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Buckling analysis of double walled carbon nanotubes embedded in Kerr elastic medium under axial compression using the nonlocal Donnell shell theory

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Abstract. In this paper, a new explicit analytical formula is derived for the critical buckling load of Double Walled Carbon Nanotubes (DWCNTs) embedded in Winkler elastic medium without taking into account the effects of the nonlocal parameter, which indicates the effects of the surrounding elastic matrix combined with the intertube Van der Waals (VdW) forces. Furthermore, we present a model which predicts that the critical axial buckling load embedded in Winkler, Pasternak or Kerr elastic medium under axial compression using the nonlocal Donnell shell theory, this model takes into account the effects of internal small length scale and the VdW interactions between the inner and outer nanotubes. The present model predicts that the critical axial buckling load of embedded DWCNTs is greater than that without medium under identical conditions and parameters. We can conclude that the embedded DWCNTs are less susceptible to axial buckling than those without medium.

Keywords: buckling; Double Walled Carbon Nanotubes (DWCNTs); elastic medium; Van der Waals (VdW) interaction; axial compression; small scale effect; nonlocal elasticity theory; Donnell shell theory

1. Introduction

Carbon nanotubes (CNTs) have been discovered by Iijima (1991), the studies showed that CNTs possess exceptionally high elastic modulus which is higher than 1 TPa (Treacy *et al*. 1996, Bernholc *et al*. 1998, Yao and Lordi 1998, Lourie and Wagner 1998), and sustain large elastic strain up to 5% and breaking strain up to 20% (Yacobson *et al*. 1996, Wagner *et al*. 1997, Bernholc *et al*. 1998, Nardelli *et al*. 1998, Zhang *et al*. 1998). For studying the mechanical properties of CNTs, an energy-equivalent model, which combines between molecular and solid mechanics, can be used (Wu *et al*. 2006, Eltaher *et al*. 2019, Mohamed *et al*. 2020). Hence, CNTs give the advantage of exceptionally high stiffness that we can combine with excellent resilience, which gives one of the most important applications of CNTs. The study of buckling and bending of SWCNTs or MWCNTs is the subject of numerous experimental, molecular dynamics and continuum mechanics simulations (Iijima *et al*. 1996, Falvo *et al*. 1997, Rahmani and Antonov 2019, Malikan *et al*. 2019a, 2020, Mohamed *et al*. 2020, Xie *et al*. 2020, Asghar *et al*. 2020). Some research works show that all buckling studies based on the molecular dynamics simulations can be predicted by the continuum shell model (Yacobson *et al*. 1996). The laws of continuum mechanics are robust and allow to treat the discrete objects only with a few atoms in diameter (Yakobson and Smalley 1997). So, continuum models play an important role in the study of micro/nano-structures due to difficulties in nanoscale experiments and atomistic modeling which is very expensive for large-sized atomic system. Now we present some scientific research for the study of micro/nano-structures, Taj *et al*. (2020) examined the vibrational analysis in microtubules based on the nonlocal orthotropic elastic shell model, Balubaid *et al*. (2019) studied the free vibrational behavior of the simply supported functionally graded nano-plate using the nonlocal two variables integral refined plate theory, Alimirzaei *et al*. (2019) presented a nonlinear analysis of viscoelastic micro-composite beam reinforced by various distributions of boron nitrid nanotube with initial geometrical imperfection by modified strain gradient theory, Berghouti *et al*. (2019) studied the dynamic behavior of functionally graded porous nano-beams, Berghouti *et al*. (2019) developed a quasi-3D beam theory for free vibration analysis of functionally graded microbeams, Bedia *et al*. (2019) developed a novel two variable shear deformation beam theories which are applied to investigate the bending and buckling behaviors of nanobeams, Karami *et al*. (2019) studied elastic bulk wave characteristics of doubly curved nanoshell made of functionally graded anisotropic material, the same authors used a higher order shear deformation refined plate theory to model an anisotropic nanoplate, Hussain *et al*. (2020) developed a new method based on the Sander theory to predict the vibrational behavior of SWCNTs, Medani *et al*. (2019) examined the static and dynamic behavior of functionally graded CNT reinforced porous sandwich polymer plate and Draoui *et al*. (2019) studied the static and

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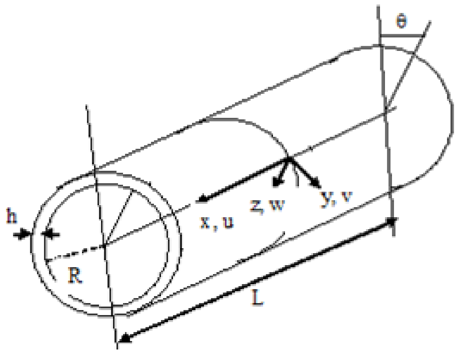
dynamic behavior of CNT reinforced composite sandwich plates.

Until now, researchers and scientists continue to study the mechanical behavior of SWCNTs or MWCNTs embedded in a polymer or a metal matrix (Kuzumaki *et al.* 1998, Lourie *et al.* 1998, Lourie and Wagner 1998, Wagner *et al.* 1998, Salvetat *et al.* 1999, Bower *et al.* 1999, Calvert 1999, Qian *et al.* 2000, Hedayati *et al.* 2012, Schadler *et al.* 2018, Wu *et al.* 2018, Yazid *et al.* 2018, Gul and Aydogdu 2018, Shahsavari *et al.* 2018, Bensattalah *et al.* 2018, Karami *et al.* 2018, Zhou *et al.* 2019, Malikan *et al.* 2020, Jena *et al.* 2019). The stresses are effectively transferred between the embedded nanotubes and the surrounding elastic medium where many studies show that the bond strength is reasonably strong between them. In the continuum modeling and due to multi-walled structure of CNTs, we can distinguish them by classical elastic shells or tubes and the associated inter-tube VdW forces. Hence, the objective of several researchers is the development of continuum models for MWCNTs embedded in an elastic medium in the presence of the inter-tube VdW forces. Especially, the study of the effects of the surrounding elastic medium and the VdW forces on axially compressed buckling behavior are of great interest to understand and develop the mechanical properties of CNTs. Now we discuss some applications of DWCNTs, Mohammadimehr *et al.* (2019) developed Timoshenko beam model, based on the nonlocal elasticity theory, to study the elastic buckling of DWCNTs embedded in Winkler elastic medium taking into account the VdW forces between the inner and outer nanotubes, Ru *et al.* (2001) derived an explicit formula for the critical axial strain of a DWCNT embedded in Winkler elastic matrix using a simplified model for the VdW interaction, Zhanga *et al.* (2006) used the energy method to analyze the elastic buckling of a long DWCNT, where the effect of elastic medium and VdW forces on the buckling are considered, Wang *et al.* (2006) proposed a solid shell element model, for the elastic bifurcation buckling analysis of DWCNTs, which the effect of transverse shear deformation becomes significant in a stocky DWCNT with small aspect ratio (radius-to-thickness), Ranjbartoreh *et al.* (2007) investigated the buckling behavior of DWCNTs with surrounding elastic medium via a circular cylindrical shell model, Yao *et al.* (2007) derived an explicit formula for the critical axial stress of DWCNTs taking into account the effects of temperature change, surrounding elastic medium and VdW forces between the inner and outer nanotubes, Lu *et al.* (2007a) studied the buckling phenomena and mechanical behavior of SWCNTs and DWCNTs using molecular dynamics simulations, Lu *et al.* (2007b) studied buckling of DWCNTs on the basis of the finite deformation shell theory which is established from the interatomic potential and the continuum model for VdW interactions, Elishakoff and Bucas (2007) used the Bubnov-Galerkin method to treat the buckling of clamped-free DWCNTs subjected to a concentrated compressive load at the free end, Chemi *et al.* (2018) implemented the nonlocal Timoshenko beam theory to determine the nonlocal critical buckling loads of chiral DWCNTs embedded in an elastic medium, they discussed the effect of the buckling mode number, the elastic medium, aspect ratio, and chirality on the nonlocal critical buckling loads.

