n}}\bar{A}_{12n},\quad \alpha_{5wn}^{bc}=\frac{m_{2n}}{m_{3n}}\bar{A}_{12n},\quad \alpha_{6wn}^{bc}=\frac{1}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)-\bar{N}_{xp2}\right)$$

$$\alpha_{7wn}^{bc}=\frac{1}{m_{1n}^2m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{11n}),\quad \alpha_{8wn}^{bc}=\frac{m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{12n}-2\bar{D}_{66n}),\quad \alpha_{9wn}^{bc}=\frac{1}{2m_{1n}^2m_{3n}}\left(\bar{A}_{11n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right)$$

$$\alpha_{10wn}^{bc}=\frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{12n}+2\bar{A}_{66n}),\quad \alpha_{11wn}^{bc}=-\frac{1}{2m_{1n}^2m_{3n}}\bar{G}_{11n}^*,\quad \alpha_{12wn}^{bc}=\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},\quad \alpha_{13wn}^{bc}=\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n}$$

$$\alpha_{14wn}^{bc}=\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},\quad \alpha_{15wn}^{bc}=\frac{m_{0n}^2\,m_{2n}}{m_{3n}}\bar{A}_{22n},\quad \alpha_{16wn}^{bc}=\frac{m_{0n}^2\,m_{2n}}{m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right)$$

$$\alpha_{17wn}^{bc}=\frac{m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{21n}-2\bar{D}_{66n}),\quad \alpha_{18wn}^{bc}=\frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{21n}+2\bar{A}_{66n}),\quad \alpha_{19wn}^{bc}=\frac{m_{0n}^2}{m_{3n}}\left(2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)-\bar{N}_{\theta p2}\right)$$

$$\alpha_{20wn}^{bc}=\frac{m_{0n}^2m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^*-\bar{D}_{22n}),\quad \alpha_{21wn}^{bc}=\frac{m_{0n}^2m_{2n}^2}{2m_{3n}}\left(\bar{A}_{22n}-2(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^l)\right),\quad \alpha_{22wn}^{bc}=-\frac{m_{2n}^2}{2m_{3n}}\bar{G}_{11n}^*$$

$$\alpha_{23wn}^{bc}=-\frac{1}{m_{1n}^2m_{3n}}\bar{D}_{11n},\quad \alpha_{24wn}^{bc}=-\frac{m_{2n}^2}{m_{3n}}\bar{D}_{12n},\quad \alpha_{25wn}^{bc}=-\frac{1}{m_{3n}}\bar{M}_{xp2},\quad \alpha_{26wn}^{bc}=\frac{1}{m_{1n}^2m_{3n}}\bar{E}_{11n}^*$$

$$\alpha_{27wn}^{bc}=\frac{m_{2n}^2}{m_{3n}}\bar{E}_{11n}^*,\quad \alpha_{28wn}^{bc}=-\frac{1}{2m_{1n}^2m_{3n}}\bar{G}_{11n}^*,\quad \alpha_{29wn}^{bc}=-\frac{2m_{2n}^2}{m_{3n}}\bar{D}_{66n},\quad \alpha_{30wn}^{bc}=-\frac{2m_{2n}^2}{m_{3n}}\bar{D}_{66n}$$

$$\alpha_{31wn}^{bc}=-\frac{m_{2n}^2}{m_{3n}}\bar{D}_{21n},\quad \alpha_{32wn}^{bc}=-\frac{m_{0n}^2m_{2n}^2}{m_{3n}}\bar{D}_{22n},\quad \alpha_{33wn}^{bc}=-\frac{m_{0n}^2}{m_{3n}}\bar{M}_{\theta p2},\quad \alpha_{34wn}^{bc}=\frac{m_{2n}^2}{m_{3n}}\bar{E}_{11n}^*$$

$$\alpha_{35wn}^{bc}=\frac{m_{0n}^2m_{2n}^2}{m_{3n}}\bar{E}_{11n}^*,\quad \alpha_{36wn}^{bc}=-\frac{m_{2n}^2}{2m_{3n}}\bar{G}_{1n1}^*$$

Appendix 4

$$(M)_{un}^{u}=\iint\left(\chi_e\chi_i\vartheta_f\vartheta_j-\bar{\mu}\left(\chi_e\chi_i''\vartheta_f\vartheta_j+m_0^2\chi_e\chi_i\vartheta_f\vartheta_j''\right)\right)_n d\xi\,d\theta$$

$$(K)_{un}^{u}=\iint\left(\begin{array}{c}\alpha_{1u}\chi_e\chi_i''\vartheta_f\vartheta_j+\alpha_{2u}\chi_e\chi_i\vartheta_f\vartheta_j''\\-\bar{\eta}\left(\alpha_{1u}\left(\chi_e\chi_i''''\vartheta_f\vartheta_j+m_0^2\chi_e\chi_i''\vartheta_f\vartheta_j''\right)+\alpha_{2u}\left(\chi_e\chi_i''\vartheta_f\vartheta_j''+m_0^2\chi_e\chi_i\vartheta_f\vartheta_j''''\right)\right)\end{array}\right)_n d\xi\,d\theta$$

