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Free vibration responses of nonlinear FG-CNT distribution in a polymer matrix

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Abstract. The object of this paper is to investigate the free vibration behavior under the effect of carbon nanotube distribution in functionally graded carbon nanotube-reinforced composite (FG-CNTRC) by using higher-order shear deformation theories. In this work, we present a novel distribution method for carbon nanotubes in the polymer matrix by using a new exponential power law distribution of carbon nanotube volume fraction. It is assumed that the SWCNTs are aligned along the beam axial direction and the distribution of the SWCNTs may vary through the thickness of the beam with different patterns of reinforcement. The rule of mixtures is used in order to obtain material properties of the CNTRC beams. Hamilton's principle is used in deriving the equations of motion. The validity of the free Vibration results is examined by comparing them with those of the known data in the literature. The results that obtained indicate that the carbon nanotube volume fraction distribution play a very important role on the free vibrations characteristics of the CNTRC beam.

Keywords: beam; free vibration; nanotube; nonlinear distribution; shear deformation; volume fraction

1. Introduction

In recent years, the nano-structures and their technological applications attracted many of intent. The carbon structures were one of the priorities of scientific research to their relationship with several areas of contemporary life. The carbon nanotube, discovered in Japan by Iijima (1991), have continued to attract many researchers in various fields in order to exploit to the maximum and in various fields, due the excellent electrical, thermal, and mechanical properties of these new materials (Dresselhaus and Avouris 2001, Bachtold *et al*. 2001). CNTs are expected to find applications in all industrial areas, also provide rich research subjects (Bensattalah *et al*. 2019, 2020, Belmahi *et al*. 2018, 2019).

The molecular dynamic (MD) and continuum mechanics are considered as an alternative means for modeling materials at the nano-scale (Liu *et al*. 2012). Indeed, the effects of scales are important in the mechanical behavior in nano-composite. Recently, a new class of composite materials known as functionally gradient materials (FGM) has drawn considerable attention from researchers. Many studies on FGMs have been conducted in a wide range of areas, since the concept of FGM was first proposed in 1984 (Koizumi 1997). Zghal *et al*. (2021) studied free vibration behavior of porous beams with gradually varying mechanical properties based on a robust finite beam element. Various researches exist in the literature, in terms of FGM (Wali *et al*. 2015, Zghal *et al*. 2022, Zghal and Dammak 2020).

Stimulated by the notion of FGM (Ahmed *et al*. 2018, Abdelmalek *et al*. 2019), the functionally graduated (FG) reinforcement distribution model has been applied successfully for composite materials reinforced by the CNT (FG-CNTRC) (Ajayan *et al*. 1994). Various researches exist in the literature, in terms of FG-CNTRC vibration behavior (Mehar and Panda 2016, Allahkarami *et al*. 2017, Avcar and Alwan 2017, Ebrahimi and Daman 2017, Baltacioglu and Civalek 2018, Arani *et al*. 2018, Avcar 2019, Rostami *et al*. 2020, Mahesh and Harursampath 2020a, Farokhian and Kolahchi 2020), buckling (Bouazza *et al*. 2014, Kolahchi *et al*. 2015, Bouazza *et al*. 2015, Boulal *et al*. 2020, Tayeb *et al*. 2020, Mohammadimehr and Alimirzaei 2017, Eltaher *et al*. 2020, Mehar and Panda 2019, Do *et al*. 2019), bending (Alankaya and Erdonmez 2017, Mehar and Panda 2016, Mahesh and Harursampath 2020b), Dynamic Responses (Kolahchi and Moniri 2016, Mehar and Panda 2018), Wave propagation (Karami *et al*. 2018, 2019).