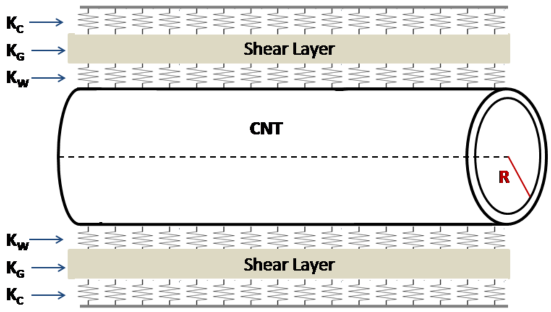
Motivated by the considerations and works cited above on DWCNTs, an elastic double shell model for buckling of DWCNTs embedded in an elastic medium under axial compression is presented. The first point studied concerns the development of a new explicit analytical formula which is derived for the critical buckling load of embedded DWCNTs in Winkler elastic in the presence of inter-tube VdW forces, but without taking into account the effects of the nonlocal parameter. The second point studied is the analysis of the critical buckling load using Winkler, Pasternak and Kerr models for the surrounding elastic medium, where we take into account the effects of internal small length scale and the VdW interactions between the inner and outer nanotubes. Compared to the existing literature, for buckling of DWCNTs embedded in an elastic medium under axial compression taking into account the effects of internal small length scale and the VdW interactions, the use of Kerr foundation may be one of the novelties of the present paper. The present model predicts that the critical axial buckling load of embedded DWCNTs is greater than that without medium under identical conditions and parameters.

2. Model of CNT embedded in an elastic medium based on the nonlocal Donnell shell theory

The relations of classical Donnell shell theory are used to describe mechanical behavior of CNTs. To obtain the continuum model of CNT structure, it is necessary to introduce the equivalent Young modulus E and the equivalent shell thickness h. In general, there are different approaches to determine the above-mentioned values. Among the approaches in the literature is to obtain the effective Young modulus and the wall thickness of the CNT by molecular dynamics simulation and the continuum mechanics shell model (Bao *et al.* 2004). On the basis of the simplifying shallow-shell hypothesis, Donnell (1934) developed the theory of circular cylindrical shells. This theory has been widely used because it is relatively simple and practically accurate. The most used form of Donnell shell theory introduces a stress function, the objective is to combine the three equations of equilibrium involving the shell displacements in the radial, circumferential and axial directions to obtain two equations involving only the radial displacement w and the stress function ϕ. This theory is accurate only for a circumferential half wavenumber n greater than 4 which allows us to satisfy the assumption $1/n^2 << 1$. Moreover, Donnell shell equations are obtained by neglecting the in-plane inertia, transverse shear deformation and rotary inertia where the thin shell assumption $\left(\frac{thickness}{radius}\right)^2 \ll 1$ must be satisfied to obtain accurate results. In addition, the other assumption of the large aspect ratio $\frac{length}{radius} \geq 10$ has been used to neglect the effect of the shear deformation which is more significant in the low aspect ratio $\frac{length}{radius}$. Using Donnell shell theory (Donnell 1934, Timesli 2020), the median surface of the



(a) Thin-walled circular cylindrical shell



(b) CNT embedded in Kerr foundation

Fig. 1 Schematic for simply-supported CNT embedded in Kerr elastic foundation

shell is used to calculate the induced stresses and the effects of transverse shear and rotary inertia have been neglected using the thin shell assumption in the derivation.

Consider a thin-walled circular cylindrical shell of a middle surface of radius R, wall thickness h which is much lower than R, and length L (see Fig. 1(a)). This shell is made of an elastic homogeneous isotopic material of Young modulus E and Poisson ratio v, which represents a CNT. The CNT is embedded in an elastic foundation as shown in Fig. 1(b), which allows us to study the effect of various elastic foundation parameters on the axial buckling load λ of CNT. These parameters are the lower spring modulus K_W called also Winkler modulus, the shear layer modulus K_G and the upper spring modulus K_C.

Using Donnell's model of the thin-walled circular cylindrical shell and the foundation model (He *et al.* 2005), the nonlinear equilibrium equation can be written in the following form

$$\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{2}{R}\frac{\partial^2 M_{x\theta}}{\partial x\partial\theta}+\frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial\theta^2}+\frac{N_{\theta\theta}}{R}+N_{xx}\frac{\partial^2 w}{\partial x^2} +2\frac{N_{x\theta}}{R}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{N_{\theta\theta}}{R^2}\frac{\partial^2 w}{\partial\theta^2}+p-f=0 \tag{1}$$

where N_{xx} and $N_{\theta\theta}$ are the normal forces, $N_{x\theta}$ is the internal shear force, M_{xx} and $M_{\theta\theta}$ are the bending moments and $M_{x\theta}$ is the twisting moment, w is the transverse displacement of the reference surface, p is an external axial pressure and f is a pressure which is related to elastic medium model. The nonlocal elasticity theory introduced by Eringen (Eringen 1972, Eringen and Edelen 1972) is still used by the researchers in recent years (Kiani 1998, Hussain *et al.* 2005, Semmah *et al.* 2019, Boutaleb *et al.* 1999, Timesli 2020). This theory is used to take small scale effects in the buckling analysis of the Donnell cylindrical shell. So, the nonlocal Hooke's law for the stress and strain relation can be expressed by

$$\begin{cases}\sigma_{xx}-(e_0a)^2\nabla^2\sigma_{xx}=\dfrac{E}{1-\nu^2}\varepsilon_{xx}+\dfrac{\nu E}{1-\nu^2}\varepsilon_{\theta\theta}\\ \sigma_{\theta\theta}-(e_0a)^2\nabla^2\sigma_{\theta\theta}=\dfrac{\nu E}{1-\nu^2}\varepsilon_{xx}+\dfrac{E}{1-\nu^2}\varepsilon_{\theta\theta}\\ \sigma_{x\theta}-(e_0a)^2\nabla^2\sigma_{x\theta}=\dfrac{(1-\nu)}{2}\dfrac{E}{1-\nu^2}\gamma_{x\theta}\end{cases} \tag{2}$$

where e_0 is the characteristic length called the nonlocal parameter of the nonlocal theory and the nonlocal constitutive relations of Donnell shell become