$$(K_{bc})_{un}^{u}=\alpha_{1un}^{bc}(\chi_e\chi_i')|_0^1\int\vartheta_f\vartheta_j d\theta+\alpha_{7un}^{bc}\left(\vartheta_f\vartheta_j'\right)\Big|_0^{2\pi}\int\chi_e\chi_i d\xi$$

$$(K)_{un}^{v}=\iint\alpha_{3un}\left(\chi_e\phi_k'\vartheta_f\alpha_l'-\bar{\eta}\left(\chi_e\phi_k'''\vartheta_f\alpha_l'+m_0^2\chi_e\phi_k'\vartheta_f\alpha_l'''\right)\right)_n d\xi\,d\theta$$

$$(K_{bc})_{un}^{v}=\alpha_{2un}^{bc}(\chi_e\phi_k)|_0^1\int\vartheta_f\alpha_l' d\theta+\alpha_{8un}^{bc}\left(\vartheta_f\alpha_l\right)\Big|_0^{2\pi}\int\chi_e\phi_k' d\xi$$

$$(K)_{un}^{w}=\iint\alpha_{4un}\left(\chi_e\beta_o'\vartheta_f\psi_l-\bar{\eta}\left(\chi_e\beta_o'''\vartheta_f\psi_l+m_0^2\chi_e\beta_o'\vartheta_f\psi_l''\right)\right)_n d\xi\,d\theta$$

$$(K_{bc})_{un}^{w}=\alpha_{3u}^{bc}(\chi_e\beta_0)|_0^1\int\vartheta_f\psi_l d\theta$$

$$(NL)_{un}^{w}=\iint\left(\begin{array}{c}\alpha_{5u}\chi_e\beta_0'\beta_t''\vartheta_f\psi_p\psi_v+\alpha_{6u}\chi_e\beta_0'\beta_t\vartheta_f\psi_p\psi_v''+\alpha_{7u}\chi_e\beta_0'\beta_t\vartheta_f\psi_p'\psi_v'\\-\bar{\eta}\left(\begin{array}{c}\alpha_{5u}\left(\chi_e\beta_0'''\beta_t''\vartheta_f\psi_p\psi_v+\chi_e\beta_0'\beta_t''''\vartheta_f\psi_p\psi_v+m_0^2\left(\chi_e\beta_0'\beta_t''\vartheta_f\psi_p''\psi_v+\chi_e\beta_0'\beta_t''\vartheta_f\psi_p\psi_v''\right)\right)\\+\alpha_{6u}\left(\chi_e\beta_0'''\beta_t\vartheta_f\psi_p\psi_v''+\chi_e\beta_0'\beta_t''\vartheta_f\psi_p\psi_v''+m_0^2\left(\chi_e\beta_0'\beta_t\vartheta_f\psi_p''\psi_v''+\chi_e\beta_0'\beta_t\vartheta_f\psi_p\psi_v''''\right)\right)\\+\alpha_{7u}\left(\chi_e\beta_0'''\beta_t\vartheta_f\psi_p'\psi_v'+\chi_e\beta_0'\beta_t''\vartheta_f\psi_p'\psi_v'+m_0^2\left(\chi_e\beta_0'\beta_t\vartheta_f\psi_p'''\psi_v'+\chi_e\beta_0'\beta_t\vartheta_f\psi_p'\psi_v'''\right)\right)\end{array}\right)\end{array}\right)_n d\xi\, d\theta$$

$$(NL_{bc})_{un}^{w}=\alpha_{4un}^{bc}(\chi_e\beta_0'\beta_t')|_0^1\int\vartheta_f\psi_p\psi_v d\theta+\alpha_{5un}^{bc}(\chi_e\beta_o\beta_t)|_0^1\int\vartheta_f\psi_p'\psi_v' d\theta+\alpha_{9un}^{bc}\left(\vartheta_f\psi_p\psi_v'\right)\Big|_0^{2\pi}\int\chi_e\beta_0'\beta_t d\xi$$

$$\bar{F}_{upn}^{bc}=\alpha_{6un}^{bc}(\chi_e)|_0^1\int\vartheta_f d\theta$$

$$(M)_{vn}^{v}=\iint\left(\phi_q\phi_k a_f a_l-\bar{\mu}\left(\phi_q\phi_k'' a_f a_l+m_0^2\phi_q\phi_k a_f a_l''\right)\right)_n d\xi\, d\theta$$

$$(K)_{vn}^{u}=\iint\alpha_{1vn}\left(\phi_q\chi_i'\alpha_f\vartheta_l'-\bar{\eta}\left(\phi_q\chi_i'''\alpha_f\vartheta_l'+m_0^2\phi_q\chi_i'\alpha_f\vartheta_l'''\right)\right)_n d\xi\, d\theta$$