A lot of the previous research was based on the uniform and linear distribution of carbon nanotube volume fraction in the FG-CNTRC beam. Shahrbabaki and Alibeigloo (2014) studied three-dimensional free vibrations of rectangular CNTRC plates under different boundary conditions. It was found that non-dimensional frequencies vary from minimum values for the plate with CFFF to the

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maximum values for the plate with CCCC boundary conditions. Lin and Xiang (2014) studied Vibration of single-walled carbon nanotube reinforced composite beams based on the first and third order beam theories. They used the classic variational Hamilton method to obtain the virtual deformation energy and kinetic energies of the FG-CNTRC beam and then resolved by the p-Ritz method, it was found that FGX-CNT beams always have the highest frequency parameters while FGV-CNT beams give the lowest ones. Zghal *et al.* (2018) used the discrete double directors shell model for free vibration analysis of FG-CNTRC shell structures, based on the high -order shear deformation theory. They studied the influences of CNTs volume fractions, the profiles of FG-CNT distributions, boundary conditions, length-to-thickness, edge-to-radius and many other aspects ratios for various FG-CNTRC shells. Arefi *et al.* (2018) investigate the free vibration response of FG composite nanoplates resting on a Pasternak foundation, and reinforced with GNPs, based on the nonlocal elasticity theory. They found that the number of layers in the macro-structure, value of the nonlocal parameter, and weight fractions of GNPs and their geometry and size play an important role on the fundamental frequency of the structure.

Kiani *et al.* (2018) studied the free vibration behavior of FG-CNTRC conical panels by applying the First Shear deformation Theory FSDT shell theory, combined with the Donnell's kinematics and the Hamilton's principle. It is found that boundary conditions and angles of embrace of the conical shell play an important role on the fundamental frequency of the structure. Fariborz and Batra (2019) studied the free vibrations of a bi-directional graded material circular beam by using a high shear deformable beam theory that has logarithmic variation in the radial direction of the tangential displacement. Zhao *et al.* (2019) presents the free vibration analysis of functionally graded carbon nanotube reinforced composite truncated conical panels with general boundary conditions, based on the first-order shear deformation theory, the modified Fourier series method and Ritz method to obtain the frequency parameters. They concluded that the volume fractions of CNTs, distribution types of CNTs, boundary restraint parameters and geometrical parameter have an obvious effect on the vibration behavior of the FG-CNTRC truncated conical. Mellouli *et al.* (2020) studied the free vibration behavior of FG-CNTRC shell structures using the meshfree radial point interpolation method (RPIM). They concluded that the changes of the CNT volume fractions and CNT profiles have pronounced effects on the free vibration behavior of various types of FG-CNTRC shell structures.

The previous researches were based on the linear distribution of carbon nanotubes for FG-CNTRC beams. This paper constitutes a first attempt to explore the influence of nonlinear distribution of carbon nanotube volume fraction on free vibration analysis of functionally graded beams using higher order shear deformation theory (HSDT) based on Hamilton's principle is presented. To obtain the optimal CNTs distribution in the polymer for the free vibration analysis, we introduce an exponential power law distribution of carbon nanotube volume fraction; the effect of various parameters on the free vibration analysis has been studied.

2. Geometrical and FG-CNTRC Material Properties

As shown in Fig. 1, a FG-CNTRC beam with length L, and thickness h is considered, in Fig. 2 the density of CNTs within the area is constant and the CNT volume fraction varies through the thickness (in z direction) of the beam with different symmetric patterns. For the case of (n = 0) the distribution of carbon nanotube is uniform in the matrix (UD-CNTs).

In this section, the CNTRC beam is made an embedded carbon nanotube (SWCNTs) in a polymer (PMMA) matrix. The effective material properties of CNTRC beams (Young's modulus E, shear modulus G, Poisson's ratio V12, mass density ρ) can be expressed using the rule of mixture (Shen 2009)

$$\begin{cases} E_{11} = (\eta_1 E_{11}^{cnt} - E_p)V_{cnt} + V_p E^p \\ \dfrac{\eta_2}{E_{22}} = \dfrac{V_{cnt}}{E_{22}^{cnt}} + \dfrac{V_p}{E^p} \\ G_{12}\dfrac{\eta_3}{G_{12}} = \dfrac{V_{cnt}}{G_{12}^{cnt}} + \dfrac{V_p}{G^p} \end{cases} \tag{1}$$

$$\begin{cases} V_{12} = V_{cnt}\nu_{12}^{cnt} + V_p \nu^p \\ \rho = V_{cnt}\rho^{cnt} + V_p \rho^p \end{cases} \tag{2}$$