$$\begin{cases}(1-(e_0a)^2\nabla^2)N_{xx}=C(\varepsilon_{xx}+\nu\varepsilon_{\theta\theta})\\ (1-(e_0a)^2\nabla^2)N_{\theta\theta}=C(\varepsilon_{\theta\theta}+\nu\varepsilon_{xx})\\ (1-(e_0a)^2\nabla^2)N_{x\theta}=\dfrac{C}{2}(1-\nu)\gamma_{x\theta}\end{cases} \tag{3}$$

$$\begin{cases}(1-(e_0a)^2\nabla^2)M_{xx}=-D\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\nu}{R^2}\dfrac{\partial^2 w}{\partial\theta^2}\right)\\ (1-(e_0a)^2\nabla^2)M_{\theta\theta}=-D\left(\nu\dfrac{\partial^2 w}{\partial x^2}+\dfrac{1}{R^2}\dfrac{\partial^2 w}{\partial\theta^2}\right)\\ (1-(e_0a)^2\nabla^2)M_{x\theta}=-\dfrac{D}{R}(1-\nu)\dfrac{\partial^2 w}{\partial x\partial\theta}\end{cases} \tag{4}$$

where $C=\frac{Eh}{1-\nu^2}$ and $D=\frac{Eh^3}{12(1-\nu^2)}$ are respectively extensional and bending stiffness of the shell. To introduce the nonlocal effect in the nonlinear equilibrium equations, we multiply the operator $(1-(e_0a)^2\nabla^2)$ by Eq. (1) and we obtain

$$\frac{\partial^2\left((1-(e_0a)^2\nabla^2)M_{xx}\right)}{\partial x^2}+\frac{2}{R}\frac{\partial^2\left((1-(e_0a)^2\nabla^2)M_{x\theta}\right)}{\partial x\partial\theta} +\frac{1}{R^2}\frac{\partial^2\left((1-(e_0a)^2\nabla^2)M_{\theta\theta}\right)}{\partial\theta^2}+(1-(e_0a)^2\nabla^2) \left(\frac{N_{\theta\theta}}{R}+N_{xx}\frac{\partial^2 w}{\partial x^2}+2\frac{N_{x\theta}}{R}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{N_{\theta\theta}}{R^2}\frac{\partial^2 w}{\partial\theta^2}+p-f\right) =0 \tag{5}$$

which leads to

$$k^2\Delta^2 w-(1-(e_0a)^2\nabla^2)\left(\rho\frac{N_{\theta\theta}}{Eh}+\frac{1}{Eh}\left(N_{xx}\frac{\partial^2 w}{\partial x^2} +2\frac{N_{x\theta}}{R}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{N_{\theta\theta}}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{p-f}{Eh}\right)=0 \tag{6}$$

where $k^2=D/Eh$, $\rho=1/R$ is the curvature.

According to the shell theory, the membrane forces are connected to the stress function ϕ and we can write them as follows $N_{xx}=\frac{Eh}{R^2}\frac{\partial^2\phi}{\partial\theta^2}$, $N_{\theta\theta}=Eh\frac{\partial^2\phi}{\partial x^2}$ and $N_{x\theta}=\frac{Eh}{R}\frac{\partial^2\phi}{\partial x\partial\theta}$. We use the adjacent equilibrium criterion (Brush and Almroth 1975) to study the possible existence of adjacent equilibrium configurations. We consider that the indices 0 and b indicate respectively pre-buckling and post-buckling quantities and we neglect the terms of second order in index b to obtain the following equation.

$$k^2\Delta^2 w_b - (1-(e_0a)^2\nabla^2)\left(\rho\frac{\partial^2\phi}{\partial x^2} + \frac{1}{E\hbar}\left(N_{xx0}\frac{\partial^2 w_b}{\partial x^2} + 2\frac{N_{x\theta0}}{R}\frac{\partial^2 w_b}{\partial x\partial\theta} + \frac{N_{\theta\theta0}}{R^2}\frac{\partial^2 w_b}{\partial\theta^2}\right) + \frac{p_b - f_b}{E\hbar}\right) = 0 \tag{7}$$

The stress function $\phi(x,\theta)$ verifies the following compatibility condition.

$$\Delta^2\phi + \rho\frac{\partial^2 w_b}{\partial x^2} - (e_0a)^2\left(\frac{\partial^6\phi}{\partial x^6} + 3\rho^2\frac{\partial^6\phi}{\partial x^4\partial\theta^2} + 3\rho^4\frac{\partial^6\phi}{\partial x^2\partial\theta^4} + \rho^6\frac{\partial^6\phi}{\partial\theta^6}\right) = 0 \tag{8}$$

If the shear membrane forces are neglected $N_{x\theta0} = 0$, the axial compression is $N_{xx0} = P$ and the circumferential membrane force is $N_{\theta\theta0} = F$, the systems (7)-(8) become

$$\begin{aligned} &k^2\Delta^2 w_b - (1-(e_0a)^2\nabla^2)\left(\rho\frac{\partial^2\phi}{\partial x^2} + \left(\lambda\frac{\partial^2 w_b}{\partial x^2} + \frac{F}{E\hbar R^2}\frac{\partial^2 w_b}{\partial\theta^2}\right) + \frac{p_b - f_b}{E\hbar}\right) = 0 \\ &\Delta^2\phi + \rho\frac{\partial^2 w_b}{\partial x^2} - (e_0a)^2\left(\frac{\partial^6\phi}{\partial x^6} + 3\rho^2\frac{\partial^6\phi}{\partial x^4\partial\theta^2} + 3\rho^4\frac{\partial^6\phi}{\partial x^2\partial\theta^4} + \rho^6\frac{\partial^6\phi}{\partial\theta^6}\right) = 0 \end{aligned} \tag{9}$$

where $\lambda = \frac{P}{Eh}$ is the load parameter.

3. Elastic medium models

3.1 Single-parameter Winkler-type model

Using Winkler model (Winkler *et al.* 1867), we assume that the reaction force of CNT is proportional to the foundation deflection at each point in the foundation. This hypothesis allows to model the foundation by a juxtaposition of elastic springs. The proportionality constant K_W of these springs is known as the modulus of the reaction force of CNT.

$$f = K_W w \tag{10}$$

Winkler's model (10) is simple and combines well with numerical and analytical methods. Despite the simplicity and efficiency of Winkler foundation model, it has two major drawbacks. Firstly, it does not take into account the interaction between the springs, which amounts to neglecting the shear effect in the surrounding elastic medium. Consequently, a displacement discontinuity is created between the loaded area and the unloaded area under the foundation. Secondly, it does not take into account the plasticity that can occur in the elastic medium. Recently, Shariati *et al.* (2020) presented the stability analysis of cantilevered curved microtubules in axons regarding various size elements, the Winkler elastic medium is used to take into account the impacts of covering MAP Tau proteins along with cytoplasm, Chaabane *et al.* (2019) presented a study of the static and dynamic behaviors of functionally graded beams using a hyperbolic shear deformation theory which they discussed the effect of Winkler spring constant on the center deflection, fundamental frequency, shear and normal stress of functionally graded beam. This model is also used in different works to study the CNTs embedded in Winkler elastic medium (Rahmanian *et al.* 2016, Teifouet *et al.* 2018, Jena *et al.* 2019, Mohamed *et al.* 2020).