$$(K_{bc})_{vn}^{u}=\alpha_{1vn}^{bc}\left(\phi_q\chi_i\right)\Big|_0^1\int\alpha_f\vartheta_l' d\theta+\alpha_{4vn}^{bc}\left(\alpha_f\vartheta_l\right)\Big|_0^{2\pi}\int\phi_q\chi_i' d\xi$$

$$(K)_{vn}^{v}=\iint\left(\begin{array}{c}\alpha_{2v}\phi_q\phi_k''\alpha_f\alpha_l+\alpha_{3v}\phi_q\phi_k\alpha_f\alpha_l''\\-\bar{\eta}\left(\alpha_{2v}\left(\phi_q\phi_k''''\alpha_f\alpha_l+m_0^2\phi_q\phi_k''\alpha_f\alpha_l''\right)+\alpha_{3v}\left(\phi_q\phi_k''\alpha_f\alpha_l''+m_0^2\phi_q\phi_k\alpha_f\alpha_l''''\right)\right)\end{array}\right)_n d\xi\, d\theta$$

$$(K_{bc})_{vn}^{v}=\alpha_{2vn}^{bc}\left(\phi_q\phi_k'\right)\Big|_0^1\int\alpha_f\alpha_l d\theta+\alpha_{5vn}^{bc}\left(\alpha_f\alpha_l'\right)\Big|_0^{2\pi}\int\phi_q\phi_k d\xi$$

$$(K)_{vn}^{w}=\alpha_{6vn}\iint\left(\phi_q\beta_o\alpha_f\psi_l'-\bar{\eta}\left(\phi_q\beta_o''\alpha_f\psi_l'+m_0^2\phi_q\beta_o\alpha_f\psi_l'''\right)\right)_n d\xi\, d\theta$$

$$(K_{bc})_{vn}^{w}=\alpha_{6vn}^{bc}\left(\alpha_f\psi_l\right)\Big|_0^{2\pi}\int\phi_q\beta_o d\xi$$

$$(NL)_{vn}^{w}=\iint\left(\begin{array}{c}\alpha_{4v}\phi_q\beta_o'\beta_t'\alpha_f\psi_p\psi_v'+\alpha_{5v}\phi_q\beta_0''\beta_t\alpha_f\psi_p\psi_v'+\alpha_{7v}\phi_q\beta_o\beta_t\alpha_f\psi_p'\psi_v''\\-\bar{\eta}\left(\begin{array}{c}\alpha_{4v}\left(\phi_q\beta_o'''\beta_t'\alpha_f\psi_p\psi_v'+\phi_q\beta_o'\beta_t'''\alpha_f\psi_p\psi_v'+m_0^2\left(\phi_q\beta_o'\beta_t'\alpha_f\psi_p''\psi_v'+\phi_q\beta_o'\beta_t'\alpha_f\psi_p\psi_v'''\right)\right)\\+\alpha_{5v}\left(\phi_q\beta_0''''\beta_t\alpha_f\psi_p\psi_v'+\phi_q\beta_0''\beta_t''\alpha_f\psi_p\psi_v'+m_0^2\left(\phi_q\beta_0''\beta_t\alpha_f\psi_p''\psi_v'+\phi_q\beta_0''\beta_t\alpha_f\psi_p\psi_v'''\right)\right)\\+\alpha_{7v}\left(\phi_q\beta_o''\beta_t\alpha_f\psi_p'\psi_v''+\phi_q\beta_o\beta_t''\alpha_f\psi_p'\psi_v''+m_0^2\left(\phi_q\beta_o\beta_t\alpha_f\psi_p'''\psi_v''+\phi_q\beta_o\beta_t\alpha_f\psi_p'\psi_v''''\right)\right)\end{array}\right)\end{array}\right)_n d\xi\, d\theta$$

$$(NL_{bc})_{vn}^{w}=\alpha_{3vn}^{bc}\left(\phi_q\beta_o'\beta_t\right)\Big|_0^1\int\alpha_f\psi_p\psi_v' d\theta+\alpha_{7vn}^{bc}\left(\alpha_f\psi_p\psi_v\right)\Big|_0^{2\pi}\int\phi_q\beta_o'\beta_t' d\xi+\alpha_{8vn}^{bc}\left(\alpha_f\psi_p'\psi_v'\right)\Big|_0^{2\pi}\int\phi_q\beta_o\beta_t d\xi$$

$$\bar{F}_{vpn}^{bc}=\alpha_{9vn}^{bc}\left(\alpha_f\right)\Big|_0^{2\pi}\int\phi_q d\xi$$

$$(M)_{wn}^{w}=\iint\left(\beta_r\beta_o\psi_s\psi_p+\alpha_{31w}\beta_r\beta_o''\psi_s\psi_p+\alpha_{32w}\beta_r\beta_o\psi_s\psi_p''-\bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\right)_n d\xi\, d\theta$$