$E_{11}^{cnt}, E_{22}^{cnt}, E^p$ and G_{12}^{cnt}, G^p are defined as the Young's modulus and shear modulus of SWCNTs and polymer matrix, respectively. Also, V_{cnt} and V_p are the volume fractions for carbon nanotube and the polymer matrix with the relation of $V_p + V_{\mathrm{cnt}} = 1$. The CNT efficiency parameters (η) related to the volume fraction are given from

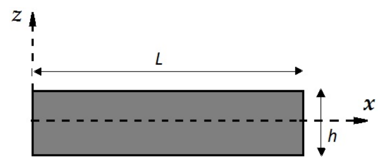

Fig. 1 Geometry of carbon nanotube-reinforced composite CNTRC beam

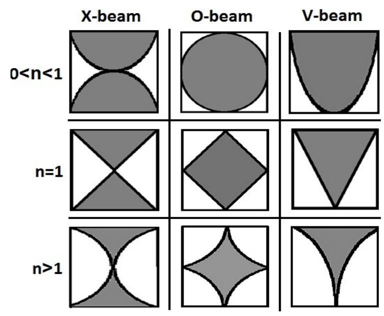

Fig. 2 Cross sections of different types of carbon nanotube reinforcement

Yas and Samadi (2012).

Four different symmetric patterns of distribution are considered in this study, including a new exponential power law distribution of carbon nanotube volume fraction. The FG carbon nanotube along the thickness direction of the beam is assumed to be as following

$$V_{cnt}=\begin{cases}V_{cnt}^* & \text{for UD-beam}\\ (n+1)(1-2\frac{|z|}{h})^n V_{cnt}^* & \text{for O-beam}\\ (n+1)(2\frac{|z|}{h})^n V_{cnt}^* & \text{for X-beam}\\ (n+1)(\frac{1}{2}+\frac{z}{h})^n V_{cnt}^* & \text{for V-beam}\end{cases} \tag{3}$$

Where n is the exponent degree of V_{cnt} law distribution, and V_{cnt}^* is the volume fraction of CNTs, which can be obtained from the equation

$$V_{cnt}^*=\frac{W_{cnt}}{W_{cnt}+(\rho^{cnt}/\rho^{p})(1-W_{cnt})} \tag{4}$$

Where W_{cnt} is the mass fraction of the CNTs.

3. Theoretical approach

Based on the higher-order shear deformation beam theory, the displacement field consisting of the axial displacement, u, and the transverse displacement, w, can be written in the following forms (Şimşek 2010)

$$\begin{cases}u(\text{x,z,t})=u_0-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial ws}{\partial x}\\ w(x,z,t)=w_b(x,t)+w_s(x,t)\end{cases} \tag{5}$$

Where u_0 is the mid-plane displacements of the beam in the x direction; w_b and w_s are the bending and shear components of transverse displacement, respectively.

The shape function $f(z)$ is chosen based on a third order shear deformation theory as Reissner (1945).

$$f(z)=z-\frac{5}{4}z(1-\frac{4z^2}{3h^2}) \tag{6}$$

The normal strain ε_x and shear strain γ_{xz} associated with the displacements in Eq. (5) are

$$\begin{cases}\varepsilon_x=\frac{\partial u}{\partial x}=\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w_b}{\partial x^2}-f(z)\frac{\partial^2 w_s}{\partial x^2}\\ \gamma_{xz}=g(z)\frac{\partial w_s}{\partial x}\end{cases} \tag{7a}$$

Where

$$g(z)=1-\frac{df(z)}{dz} \tag{7b}$$

The stress-strain relations are given as

$$\begin{cases}\sigma_x=Q_{11}(z)\varepsilon_x\\ \tau_{xz}=Q_{55}(z)\gamma_{xz}\end{cases} \tag{8a}$$

Where σ_x and γ_{xz} are the normal and shear stress, respectively.

Q_{11} and Q_{55} are given by

$$\begin{cases}Q_{11}(z)=\frac{E_{11}(z)}{1-\nu^2}\\ Q_{55}(z)=G_{12}(z)\end{cases} \tag{8b}$$

To obtain the equations of motion, the Hamilton's principle is employed as follows

$$\int_{t_1}^{t_2}(\delta U-\delta K)=0 \tag{9}$$

Where δ represents the virtual variational symbol, U is the strain energy of the beam, and K is the kinetic energy.