3.2 Double-parameter Pasternak-type model

Pasternak (1954) also proposed a model based on the Winkler model. This model assumes that there is a shear interaction between the springs. The idea is to connect the springs to a horizontal layer which only deforms in the direction of transverse shear (see Fig. 1(b)). The relationship between contact pressure and DWCNT can be expressed according to the shear layer modulus K_G as follows.

$$f = K_W w - K_G\nabla^2 w \tag{11}$$

Several research works are based on this model. Recently, Chikr *et al.* (2020) used a new theory of refined trigonometric sear deformation to study the buckling analysis of material sandwich plates based on a two-parameter elastic foundation under various boundary conditions, Refrafi *et al.* (2020) used a novel shear deformation theory to study the mechanical buckling and hygrothermal responses of simply supported functionally graded sandwich plate seated on elastic foundation, Bousahla *et al.* (2020) studied the buckling and vibrational behavior of the composite beam armed with SWCNTs resting on Winkler-Pasternak elastic foundation, Bellal *et al.* (2020) used nonlocal four-unknown integral model to analyze the buckling behavior of a single-layered graphene sheet embedded in visco-Pasternak's medium, Tounsi *et al.* (2020) proposed a simple four-variable trigonometric integral shear deformation model to study the static behavior of advanced functionally graded ceramic-metal plates, these latter are supported by an elastic foundation and subjected to a nonlinear hygro-thermo-mechanical load, Boukhlif *et al.* (2019) presented a dynamic investigation of functionally graded plates resting on elastic foundation on the basis of a simple quasi-3D higher shear deformation theory, Boulefrakh *et al.* (2019) employed a simple quasi mechanical behavior of functionally graded plates resting on visco-Pasternak foundations, Malikan *et al.* (2018) analyzed the damped forced vibration of SWCNTs resting on viscoelastic foundation in thermal environment using nonlocal strain gradient theory, Malikan *et al.* (2019b) analysed the transient response of oscillated CNTs with an internal and external damping using a viscoelastic nanobeam resting on a visco-Pasternak foundation. There are many other recent references in the literature which use

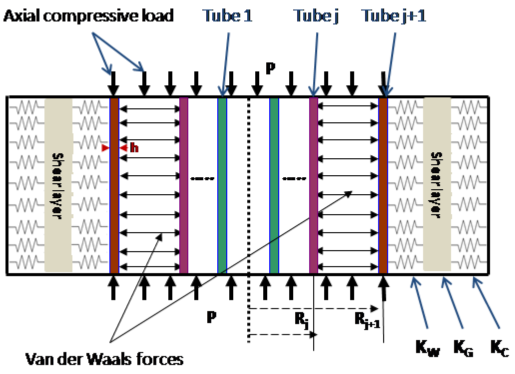


Fig. 2 MWCNTs under axial compression embedded in Kerr elastic medium

the Pasternak model such as Mahmoudi *et al.* (2019), Karami *et al.* (2019), Zaouia *et al.* (2019), Jena *et al.* (2020b).

3.3 Three-parameter Kerr-type model

Kerr (1965) had added the third parameter for the enrichment of the Pasternak model. This model provides more flexibility for the continuity of the foundation between loaded and unloaded area of the shell elastic, where the surrounding elastic medium is envisaged as consisted of lower and upper spring beds sandwiching a shear layer (see Fig. 1(b)) and the foundation model is given by

$$f=\frac{1}{1+\frac{K_W}{K_C}}\left(K_W w-K_G\nabla^2 w-\frac{DK_G}{K_C}\nabla^6 w\right) \tag{12}$$

where K_W and K_C are respectively the lower and upper spring modules, K_G is the intermediate shear layer modulus. The studies of the CNTs resting or incorporated into the Kerr foundation are rare in the literature. As examples for the use of this type of foundation, Kaddari *et al.* (2020) proposed a new type of quasi-3D hyperbolic shear deformation theory to discuss the statics and free vibration of functionally graded porous plates resting on Kerr elastic foundation, Jena *et al.* (2020a) used Navier's technique and shifted Chebyshev polynomial-based Rayleigh-Ritz method for vibration analysis of a functionally graded porous beam embedded in Kerr foundation. Other researchers have discussed the effect of all models as Addou *et al.* (2019), in this work a simple quasi-3D hyperbolic theory is used to investigate the effect of porosity on dynamic behavior of functionally graded plates and all types of elastic foundations cited above.

4. Multiple Donnell shells continuum approach

Consider N tubes of radius R_1, R_2, ... and R_N. Other geometrical and mechanical properties (length L, thickness h, Young's modulus E and Poisson's ratio ν) are the same for all tubes. These tubes represent the MWCNTs which subjected to an axial compression P. The walls of adjacent tubes interact through VdW forces and MWCNTs is embedded in Kerr elastic medium as shown in Fig. 2.

Using the system (9), the transverse displacement $w_j(x,\theta)$ and the corresponding stress functions $\phi_j(x,\theta)$ are solutions of the following nonlocal Donnell shell equilibrium equations.

$$\begin{cases} k^2\Delta_j^2 w_j-\left(1-\mu^2\nabla_j^2\right)\left(\lambda\frac{\partial^2 w_j}{\partial x^2}+\rho_j\frac{\partial^2\phi_j}{\partial x^2}+\frac{1}{Eh}\right.\\ \left(\frac{F_j}{R_j^2}\frac{\partial^2 w_j}{\partial\theta^2}+\sum_{k=1}^{N}\left(w_j-w_k\right)c_{jk}\right)-\frac{\delta_{jN}}{Eh}\left(1+\frac{K_W}{K_C}\right)^{-1}\\ \left.\left(K_W w_j-K_G\nabla^2 w_j-\frac{DK_G}{K_C}\nabla^6 w_j\right)\right)=0\\ \Delta_j^2\phi_j+\rho_j\frac{\partial^2 w_j}{\partial x^2}-\mu^2\left(\frac{\partial^6\phi}{\partial x^6}+3\rho^2\frac{\partial^6\phi}{\partial x^4\partial\theta^2}\right.\\ \left.+3\rho^4\frac{\partial^6\phi}{\partial x^2\partial\theta^4}+\rho^6\frac{\partial^6\phi}{\partial\theta^6}\right)=0 \end{cases} \tag{13}$$

where $\mu=e_0a$, $j=1,2,\ldots,N$, $\rho_j=1/R_j$ is the curvature radius of j^{th} tube, $\Delta_j^2=\left(\frac{\partial^2}{\partial x^2}+\rho_j^2\frac{\partial^2}{\partial\theta^2}\right)^2$ is the bi-laplacian operator, ϕ_j is the stress function of j^{th} tube, F_j are the forces by the length unit, prior buckling, in the circumferential direction of tube j, c_{jk} are VdW coefficients.