$$(M_{bc})_{wn}^{w}=\alpha_{11wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p d\theta+\alpha_{28wn}^{bc}(\beta_r\beta_o)|_0^1\int\psi_s\psi_p d\theta+\alpha_{36wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi$$

$$(C)_{wn}^{w}=\iint\bar{C}_{wn}\left(\beta_r\beta_o\psi_s\psi_p-\bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\right)_n d\xi\, d\theta$$

$$(K)_{wn}^{u}=\iint\alpha_{1wn}\left(\beta_r\chi_i'\psi_s\vartheta_j-\bar{\eta}\left(\beta_r\chi_i'''\psi_s\vartheta_j+m_0^2\beta_r\chi_i'\psi_s\vartheta_j''\right)\right)_n d\xi\, d\theta$$

$$(K)_{wn}^{v}=\iint\alpha_{12wn}\left(\beta_r\phi_k\psi_s\alpha_l'-\bar{\eta}(\beta_r\phi_k''\psi_s\alpha_l'+m_0^2\beta_r\phi_k\psi_s\alpha_l''')\right)_n d\xi\, d\theta$$

$$(K)_{wn}^{w}=\iint\left(\begin{array}{c}\alpha_{16w}\beta_r\beta_o\psi_s\psi_p+\alpha_{19w}\beta_r\beta_0''\psi_s\psi_p+\alpha_{23w}\beta_r\beta_o''''\psi_s\psi_p\\+\alpha_{25w}\beta_r\beta_o''\psi_s\psi_p''+\alpha_{27w}\beta_r\beta_o\psi_s\psi_p''+\alpha_{28w}\beta_r\beta_o\psi_s\psi_p''''\\-\bar{\eta}\left(\begin{array}{c}\alpha_{16w}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)+\alpha_{19w}\left(\beta_r\beta_0''''\psi_s\psi_p+m_0^2\beta_r\beta_0''\psi_s\psi_p''\right)\\+\alpha_{23w}\left(\beta_r\beta_o''''''\psi_s\psi_p+m_0^2\beta_r\beta_o''''\psi_s\psi_p''\right)+\alpha_{25w}\left(\beta_r\beta_o''''\psi_s\psi_p''+m_0^2\beta_r\beta_o''\psi_s\psi_p''''\right)\\+\alpha_{27w}\left(\beta_r\beta_o''\psi_s\psi_p''+m_0^2\beta_r\beta_o\psi_s\psi_p''''\right)+\alpha_{28w}\left(\beta_r\beta_o''\psi_s\psi_p''''+m_0^2\beta_r\beta_o\psi_s\psi_p''''''\right)\end{array}\right)\end{array}\right)_n d\xi\, d\theta$$

$$(K_{bc})_{wn}^{w}=\alpha_{6wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p d\theta+\alpha_{7wn}^{bc}(\beta_r\beta_o''')|_0^1\int\psi_s\psi_p d\theta+\alpha_{8wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p'' d\theta$$
$$+\alpha_{17wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi}\int\beta_r\beta_o'' d\xi+\alpha_{19wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi+\alpha_{20wn}^{bc}\left(\psi_s\psi_p'''\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi+\alpha_{23wn}^{bc}(\beta_r\beta_o'')|_0^1\int\psi_s\psi_p d\theta$$
$$+\alpha_{24wn}^{bc}(\beta_r\beta_o)|_0^1\int\psi_s\psi_p'' d\theta+\alpha_{26wn}^{bc}(\beta_r\beta_o'')|_0^1\int\psi_s\psi_p d\theta+\alpha_{27wn}^{bc}(\beta_r\beta_o)|_0^1\int\psi_s\psi_p'' d\theta+\alpha_{29wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi}\int\beta_r\beta_o' d\xi$$
$$+\alpha_{30wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p' d\theta+\alpha_{31wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi}\int\beta_r\beta_o'' d\xi+\alpha_{32wn}^{bc}\left(\psi_s\psi_p''\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi+\alpha_{34wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi}\int\beta_r\beta_o'' d\xi$$
$$+\alpha_{35wn}^{bc}\left(\psi_s\psi_p''\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi$$

$$(K_e)_{wn}^{w}=\iint\left(\beta_r\beta_o\psi_s\psi_p-\bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\right)_n d\xi\, d\theta,(K_{e2})_{wn}^{w}=-\bar{C}_2\bar{F}_e\bar{V}_{DC}^2(K_e)_{wn}^{w}$$