The Variation of virtual strain energy of the beam is

$$\begin{aligned}\delta U&=\int_0^L\int_{-h/2}^{h/2}(\sigma_x\delta\varepsilon_x+\tau_{xz}\delta\gamma_{xz})dzdx\\ &=\int_0^L\Bigg(N\frac{d\delta u_0}{dx}-M_b\frac{d^2\delta u_b}{dx^2}\\ &\qquad -M_s\frac{d^2\delta u_s}{dx^2}+Q\frac{d\delta w_s}{dx}\Bigg)\end{aligned} \tag{10}$$

Where N, Mb, Ms and Q are the stress resultants defined as

$$(N,M_b,M_s)=\int_0^L(1,z,f)\sigma_x dz, Q=\int_{-h/2}^{h/2}g\tau_{xz}dz \tag{11}$$

By assuming that the superposed dot on a variable indicates time derivative, the variation of the virtuel kinetic energy for CNTRC beam can be written as

$$\begin{aligned}\delta K&=\int_0^L\int_{-h/2}^{h/2}\rho(z)[\dot u\delta\dot u+\dot w\delta\dot w]\,dzdx\\ &=\int_0^L\Bigg[I_0\big(\dot u_0\delta\dot u_0+(\dot w_b+\dot w_s)(\delta\dot w_b+\delta\dot w_s)\big)\\ &\quad -I_1\left(\dot u_0\frac{d\delta w_b}{dx}+\frac{dw_b}{dx}\delta\dot u_0\right)\Bigg)+I_2\left(\frac{dw_b}{dx}\frac{d\delta w_b}{dx}\right)\\ &\quad -J_1\left(\dot u_0\frac{d\delta\dot w_s}{dx}+\frac{d\dot w_s}{dx}\delta\dot u_0\right)\\ &\quad +J_2\left(\frac{d\dot w_b}{dx}\frac{d\delta\dot w_s}{dx}+\frac{d\dot w_s}{dx}\frac{d\delta\dot w_b}{dx}\right)\\ &\quad +K_1\left(\frac{d\dot w_s}{dx}\frac{d\delta\dot w_s}{dx}\right)\Bigg]dx\end{aligned} \tag{12}$$

In which $\rho(z)$ is the mass density, Ii (i = 0, 1, 2), Ji (i = 1, 2) and K_1 are the mass inertias defined as

$$\begin{aligned}(I_0,I_1,I_2)&=\int_{-h/2}^{h/2}(1,z,z^2)\,\rho(z)dz\\ (J_1,J_2)&=\int_{-h/2}^{h/2}(f,zf)\,\rho(z)dz\\ K_1&=\int_{-h/2}^{h/2}f^2\,\rho(z)dz\end{aligned} \tag{13}$$

Substituting the expressions of δU, δK from Eq. (10) and (12) in Eq. (9) and integrating by parts yields the following governing equations

$$\begin{cases} \delta u_0: \dfrac{dN}{dx} = I_0 \ddot{u}_0 - I_1 \dfrac{d\ddot{w}_b}{dx} - J_1 \dfrac{d\ddot{w}_s}{dx} \\ \delta w_b: \dfrac{d^2 M_b}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) \\ +I_1 \dfrac{d\ddot{u}_0}{dx} - I_2 \dfrac{d^2\ddot{w}_b}{dx^2} - J_2 \dfrac{d^2\ddot{w}_s}{dx^2} \\ \delta w_s: \dfrac{d^2 M_s}{dx^2} + \dfrac{dQ}{dx} = I_0(\ddot{w}_b + \ddot{w}_s) \\ +J_1 \dfrac{d\ddot{u}_0}{dx} - J_2 \dfrac{d^2\ddot{w}_b}{dx^2} - K_1 \dfrac{d^2\ddot{w}_s}{dx^2} \end{cases} \tag{14}$$

From above relations, all stress resultants can be written in the form of material stiffness components and displacements as follow

$$\begin{cases} N = A_{11} \dfrac{du_0}{dx} - B_{11} \dfrac{d^2 w_b}{dx^2} - B_{11}^s \dfrac{d^2 w_s}{dx^2} \\ M_b = B_{11} \dfrac{du_0}{dx} - D_{11} \dfrac{d^2 w_b}{dx^2} - D_{11}^s \dfrac{d^2 w_s}{dx^2} \\ M_s = B_{11}^s \dfrac{du_0}{dx} - D_{11}^s \dfrac{d^2 w_b}{dx^2} - H_{11}^s \dfrac{d^2 w_s}{dx^2} \\ Q = A_{55}^s \dfrac{dw_s}{dx} \end{cases} \tag{15}$$