5. Buckling analysis of DWCNTs embedded in an elastic medium

5.1 Buckling load λ

The solution of the problem (13) is sought in the following form.

$$\begin{cases} w_j(x,\theta)=A_j\,exp\left(i\frac{m\pi}{L}x\right)cos(n\theta)+cc\\ j=1,2,3,\ldots,N\\ \phi_j(x,\theta)=a_j\,exp\left(i\frac{m\pi}{L}x\right)cos(n\theta)+cc \end{cases} \tag{14}$$

where A_j and a_j are arbitrary complex constants, n and m are respectively the circumferential and axial half wavenumbers of the j^{th} tube and cc represents the complex conjugate. In this paper we are interested by DWCNT consisting of two tubes ($N=2$). After the substitution of the solution (14) in the problem (13) with $N=2$, the first equation of the system (13) gives

$$\begin{cases} k^2(p^2+q_1^2)^2A_1-\lambda[p^2+\mu^2p^2(p^2+q_1^2)]A_1\\ +\frac{c_{12}}{E\hbar}\left[\left(1+\mu^2(p^2+q_1^2)\right)A_2-(1+\mu^2\right.\\ \left.(p^2+q_1^2))A_1\right]-\frac{F_1}{E\hbar}[q_1^2-\mu^2q_1^2(p^2+q_1^2)]A_1\\ +\rho_1[p^2+\mu^2p^2(p^2+q_1^2)]a_1=0\\ k^2(p^2+q_2^2)^2A_2-\lambda[p^2+\mu^2p^2(p^2+q_2^2)]A_2 \end{cases} \tag{15}$$

$$\begin{cases} +\frac{c_{21}}{E\hbar}[(1+\mu^2(p^2+q_2^2))A_2-(1+\mu^2(p^2+q_2^2)) \\ A_1]-\frac{F_2}{E\hbar}[q_2^2-\mu^2 q_2^2(p^2+q_2^2)]A_2+\ \rho_2[p^2+\mu^2p^2 \\ (p^2+q_2^2)]a_2-\frac{1}{E\hbar}\left(1+\frac{K_W}{K_C}\right)^{-1}[K_W+(K_G \\ +\mu^2K_W)(p^2+q_2^2)+\mu^2K_G(p^4+2p^2q_2^2+q_2^4) \\ +\mathrm{D}\frac{K_G}{K_C}(p^6+3p^4q_2^2+3p^2q_2^4+q_2^6)+\mu^2D\frac{K_G}{K_C} \\ (p^8+4p^6q_2^2+6p^4q_2^4+4p^2q_2^6+q_2^8)]A_2=0 \end{cases}$$

where $p=m/L$, $q_1=n/R_1$ and $q_2=n/R_2$. The second equation of the system (13) gives

$$\begin{cases} (p^2+q_1^2)^2a_1+\mu^2(p^6+3p^4q_1^2+3p^2q_1^4+q_1^6)a_1 \\ -\rho_1p^2A_1=0 \\ (p^2+q_2^2)^2a_2+\mu^2(p^6+3p^4q_2^2+3p^2q_2^4+q_2^6)a_2 \\ -\rho_2p^2A_2=0 \end{cases} \quad (16)$$

which leads to determine the constants a_1 and a_2.

$$\begin{cases} a_1=\frac{\rho_1p^2A_1}{(p^2+q_1^2)^2+\mu^2(p^6+3p^4q_1^2+3p^2q_1^4+q_1^6)} \\ a_2=\frac{\rho_2p^2A_2}{(p^2+q_2^2)^2+\mu^2(p^6+3p^4q_2^2+3p^2q_2^4+q_2^6)} \end{cases} \quad (17)$$

Using the values of the constants a_1 and a_2 in Eq. (17), we can rewrite the equations of the system (15) in the following forms.

$$\begin{cases} [\alpha_1-\lambda p^2(1+\mu^2(1+\beta_1^2)p^2)]A_1+\frac{c_{12}}{E\hbar}[1+ \\ \mu^2(1+\beta_1^2)p^2]A_2=0 \\ \frac{c_{21}}{E\hbar}[1+\mu^2(1+\beta_2^2)p^2]A_1+[\alpha_2-\lambda p^2(1+ \\ \mu^2(1+\beta_2^2)p^2)A_2=0 \end{cases} \quad (18)$$

where $\beta_1=\frac{q_1}{p}$ and $\beta_2=\frac{q_2}{p}$are the aspect ratios with $\beta_2=\left(\frac{R_1}{R_2}\right)\beta_1$, α_1 and α_2 are expressed by

$$\begin{cases} \alpha_1=k^2(1+\beta_1^2)^2p^4 \\ +\frac{\rho_1^2(1+\mu^2(1+\beta_1^2)p^2)}{(1+\beta_1^2)^2+\mu^2(1+3\beta_1^2+3\beta_1^4+\beta_1^6)p^2} \\ -\frac{c_{12}}{Eh}(1+\mu^2(1+\beta_1^2)p^2)-\frac{F_1}{Eh}\beta_1^2(1-\mu^2(1+\beta_1^2)p^4) \\ \alpha_2=k^2\left(1+\beta_2^2\right)^2p^4 \\ +\frac{\rho_2^2\left(1+\mu^2\left(1+\beta_2^2\right)p^2\right)}{\left(1+\beta_2^2\right)^2+\mu^2\left(1+3\beta_2^2+3\beta_2^4+\beta_2^6\right)p^2} \\ -\frac{c_{21}}{Eh}\left(1+\mu^2\left(1+\beta_2^2\right)p^2\right)-\frac{F_2}{Eh}\beta_2^2\left(1-\mu^2\right. \\ (1+\beta_2^2)p^4)+\frac{1}{Eh}\left(1+\frac{K_W}{K_C}\right)^{-1}[K_W+ \\ (K_G+\mu^2K_W)(1+\beta_2^2)p^2+\mu^2K_G(1+2\beta_2^2 \\ +\beta_2^4)p^4+\mathrm{D}\frac{K_G}{K_C}(1+3\beta_2^2+3\beta_2^4+\beta_2^6)p^6 \\ +\mu^2D\frac{K_G}{K_C}(1+4\beta_2^2+6\beta_2^4+4\beta_2^6+\beta_2^8)p^8] \end{cases} \quad (19)$$

The system (18) has non-trivial solution A_1 and A_2 if its determinant is zero. This requirement is the buckling condition which leads to

$$A\lambda^2+B\lambda+C=0 \quad (20)$$

where

$$\begin{cases} A=[(1+\mu^2(1+\beta_1^2)p^2)(1+\mu^2(1+\beta_2^2)p^2)]p^4 \\ B=[\alpha_1(1+\mu^2(1+\beta_2^2)p^2)+\alpha_2(1+\mu^2(1+\beta_1^2)p^2)] \\ C=\alpha_1\alpha_2-\frac{c_{12}(c_{21}-0.5K_W)}{(E\hbar)^2}(1+\mu^2(1+\beta_1^2)p^2) \\ (1+\mu^2(1+\beta_2^2)p^2) \end{cases} \quad (21)$$

The expression of the buckling load equals the root of Eq. (20), which is obtained by taking the negative sign before the square-root.

$$\lambda=\frac{B-\sqrt{B^2-4AC}}{2A} \quad (22)$$

We can rewrite Eq. (22) according to β_1, β_2 and p as follows

$$\lambda(\beta_1,\beta_2,p) =\frac{1}{2p^2\eta_1\eta_2}\left(\alpha_1\eta_2+\alpha_2\eta_1-\sqrt{(\alpha_1\eta_2+\alpha_2\eta_1)^2+D}\right) \quad (23)$$

where

$$\begin{cases} \eta_1=(1+\mu^2(1+\beta_1^2)p^2) \\ \eta_2=(1+\mu^2(1+\beta_2^2)p^2) \\ D=4\eta_1\eta_2\left(\eta_1\eta_2\frac{c_{12}(c_{21}-0.5K_W)}{(Eh)^2}-\alpha_1\alpha_2\right) \end{cases} \quad (24)$$

The critical buckling load λ_{cr}, using the Eq. (23), can be obtain according to the critical axial wave number p_{cr} for fixed values of aspect ratios β_1 and β_2. The number p_{cr} can be selected from numerical minimization of the buckling load λ with respect to the axial wave number p. In this work we use the proposed algorithm in our previous work (Timesli 2020) which allows us to determinate quickly and automatically the best value of p_{cr}. In order to find p_{cr}, the relative error ($error\,(p_n^i)=\left|\frac{\lambda_{cr}^{(n-1)}-\lambda_{cr}^{n}}{\lambda_{cr}^{(n-1)}}\right|$) of the critical buckling load decreases as the parameter p increases at the beginning. Continuously increasing p, the relative error becomes larger and the proposed algorithm gives incorrect result (see Fig. 3).