$$(NL)^u_{wn} = \iint \begin{pmatrix} \alpha_{2w}\beta_r\beta_o''\chi_i'\psi_s\psi_p\vartheta_j + \alpha_{3w}\beta_r\beta_o\chi_i'\psi_s\psi_p''\vartheta_j + \alpha_{4w}\beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j + \alpha_{5w}\beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j' \\ +\alpha_{6w}\beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j' + \alpha_{7w}\beta_r\beta_o'\chi_i\psi_s\psi_p'\vartheta_j' + \alpha_{8w}\beta_r\beta_o'\chi_i\psi_s\psi_p\vartheta_j'' \\ -\bar{\eta}\begin{pmatrix} \alpha_{2w}\left(\beta_r\beta_o''''\chi_i'\psi_s\psi_p\vartheta_j + \beta_r\beta_o''\chi_i'''\psi_s\psi_p\vartheta_j + m_0^2\left(\beta_r\beta_o''\chi_i'\psi_s\psi_p''\vartheta_j + \beta_r\beta_o''\chi_i'\psi_s\psi_p\vartheta_j''\right)\right) \\ +\alpha_{3w}\left(\beta_r\beta_o''\chi_i'\psi_s\psi_p''\vartheta_j + \beta_r\beta_o\chi_i'''\psi_s\psi_p''\vartheta_j + m_0^2\left(\beta_r\beta_o\chi_i'\psi_s\psi_p''''\vartheta_j + \beta_r\beta_o\chi_i'\psi_s\psi_p''\vartheta_j''\right)\right) \\ +\alpha_{4w}\left(\beta_r\beta_o'''\chi_i''\psi_s\psi_p\vartheta_j + \beta_r\beta_o'\chi_i''''\psi_s\psi_p\vartheta_j + m_0^2\left(\beta_r\beta_o'\chi_i''\psi_s\psi_p''\vartheta_j + \beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j''\right)\right) \\ +\alpha_{5w}\left(\beta_r\beta_o''\chi_i'\psi_s\psi_p'\vartheta_j' + \beta_r\beta_o\chi_i'''\psi_s\psi_p'\vartheta_j' + m_0^2\left(\beta_r\beta_o\chi_i'\psi_s\psi_p'''\vartheta_j' + \beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j'''\right)\right) \\ +\alpha_{6w}\left(\beta_r\beta_o''\chi_i'\psi_s\psi_p'\vartheta_j' + \beta_r\beta_o\chi_i'''\psi_s\psi_p'\vartheta_j' + m_0^2\left(\beta_r\beta_o\chi_i'\psi_s\psi_p'''\vartheta_j' + \beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j'''\right)\right) \\ +\alpha_{7w}\left(\beta_r\beta_o'''\chi_i\psi_s\psi_p'\vartheta_j' + \beta_r\beta_o'\chi_i''\psi_s\psi_p'\vartheta_j' + m_0^2\left(\beta_r\beta_o'\chi_i\psi_s\psi_p'''\vartheta_j' + \beta_r\beta_o'\chi_i\psi_s\psi_p'\vartheta_j'''\right)\right) \\ +\alpha_{8w}\left(\beta_r\beta_o'''\chi_i\psi_s\psi_p\vartheta_j'' + \beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j'' + m_0^2\left(\beta_r\beta_o'\chi_i\psi_s\psi_p''\vartheta_j'' + \beta_r\beta_o'\chi_i\psi_s\psi_p\vartheta_j''''\right)\right) \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta$$

$$(NL_{bc})^u_{wn} = \alpha^{bc}_{1wn}(\beta_r\beta_o'\chi_i')|_0^1 \int \psi_s\psi_p\vartheta_j d\theta + \alpha^{bc}_{2wn}(\beta_r\beta_o\chi_i)|_0^1 \int \psi_s\psi_p'\vartheta_j' d\theta + \alpha^{bc}_{12wn}\left(\psi_s\psi_p'\vartheta_j\right)\Big|_0^{2\pi} \int \beta_r\beta_o\chi_i' d\xi$$
$$+ \alpha^{bc}_{13wn}\left(\psi_s\psi_p\vartheta_j'\right)\Big|_0^{2\pi} \int \beta_r\beta_o'\chi_i d\xi$$