Where the beam stiffness can be written by

$$\begin{cases} (A_{11}, B_{11}, D_{11}) = \displaystyle\int_{-h/2}^{h/2} (1, z, z^2) dz \\ (B_{11}^s, D_{11}^s, H_{11}^s) = \displaystyle\int_{-h/2}^{h/2} (f(z), z f(z), f^2(z)) dz \\ A_{55}^s = \displaystyle\int_{-h/2}^{h/2} Q_{55}[g(z)] dz \end{cases} \tag{16}$$

By substituting expression from Eq. (15) of stress resultants into Eq. (14), it is obtained

$$\begin{cases} A_{11} \dfrac{d^2 u_0}{dx^2} - B_{11} \dfrac{d^3 w_b}{dx^3} - B_{11}^s \dfrac{d^3 w_s}{dx^3} = \\ I_0 \ddot{u}_0 - I_1 \dfrac{d\ddot{w}_b}{dx} - J_1 \dfrac{d\ddot{w}_s}{dx} \\ B_{11} \dfrac{d^3 u_0}{dx^3} - D_{11} \dfrac{d^4 w_b}{dx^4} - D_{11}^s \dfrac{d^4 w_s}{dx^4} = \\ I_0(\ddot{w}_b + \ddot{w}_s) + I_1 \dfrac{d\ddot{u}_0}{dx} - I_2 \dfrac{d^2\ddot{w}_b}{dx^2} - J_2 \dfrac{d^2\ddot{w}_s}{dx^2} \\ B_{11}^s \dfrac{d^3 u_0}{dx^3} - D_{11}^s \dfrac{d^4 w_b}{dx^4} - D_{11}^s \dfrac{d^4 w_s}{dx^4} + \dfrac{d^2 w_s}{dx^2} = \\ I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \dfrac{d\ddot{u}_0}{dx} - J_2 \dfrac{d^2\ddot{w}_b}{dx^2} - K_1 \dfrac{d^2\ddot{w}_s}{dx^2} \end{cases} \tag{17}$$

4. Analytical solution

The analytical solutions of the for simply supported FG-CNTRC beams was dealt here. Assuming that the beam CNTRC is simply supported, the solution functions that completely satisfy the boundary conditions in the equations below are assumed as follows

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \, cos(\lambda x) e^{iwt} \\ W_{bm} \, sin(\lambda x) e^{iwt} \\ W_{sm} \, sin(\lambda x) e^{iwt} \end{Bmatrix} \tag{18}$$

Where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigen frequency associated with mth eigen mode, and $\lambda = m\pi/L$.

Substituting the expansions of u_0, w_b, w_s from Eq. (18) into the equations of motion Eq. (17), the analytical solutions can be determined from the following equations

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \right) \begin{bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19a}$$

Where

$$\begin{aligned} &S_{11} = A_{11}\lambda^2; && S_{12} = -B_{11}\lambda^3; \\ &S_{13} = -B_{11}^s\lambda^3; && S_{22} = D_{11}\lambda^4; \\ &S_{23} = D_{11}^s\lambda^4; && S_{33} = H_{11}^s\lambda^4 + A_{55}^s\lambda^2 \\ &m_{11} = I_0, && m_{12} = -I_1\lambda, \\ &m_{13} = -J_1\lambda, && m_{22} = I_0 + I_1\lambda^2, \\ &m_{23} = I_0 + J_1\lambda^2, && m_{33} = I_0 + K_1\lambda^2. \end{aligned} \tag{19b}$$

5. Numerical results

In the following investigation, free vibration of FG-CNTRC beams is presented. The FG-CNTRC beams made from carbon nanotube and Poly methyl methacrylate PMMA is considered and their material compositions are varied exponentially across the beam thickness. The effective material properties of CNTRC beams at ambient temperature used throughout this work are given in Tables 1 and 2 (Yas and Samadi 2012):

Table 1 The CNT efficiency parameters (η)