For Winkler model, we can develop an explicit analytical formula of the local critical buckling load of DWCNTs embedded in Winkler elastic medium under axial compression. In this case, the nonlocal parameter $e_0=$

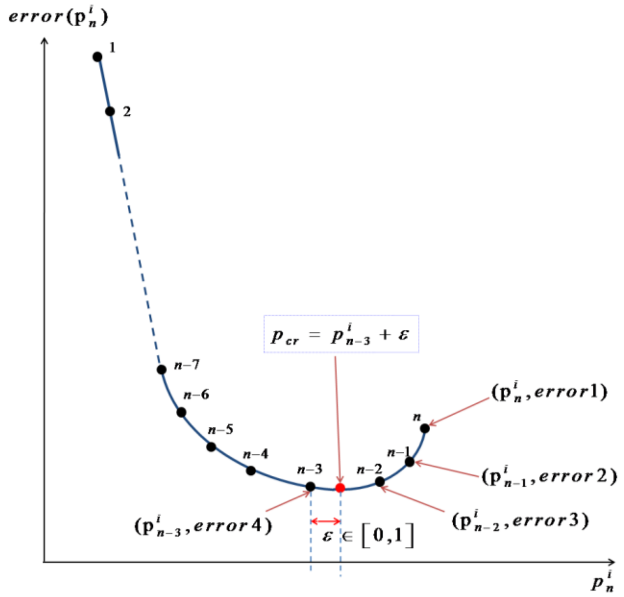


Fig. 3 Principle of finding the critical axial wave number p_{cr}

0, the foundation model of elastic medium depends only of the parameter K_W and it is represented by the simplified law $f = K_W w$. So, the local buckling load is given as follows.

$$\lambda(\beta_1,\beta_2,p) = \frac{1}{2p^2}\left(\alpha_1+\alpha_2-\sqrt{(\alpha_1+\alpha_2)^2+4\frac{c_{12}c_{21}}{(Eh)^2}}\right) \tag{25}$$

where

$$\begin{cases} \alpha_1 = k^2(1+\beta_1^2)^2p^4+\dfrac{\rho_1^2}{(1+\beta_1^2)^2}-\dfrac{\beta_1^2F_1}{Eh}p^2-\dfrac{c_{12}}{Eh} \\ \alpha_2 = k^2(1+\beta_2^2)^2p^4+\dfrac{\rho_2^2}{(1+\beta_2^2)^2}-\dfrac{\beta_2^2F_2}{Eh}p^2-\dfrac{c_{21}}{Eh} \\ \qquad +\dfrac{K_W}{Eh} \end{cases} \tag{26}$$

Taking into account the fact of $\beta_1^2F_1 = \beta_2^2F_2$ (He *et al.* 2005, Timesli *et al.* 2017), we can write the expression of the buckling loads in terms of the axial wave number p as follows.

$$\lambda(\beta_1,\beta_2,p) = \frac{1}{2p^2}\left(a_1p^4+a_2-\sqrt{(a_3^2p^8+a_5p^4+a_6)}\right)+\frac{a_7}{2} \tag{27}$$

where the coefficients $a_i\,(i=1,\ldots,7)$ are given by

$$\begin{cases} a_1 = k^2((1+\beta_1^2)^2+(1+\beta_2^2)^2) \\ a_2 = \dfrac{\rho_1^2}{(1+\beta_1^2)^2}+\dfrac{\rho_2^2}{(1+\beta_2^2)^2}-\dfrac{c_{12}+c_{21}}{Eh}+\dfrac{K_W}{Eh} \\ a_3 = k^2((1+\beta_1^2)^2-(1+\beta_2^2)^2) \\ a_4 = \dfrac{\rho_1^2}{(1+\beta_1^2)^2}-\dfrac{\rho_2^2}{(1+\beta_2^2)^2}-\dfrac{c_{12}-c_{21}}{Eh}-\dfrac{K_W}{Eh} \\ a_5 = 2a_3a_4 \\ a_6 = a_4^2+\dfrac{4c_{12}c_{21}}{(Eh)^2}-\dfrac{2c_{12}K_W}{(Eh)^2} \\ a_7 = \dfrac{-2\beta_1^2F_1}{Eh} \end{cases} \tag{28}$$

For fixed aspect ratio β_1 and β_2, we can obtain the critical buckling load λ_{cr} by minimizing the buckling load $\lambda\,(\beta_1,\beta_2,p)$ given by the Eq. (27) with respect to the axial wave number p.

$$\left.\frac{\partial\lambda(\beta_1,\beta_2,p)}{\partial p}\right|_{\beta_1 \text{ and } \beta_2 \text{ fixed}} = 0 \tag{29}$$

The Eq. (29) leads to the following polynomial of degree 16 in p.

$$b_4p^{16}+b_3p^{12}+b_2p^8+b_1p^4+b_0 = 0 \tag{30}$$

where the coefficients $b_i\,(i=0,1,2,3,4)$ are given by

$$\begin{cases} b_0 = a_6(a_2^2-a_6) \\ b_1 = a_2^2a_5-2a_1a_2a_6 \\ b_2 = a_2^2a_3^2-2a_1a_2a_5+a_1^2+2a_3^2a_6 \\ b_3 = a_1^2a_5-2a_1a_2a_3^2 \\ b_4 = a_1^2a_3^2-a_3^4 \end{cases} \tag{31}$$

By putting $\Lambda = p^4$, we can rewrite Eq. (30) in the form of a polynomial of degree 4 in Λ as follows.