$$(NL)^v_{wn} = \iint \begin{pmatrix} \alpha_{9w}\beta_r\beta_o'\phi_k'\psi_s\psi_p'\alpha_l + \alpha_{10w}\beta_r\beta_o\phi_k''\psi_s\psi_p'\alpha_l + \alpha_{11w}\beta_r\beta_o'\phi_k'\psi_s\psi_p\alpha_l' \\ +\alpha_{13w}\beta_r\beta_o''\phi_k\psi_s\psi_p\alpha_l' + \alpha_{14w}\beta_r\beta_o\phi_k\psi_s\psi_p''\alpha_l' + \alpha_{15w}\beta_r\beta_o\phi_k\psi_s\psi_p'\alpha_l'' \\ -\bar{\eta}\begin{pmatrix} \alpha_{9w}\left(\beta_r\beta_o'''\phi_k'\psi_s\psi_p'\alpha_l + \beta_r\beta_o'\phi_k'''\psi_s\psi_p'\alpha_l + m_0^2\left(\beta_r\beta_o'\phi_k'\psi_s\psi_p'''\alpha_l + \beta_r\beta_o'\phi_k'\psi_s\psi_p'\alpha_l''\right)\right) \\ +\alpha_{10w}\left(\beta_r\beta_o''\phi_k''\psi_s\psi_p'\alpha_l + \beta_r\beta_o\phi_k''''\psi_s\psi_p'\alpha_l + m_0^2\left(\beta_r\beta_o\phi_k''\psi_s\psi_p'''\alpha_l + \beta_r\beta_o\phi_k''\psi_s\psi_p'\alpha_l''\right)\right) \\ +\alpha_{11w}\left(\beta_r\beta_o'''\phi_k'\psi_s\psi_p\alpha_l' + \beta_r\beta_o'\phi_k'''\psi_s\psi_p\alpha_l' + m_0^2\left(\beta_r\beta_o'\phi_k'\psi_s\psi_p''\alpha_l' + \beta_r\beta_o'\phi_k'\psi_s\psi_p\alpha_l'''\right)\right) \\ +\alpha_{13w}\left(\beta_r\beta_o''''\phi_k\psi_s\psi_p\alpha_l' + \beta_r\beta_o''\phi_k''\psi_s\psi_p\alpha_l' + m_0^2\left(\beta_r\beta_o''\phi_k\psi_s\psi_p''\alpha_l' + \beta_r\beta_o''\phi_k\psi_s\psi_p\alpha_l'''\right)\right) \\ +\alpha_{14w}\left(\beta_r\beta_o''\phi_k\psi_s\psi_p''\alpha_l' + \beta_r\beta_o\phi_k''\psi_s\psi_p''\alpha_l' + m_0^2\left(\beta_r\beta_o\phi_k\psi_s\psi_p''''\alpha_l' + \beta_r\beta_o\phi_k\psi_s\psi_p''\alpha_l'''\right)\right) \\ +\alpha_{15w}\left(\beta_r\beta_o''\phi_k\psi_s\psi_p'\alpha_l'' + \beta_r\beta_o\phi_k''\psi_s\psi_p'\alpha_l'' + m_0^2\left(\beta_r\beta_o\phi_k\psi_s\psi_p'''\alpha_l'' + \beta_r\beta_o\phi_k\psi_s\psi_p'\alpha_l''''\right)\right) \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta$$

$$(NL_{bc})^v_{wn} = \alpha^{bc}_{3wn}(\beta_r\beta_o\phi_k')|_0^1 \int \psi_s\psi_p'\alpha_l d\theta + \alpha^{bc}_{4wn}(\beta_r\beta_o'\phi_k)|_0^1 \int \psi_s\psi_p\alpha_l' d\theta + \alpha^{bc}_{14wn}\left(\psi_s\psi_p\alpha_l\right)\Big|_0^{2\pi} \int \beta_r\beta_o'\phi_k' d\xi$$
$$+ \alpha^{bc}_{15wn}\left(\psi_s\psi_p'\alpha_l'\right)\Big|_0^{2\pi} \int \beta_r\beta_o\phi_k d\xi$$

$$(NL)^w_{w2n} = \iint \begin{pmatrix} \alpha_{17w}\beta_r\beta_o\beta_t''\psi_s\psi_p\psi_v + \alpha_{18w}\beta_r\beta_o\beta_t\psi_s\psi_p\psi_v'' + \alpha_{24w}\beta_r\beta_o'\beta_t'\psi_s\psi_p\psi_{v+\alpha_{30w}\beta_r\beta_o\beta_t\psi_s\psi_p'\psi_v'} \\ -\bar{\eta}\begin{pmatrix} \alpha_{17w}\left(\beta_r\beta_o''\beta_t''\psi_s\psi_p\psi_v + \beta_r\beta_o\beta_t''''\psi_s\psi_p\psi_v + m_0^2\left(\beta_r\beta_o\beta_t''\psi_s\psi_p''\psi_v + \beta_r\beta_o\beta_t''\psi_s\psi_p\psi_v''\right)\right) \\ +\alpha_{18w}\left(\beta_r\beta_o''\beta_t\psi_s\psi_p\psi_v'' + \beta_r\beta_o\beta_t''\psi_s\psi_p\psi_v'' + m_0^2\left(\beta_r\beta_o\beta_t\psi_s\psi_p''\psi_v'' + \beta_r\beta_o\beta_t\psi_s\psi_p\psi_v''''\right)\right) \\ +\alpha_{24w}\left(\beta_r\beta_o'''\beta_t'\psi_s\psi_p\psi_v + \beta_r\beta_o'\beta_t'''\psi_s\psi_p\psi_v + m_0^2\left(\beta_r\beta_o'\beta_t'\psi_s\psi_p''\psi_v + \beta_r\beta_o'\beta_t'\psi_s\psi_p\psi_v''\right)\right) \\ +\alpha_{30w}\left(\beta_r\beta_o''\beta_t\psi_s\psi_p'\psi_v' + \beta_r\beta_o\beta_t''\psi_s\psi_p'\psi_v' + m_0^2\left(\beta_r\beta_o\beta_t\psi_s\psi_p'''\psi_v' + \beta_r\beta_o\beta_t\psi_s\psi_p'\psi_v'''\right)\right) \end{pmatrix} \end{pmatrix} d\xi\, d\theta$$