Case	η_1	$\eta_2 = \eta_3$
$V_{cnt}^* = 0.12$	1.2833	1.0566
$V_{cnt}^* = 0.17$	1.3414	1.7101
$V_{cnt}^* = 0.28$	1.3238	1.7380

Table 2 The material properties of PMMA

v^p	ρ^p	E^p
0.3	1190 Kg/m^3	2.5 GPa

Table 3 The material properties of CNT (armchair (10,10) SWCNTs)

Vcnt	ρ^{cnt}	E_{11}^{cnt}	E_{22}^{cnt}	G_{12}^{cnt}
0.19	1400 Kg/m^3	600 GPa	10 GPa	17.2 GPa

For vibration analysis, the following non-dimensionalizations are employed

$$\bar{w} = \omega L \sqrt{\frac{I_{00}}{A_{110}}} \tag{20}$$

Where A_{110} and I_{00} are A_{11} and I_0 of beam made of pure matrix material, respectively.

6. Validation of models

In these beams, values of the elastic foundation are given as zero. Table 4 shows the comparisons between present results and the results obtained by Tagrara *et al.* (2015), Wattanasakulpong and Ungbhakorn (2013), Yas and Samadi (2012). Here, the present results agree very well with other available solution.

In the literature, the researchers (Yas and Samadi 2012, Wattanasakulpong and Ungbhakorn 2013, Tagrara *et al.* 2015) studied the FG-CNTRC beam with PMMA as polymer and *SWCNT* (10, 10) as a nano-renfort. Three cases of CNT volume fraction V_{cnt}^* (0.12, 0.17, and 0.28) were presented with a maximum of CNT volume fraction in the linear distribution for the different patterns are $2V_{cnt}^*$ (0.24, 0.34, and 0.56), respectively. According for these previous maximum values, we choose the degree of the exponent (n) so that the maximum of CNT volume fraction $(n+1)V_{cnt}^*$ in the non-linear distribution does not exceed the value of 0.56.

Table 5 shows the effects of the exponent n (i.e., compared between linear distribution and parabolic distribution) in different CNT volume fraction. It can be seen that, the fundamental frequency value obtained in the *X-beam* configurations with parabolic distribution is greater than that obtained with linear distribution. On the other hand, in the *O-beam* and *V-beam* configuration the values of fundamental frequency are lowest. Also, increasing of the $Vcnt$ in *O-beam* configuration leads to decrease the fundamental frequency. We conclude that in the free vibration, the *O-beam* configuration with parabolic distribution and low of V_{cnt} is better than in the other configurations with linear distribution.

Table 4 Comparison of Dimensionless fundamental frequency for CNTRC beam ($L/h = 15,\ V_{\text{cnt}}^* = 0.12$)

Theory	UD-beam	O-beam	X-beam
Tagrara *et al*. (2015)	0.9749	0.7446	1.1163
Wattanasakulpong and Ungbhakorn (2013) FSDT	0.9976	0.7628	1.1485
Wattanasakulpong and Ungbhakorn (2013) TSDT	0.9749	0.7446	1.1163
Yas and Samadi (2012)	0.9753	0.7527	1.1150
Present	0.9745	0.7453	1.1152

Table 5 Dimensionless fundamental frequency for CNTRC beam in linear and parabolic distribution ($L/h = 15$)

	Linear distribution (n = 1)			Parabolic distribution (n = 2)		
V_{cnt}^*	X-beam	O-beam	V-beam	X-beam	O-beam	V-beam
0.12	1.1152	0.7453	0.8447	1.1812	0.6069	0.7375
0.17	1.3761	0.9088	1.0301	1.4584	0.7338	0.8892

Table 6 Dimensionless fundamental frequency for *O-beam* configuration in exponential distribution

V_{cnt}^*	L/h	n							
		0.5	1	1.5	2	2.3	2.5	3	3.5
0.12	10	1.1186	1.0083	0.9189	0.8452	0.8069	0.7835	0.7313	0.6867
	15	0.8440	0.7453	0.6684	0.6069	0.5757	0.5569	0.5154	0.4807
	20	0.6669	0.5830	0.5190	0.4686	0.4433	0.4281	0.3948	0.3672
0.17	10	1.3893	1.2463	1.1303	1.0346	0.9850	-	-	-
	15	1.0336	0.9088	0.8115	0.7338	0.6944	-	-	-
	20	0.8111	0.7065	0.6265	0.5636	0.5320	-	-	-