$$b_4\Lambda^4+b_3\Lambda^3+b_2\Lambda^2+b_1\Lambda+b_0 = 0 \tag{32}$$

The only real root of Eq. (32) is given by

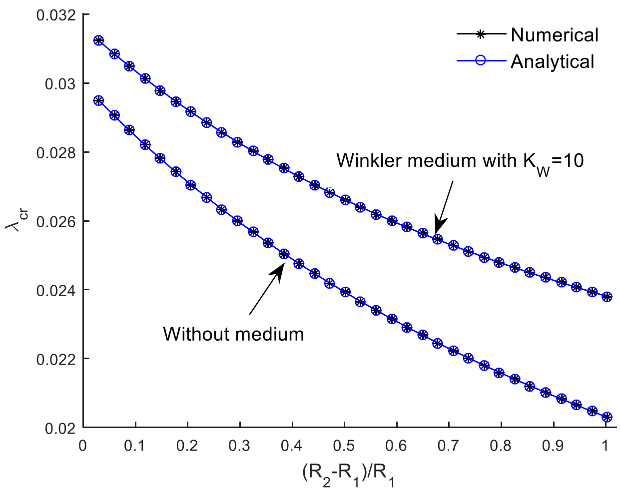
$$\Lambda = \frac{H}{2(u-S)}-\frac{b_3}{4b_4} \tag{33}$$

where

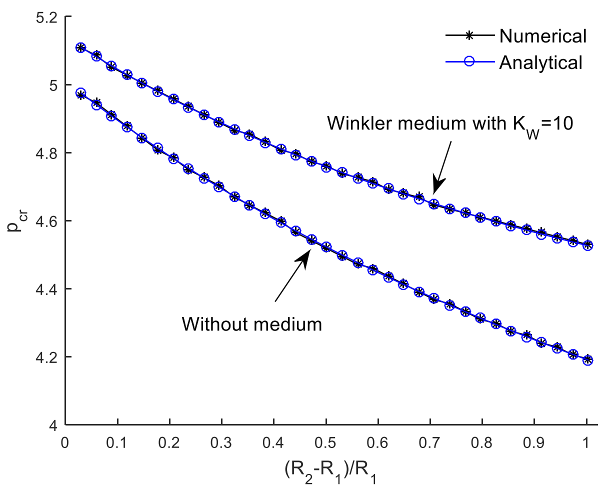
$$\begin{cases} H = \dfrac{1}{8}\dfrac{b_3^3}{b_4^3}-\dfrac{1}{2}\dfrac{b_2b_3}{b_4^2}+\dfrac{b_1}{b_4} \\ S = -\dfrac{3}{8}\dfrac{b_3^2}{b_4^2}+\dfrac{b_2}{b_4} \\ u = \dfrac{S}{3}+\sqrt[3]{-\dfrac{G_0}{2}+\sqrt{\dfrac{G_0^2}{4}+\dfrac{G_1^3}{27}}}+\sqrt[3]{-\dfrac{G_0}{2}-\sqrt{\dfrac{G_0^2}{4}+\dfrac{G_1^3}{27}}} \end{cases} \tag{34}$$

with

$$\begin{cases} G_0 = -\dfrac{2}{27}S^3+\dfrac{3}{8}SZ - H^2 \\ G_1 = -\left(\dfrac{S^2}{3}+4Z\right) \end{cases} \tag{35}$$



(a) Analytical and numerical solution of the critical buckling load λ_{cr}



(b) Analytical and numerical solution of the critical axial wave number p_{cr}

Fig. 4 Comparison between numerical and analytical results versus the ratio $(R_2\text{-}R_1)/R_1$ for both cases of presence and absence of elastic medium

$$\begin{cases} G_1 = -\left(\dfrac{S^2}{3} + 4Z\right) \\ Z = -\dfrac{3}{256}\dfrac{b_3^4}{b_4^4} + \dfrac{b_2 b_3^2}{16 b_4^3} - \dfrac{b_1 b_3}{4 b_4^2} + \dfrac{b_0}{b_4} \end{cases} \tag{35}$$

So, we can conclude the critical axial wave number p_{cr} which is equal.

$$p_{cr} = \left(\frac{H}{2(u-S)} - \frac{b_3}{4b_4}\right)^{\frac{1}{4}} \tag{36}$$

So, we find an explicit analytical formula of the critical axial wave number p_{cr}. Finally, the critical buckling load λ_{cr} of DWCNTs, taking into account VdW interaction is determined according to p_{cr} as follows $\lambda_{cr} = \lambda(p = p_{cr})$. To our knowledge, there isn't an explicit analytical formula

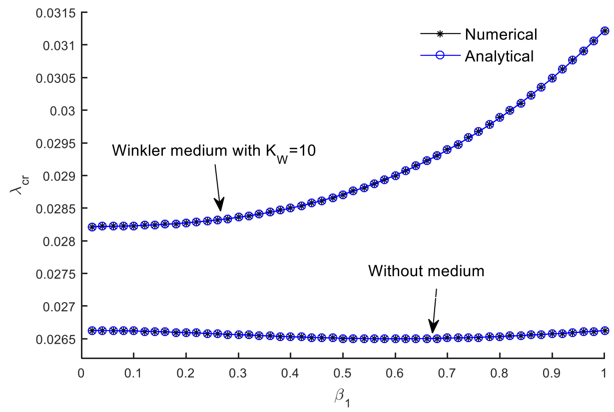


Fig. 5 Comparison between numerical and analytical solutions of the critical buckling load λ_{cr} versus the aspect ratio β_1 for both cases of presence and absence of elastic medium

of the local critical buckling load of DWCNTs under axial compression embedded in Winkler elastic medium and without the assumption on the radius of tubes where the terms containing $(R_2\text{-}R_1)/R_1$ are not neglected.

6. Numerical analysis and discussion

6.1 Validation of the explicit analytical formula of the local critical buckling load of DWCNTs embedded in Winkler elastic medium

In this numerical analysis, we consider an armchair (20, 20) (25, 25) DWCNTs (Bao *et al*. 2004, Tu *et al*. 2002, Chemi *et al*. 2015). For this reason, data of double Donnell shells model are given as follows: the inner radius R_1 = 1.356 nm, the median inter-shell spacing between carbon of DWCNTs δR = 0.34 nm the aspect ratios (length to radius) $L/R_1 = 10$, h = 0.066 nm, a = 0.142 nm, E = 852.764 GPa, v = 0.34, $c_{12} = (-320*10^{-3})/0.16a^2$ nN/nm^3, $c_{21} = (R_1/R_2)c_{12}$, F_1 = 0, e_0 = 0, β_1 = 0.5 and Winkler parameter Kw = 10 nN/nm^3. The numerical and analytical estimation of the local critical buckling load λ_{cr} of DWCNTs and the critical axial wave number in the axial direction p_{cr} versus the ratio $(R_2\text{-}R_1)/R_1$, in absence or presence of Winkler elastic medium, are given in Figs. 4(a) and (b), respectively. We can observe that the local critical buckling load of DWCNTs embedded in Winkler elastic medium is greater than that of without medium and it decreases with increasing of $(R_2\text{-}R_1)/R_1$. We can note that the elastic foundation increases the DWCNT rigidity.

Fig. 5 presents the comparison of the proposed analytical formula and minimization procedure varying the aspect ratio β_1. In the case without elastic medium, it is seen that the critical buckling load λ_{cr} in term of aspect ratio presents a minimum corresponding to a value of the aspect ratio β^*. Beyond this value, the critical buckling load decreases with increasing the aspect ratio. From this value β^*, the critical buckling load λ_{cr} increases. In the case with elastic medium, the increasing of the aspect ratio β_1 leads to the increase in the critical load.