$$(NL_{bc})^w_{w2n} = \alpha^{bc}_{5wn}(\beta_r\beta_o\beta_t')|_0^1 \int \psi_s\psi_p\psi_v d\theta + \alpha^{bc}_{16wn}\left(\psi_s\psi_p\psi_v'\right)\Big|_0^{2\pi} \int \beta_r\beta_o\beta_t d\xi$$

$$(NL_e)^w_{w2n} = \iint \left(\beta_r\beta_o\beta_t\psi_s\psi_p\psi_v - \bar{\mu}\left(\beta_r\beta_o''\beta_t\psi_s\psi_p\psi_v + \beta_r\beta_o\beta_t''\psi_s\psi_p\psi_v + m_0^2\left(\beta_r\beta_o\beta_t\psi_s\psi_p''\psi_v + \beta_r\beta_o\beta_t\psi_s\psi_p\psi_v''\right)\right)\right)_n d\xi\, d\theta$$

$$(NL_{2e})^w_{w2n} = -\bar{C}_3\bar{F}_e\bar{V}_{DC}^2(NL_e)^w_{w2n}$$

$$(NL)^w_{w3n} = \iint \begin{pmatrix} \alpha_{20w}\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b' + \alpha_{21w}\beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b + \alpha_{22w}\beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b' \\ +\alpha_{26w}\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'' + \alpha_{29w}\beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' \\ -\bar{\eta}\begin{pmatrix} \alpha_{20w}\begin{pmatrix} \beta_r\beta_o'''\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b' + \beta_r\beta_o'\beta_t'''\beta_a\psi_s\psi_p\psi_v'\psi_b' + \beta_r\beta_o'\beta_t'\beta_a''\psi_s\psi_p\psi_v'\psi_b' \\ +m_0^2(\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'''\psi_b' + \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b''') \end{pmatrix} \\ +\alpha_{21w}\begin{pmatrix} \beta_r\beta_o''''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b + \beta_r\beta_o''\beta_t'''\beta_a'\psi_s\psi_p\psi_v\psi_b + \beta_r\beta_o''\beta_t'\beta_a'''\psi_s\psi_p\psi_v\psi_b \\ +m_0^2(\beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p''\psi_v\psi_b + \beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v''\psi_b + \beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b'') \end{pmatrix} \\ +\alpha_{22w}\begin{pmatrix} \beta_r\beta_o''''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b' + \beta_r\beta_o''\beta_t''\beta_a\psi_s\psi_p\psi_v'\psi_b' + \beta_r\beta_o''\beta_t\beta_a''\psi_s\psi_p\psi_v'\psi_b' \\ +m_0^2(\beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'''\psi_b' + \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b''') \end{pmatrix} \\ +\alpha_{26w}\begin{pmatrix} \beta_r\beta_o'''\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'' + \beta_r\beta_o'\beta_t'''\beta_a\psi_s\psi_p\psi_v\psi_b'' + \beta_r\beta_o'\beta_t'\beta_a''\psi_s\psi_p\psi_v\psi_b'' \\ +m_0^2(\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p''\psi_v\psi_b'' + \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v''\psi_b'' + \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'''') \end{pmatrix} \\ +\alpha_{29w}\begin{pmatrix} \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \beta_r\beta_o\beta_t''\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \beta_r\beta_o\beta_t\beta_a''\psi_s\psi_p''\psi_v'\psi_b' \\ +m_0^2(\beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''''\psi_v'\psi_b' + \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'''\psi_b' + \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b''') \end{pmatrix} \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta$$

$$(NL_{bc})^w_{w3n} = \alpha^{bc}_{9wn}(\beta_r\beta_o'\beta_t'\beta_a')|_0^1 \int \psi_s\psi_p\psi_v\psi_b d\theta + \alpha^{bc}_{10wn}(\beta_r\beta_o'\beta_t\beta_a)|_0^1 \int \psi_s\psi_p\psi_v'\psi_b' d\theta + \alpha^{bc}_{18wn}\left(\psi_s\psi_p\psi_v\psi_b'\right)\Big|_0^{2\pi} \int \beta_r\beta_o'\beta_t'\beta_a d\xi$$
$$+ \alpha^{bc}_{21wn}\left(\psi_s\psi_p'\psi_v'\psi_b'\right)\Big|_0^{2\pi} \int \beta_r\beta_o\beta_t\beta_a d\xi$$