Table 7 Dimensionless fundamental frequency for *X-beam* configuration in exponential distribution

V_{cnt}^{*}	L/h	n							
		0.5	1	1.5	2	2.3	2.5	3	3.5
0.12	10	1.3395	1.3890	1.4232	1.4486	1.4611	1.4685	1.4847	1.4982
	15	1.0609	1.1152	1.1530	1.1812	1.1950	1.2031	1.2208	1.2353
	20	0.8607	0.9122	0.9485	0.9757	0.9890	0.9968	1.0138	1.0277
0.17	10	1.6734	1.7363	1.7791	1.8106	1.8260	-	-	-
	15	1.3076	1.3760	1.4234	1.4584	1.4754	-	-	-
	20	1.0527	1.1168	1.1618	1.1952	1.2115	-	-	-

Table 8 Dimensionless fundamental frequency for *V-beam* configuration type in exponential distribution

V_{cnt}^{*}	L/h	n							
		0.5	1	1.5	2	2.3	2.5	3	3.5
0.12	10	1.1915	1.1276	1.0691	1.0159	0.9864	0.9678	0.9245	0.8859
	15	0.9087	0.8447	0.7875	0.7375	0.7107	0.6941	0.6565	0.6238
	20	0.7220	0.6651	0.6151	0.5722	0.5496	0.5356	0.5044	0.4778
0.17	10	1.4804	1.3930	1.3122	1.2386	1.1979	-	-	-
	15	1.1138	1.0301	0.9550	0.8892	0.8539	-	-	-
	20	0.8788	0.8060	0.7418	0.6865	0.6572	-	-	-

The influence of the degree of exponent (n), CNT volume fraction; aspect ratio and configuration types O-beam, *X-beam* and *V-beam* on the dimensionless fundamental frequency as shows in Tables 6, 7 and 8, respectively. It is observed that in the different types of the beams when the aspect ratio increases, the dimensionless fundamental frequency decreases, but when the carbon nanotube volume fraction increases, the dimensionless fundamental frequency increases. On the other hand, we note that the increase in the degree of exponent (n) lead to decreases the dimensionless fundamental frequency in *O-beam* and *V-beam*, but increases it in *X-beam*. It can be concluded that the *X-beam* has the highest fundamental frequency and on the contrary, the *O-beam* has the lowest fundamental frequency.

Fig. 3 shows the variations of dimensionless fundamental frequency under the effect of degree of exponent in different value of CNT volume fraction. It is clear that the increase of the degree of exponent decrease the fundamental frequency. On the other hand, the fundamental frequency is highest in the case of $V_{cnt}^{*} = 0.17$, and lowest in the case of $V_{cnt}^{*} = 0.12$.

The effects of volume fraction exponent and type of distribution on the fundamental frequency are investigated in Fig. 4. As observed in this figure, increasing of the volume fraction exponent leads to decreasing the fundamental frequency in *O-beam* and *V-beam*, and increasing them in *X-beam*. This means the importance of the type of distribution and the degree of exponent to improve the vibration resistance.

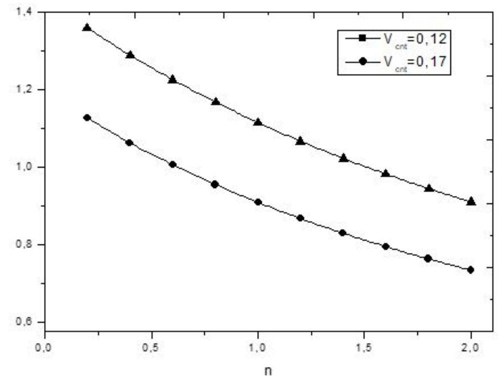

Fig. 3 Effect of degree of exponent (n) and the volume fraction on the dimensionless fundamental frequency (*O-beam*, *L/h* = 15)

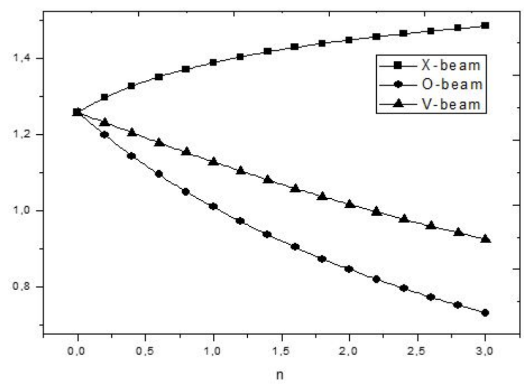

Fig. 4 Effect of degree of exponent n and CNTRC beam configuration type in fundamental frequency ($L/h = 10$, $V_{cnt}^{*} = 0.12$)

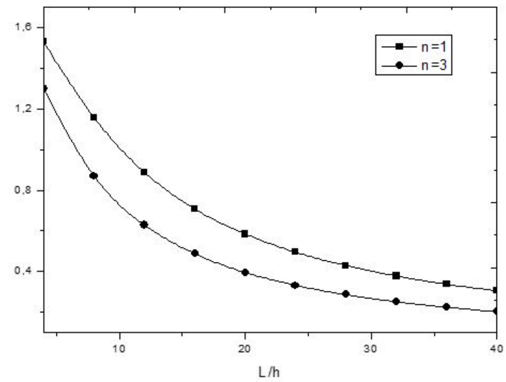

Fig. 5 Comparison between the effect of linear and nonlinear (CNT) distribution on the fundamental frequency (O-beam, $V_{cnt}^{*} = 0.12$)

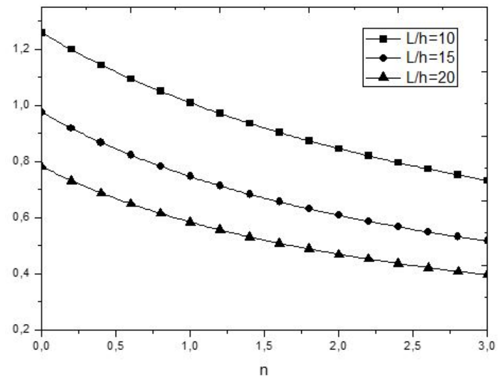

Fig. 6 Effect of degree of exponent *n* and aspect ratio on fundamental frequency (O-beam, $V_{cnt}^{*} = 0.12$)

In Fig. 5, the variations of dimensionless fundamental frequency in *O-beam* configuration type due the aspect ratio in two cases of distributions, linear distribution (n = 1) and non-linear distribution (n = 3). It is seen from the figure that, increasing the aspect ratio decreases the fundamental frequency. It is also seen from the figure that, increasing value of degree of exponent decreases the fundamental frequency. The figure shows also, the Great influence of the non-linearity of CNT volume fraction distribution on the FG-CNTRC beam performances. It can be observed, in the case of non-linear distribution (n = 3) the dimensionless fundamental frequency is less than the dimensionless fundamental frequency that observed in the case of linear distribution.

Fig. 6 shows that increasing of the value aspect ratio *L/h* and increasing of the volume fraction exponent decreases the natural frequency. Finally, the exponential laws of carbon nanotube volume fraction play a major role to develop the parameters of the FG-CNTRC beam.

7. Conclusions

The aim of this paper is to study the influence of the non-linear distribution of carbon nanotube volume fraction in the vibration behavior. By using higher-order shear deformation theories, the effects of different parameters were investigated on the fundamental frequency of the FG-CNTRC beam. The beam was simply supported. The carbon nanotubes were considered aligned along the beam axial direction. The equation of motion was derived by Hamilton's principle and Navier solution method solved by this equation. A further focus of the work is to explore the sensitivity of the free vibration response of the reinforced beam to the different parameters as the aspect ratio, configuration type, volume fraction, and the effect of the degree of the exponent (n). The following results were obtained from this work:

- The Great importance of the non-linearity on the Strength and stiffness of the FG-CNTRC beam
- The increase in the degree of exponent (n) leads to decreases the dimensionless fundamental frequency in *O-beam* and *V-beam*, but increases it in *X-beam*
- Increasing of the aspect ratio also decrease the fundamental frequency.
- Increasing of the carbon volume fraction increase the fundamental frequency.
- *O-beam* Configuration is the best configuration to improve the fundamental frequency.

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