$$(NL_3)^w_{w3n} = \iint \begin{pmatrix} \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b \\ -\bar{\mu}\begin{pmatrix} \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b + \beta_r\beta_o\beta_t''\beta_a\psi_s\psi_p\psi_v\psi_b + \beta_r\beta_o\beta_t\beta_a''\psi_s\psi_p\psi_v\psi_b \\ +m_0^2(\beta_r\beta_o\beta_t\beta_a\psi_s\ \psi_p''\psi_v\psi_b + \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v''\psi_b + \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b'') \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta$$

$$(NL_{3e})^w_{w3n} = -\bar{C}_4\bar{F}_e\bar{V}_{DC}^2(NL_{3n})^w_{w3}$$

$$(NL)_{w1}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ -\mu^2\left(\beta_{2r}\beta_{2o}\beta_{2t}\beta''_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} + m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi''_{bn}\right) \end{pmatrix} d\xi\, d\theta$$

$$(NL)_{w2}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{1t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{1v}\psi_{bn} \\ -\mu^2\left(\beta_{2r}\beta_{2o}\beta_{2t}\beta''_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} + m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi''_{bn}\right) \end{pmatrix} d\xi\, d\theta$$

$$(NL)_{w3}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{1o}\beta_{1t}\beta_{an}\psi_{2s}\psi_{1p}\psi_{1v}\psi_{bn} \\ -\mu^2\left(\beta_{2r}\beta_{2o}\beta_{2t}\beta''_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} + m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi''_{bn}\right) \end{pmatrix} d\xi\, d\theta$$

$$(NL)_{w4}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{1r}\beta_{1o}\beta_{1t}\beta_{an}\psi_{1s}\psi_{1p}\psi_{1v}\psi_{bn} \\ -\mu^2\left(\beta_{2r}\beta_{2o}\beta_{2t}\beta''_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} + m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi''_{bn}\right) \end{pmatrix} d\xi\, d\theta$$

$$\bar{F}_{wpn} = \iint (\alpha_{33wn}\beta_r\psi_s)d\xi\, d\theta, \bar{F}_{wpn}^{bc} = \alpha_{22wn}^{bc}(\psi_s)|_0^{2\pi}\int \beta_r d\xi + \alpha_{25wn}^{bc}(\beta_r)|_0^1\int \psi_s d\theta + \alpha_{33wn}^{bc}(\psi_s)|_0^{2\pi}\int \beta_r d\xi$$

$$\bar{F}_1 = \iint (\beta_r\psi_s)d\xi\, d\theta\, , \bar{F}_{we} = \bar{C}_1\bar{F}_e\bar{V}_{DC}^2\bar{F}_1, \bar{F}_{Ha} = \bar{F}\iint (\beta_r\beta_s \cos\Omega\tau)d\xi d\theta$$

Nomenclatures

Symbol	Description	Symbol	Description
h_N	Thickness of nanoshell (NS)	h_p	Thickness of piezoelectric layer (PL)
L	Length of PENS	E_p	Young modulus of PL
R	The mid-surface radius	υ_p	Poisson ratio of PL
x	Axial direction	ρ_p	Mass density of PL
θ	Circumferential direction	e_{31p}, e_{32p}	Piezoelectric constants
z	Radius direction	η_{33p}	Dielectric constant
E_N	Young modulus of NS	s_k	Inner and outer surface of PL
υ_N	Poisson ratio of NS	λ^{s_k}, μ^{s_k}	Surface Lamé's constants of PL
ρ_N	Mass density of NS	$\bar{E}_p$	Electric field
I_k	Inner and outer interface	D_{zp}	Electric displacement
λ^{I_k}, μ^{I_k}	Interface Lamé's constants	$\tau_0^{s_k}$	Residual stress of PL
$\tau_0^{I_k}$	Residual stress of NS	$e_{31p}^{s_k}, e_{32p}^{s_k}$	Surface piezoelectric constants
ρ^{I_k}	Interface mass density of NS	ρ^{s_k}	Surface mass density of PL
C_{ijN}	Elastic constant of NS	C_{ijp}	Elastic constant of PL
σ_{ijN}	Middle stress of NS	σ_{ijp}	Middle stress of PL
$\kappa_{(x,\theta)}$	Curvature components	V_p	Piezoelectric voltage
$\varepsilon_{(x,\theta)}^0$, $\gamma_{x\theta}^0$	Middle surface strains	π	Total strain energy
u	Displacement of x direction	T	Total kinetic energy
v	Displacement of θ direction	I	Mass moments of inertia
w	Displacement of z direction	C_w	Damping coefficient
∇	Laplace operator	K_w	Winkler modulus
ω	Natural frequency	K_p	pasternak Shear modulus
M	Total mass matrix	W	Total work
C	Total damping coefficient	K	Total stiffness matrix
$\bar{F}$	Piezoelectric voltage loads	b	Gap width of the nanoshell
V_{DC}	Direct electric voltage	f	Amplitude of harmonic excitation
μ	Nonlocal parameter	η	Material length scale parameter