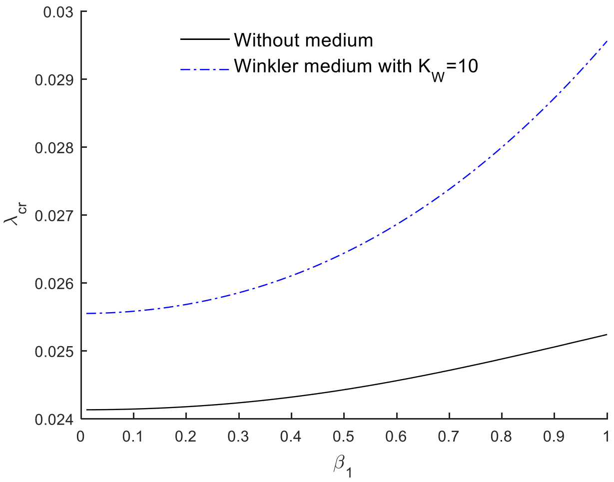


Fig. 6 Critical buckling load λ_{cr} versus the aspect ratio β_1 of DWCNTs embedded in Winkler elastic medium for the nonlocal parameter

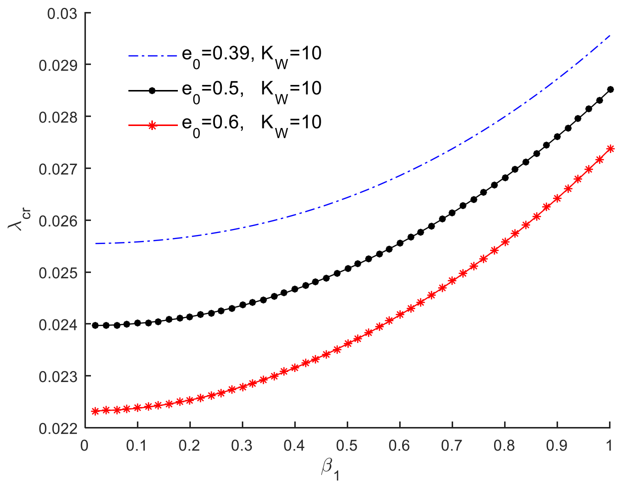
6.2 Numerical analysis for the nonlocal critical buckling load of DWCNTs embedded in an elastic medium

We consider the same data of the previous section 6.1 with the nonlocal parameter $e_0 = 0.39$. We observe in Fig. 6 the same remark as in the local case where the nonlocal nonlocal critical buckling load of DWCNTs embedded in Winkler elastic medium is greater than that without elastic medium for the same nonlocal parameter $e_0 = 0.39$. We also note that λ_{cr} increases with increasing values of aspect ratio β_1 for both cases of presence and absence of elastic medium.

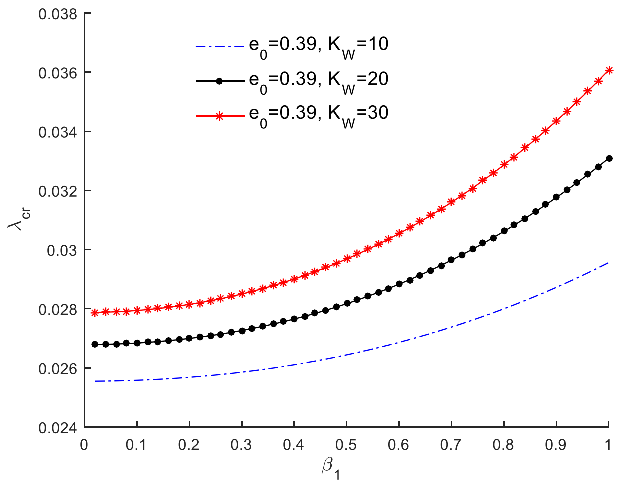
To show the effects of the Winkler modulus K_W and the nonlocal parameter e_0 on the critical buckling load λ_{cr} λ_{cr} of DWCNTs, we determine λ_{cr} by varying one of the parameters. From Fig. 7(a), we can conclude that λ_{cr} of DWCNTs embedded in an elastic medium decreases when the nonlocal parameter value increases. But if we reverse numerical experience, we obtain opposite results where λ_{cr} increases when Winkler parameter value increases (see Fig. 7(b)).

In the second numerical test, we study the nonlocal critical buckling load λ_{cr} of DWCNTs embedded in Pasternak elastic medium. We consider that the two parameters of the Pasternak model are: $K_W = 10\ nN/nm^3$ and $K_G = 1\ nN/nm$. The curves of the nonlocal critical buckling load λ_{cr} versus the aspect ratio β_1 for Winkler and Pasternak models were compared in Fig. 8, λ_{cr} of the Pasternak model is more important than that of Winkler, this result confirms the importance of taking into account the deformations associated with shearing in the elastic medium for the study of the buckling load of a DWCNT under axial compression.

From the obtained results in Fig. 9(a), it can be seen that the values of the critical buckling load λ_{cr} of DWCNTs embedded in Pasternak elastic medium decreases with increasing in the values of the nonlocal parameter. In the other hand, the values of λ_{cr} increases with increasing in the



(a) Effect of the nonlocal parameter e_0



(b) Effect of the Winkler parameter K_W

Fig. 7 The effects of the nonlocal parameter e_0 and the Winkler modulus K_W on the critical buckling load λ_{cr} of DWCNTs

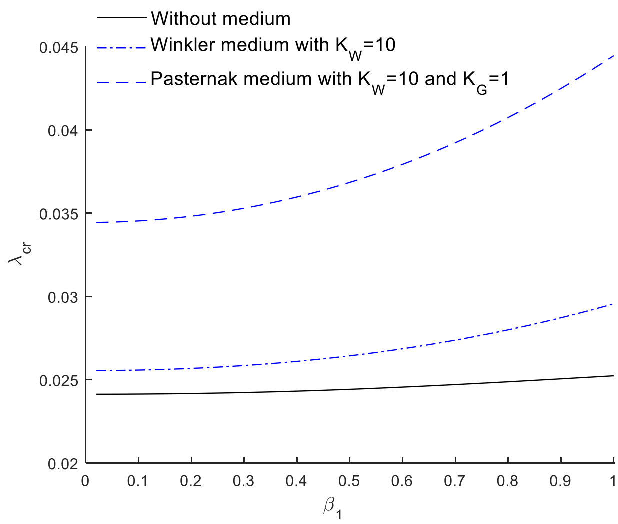
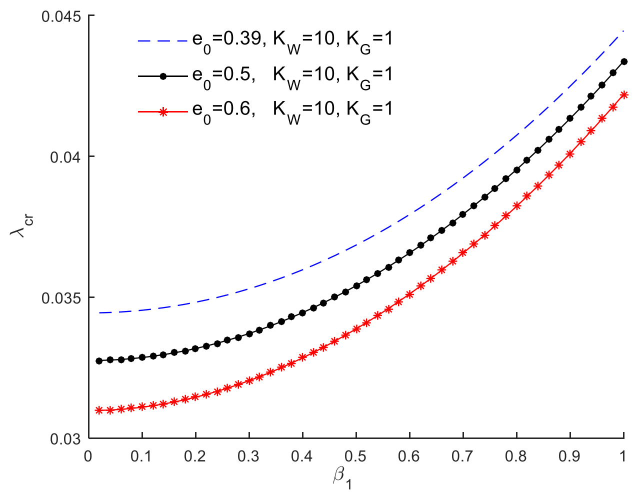
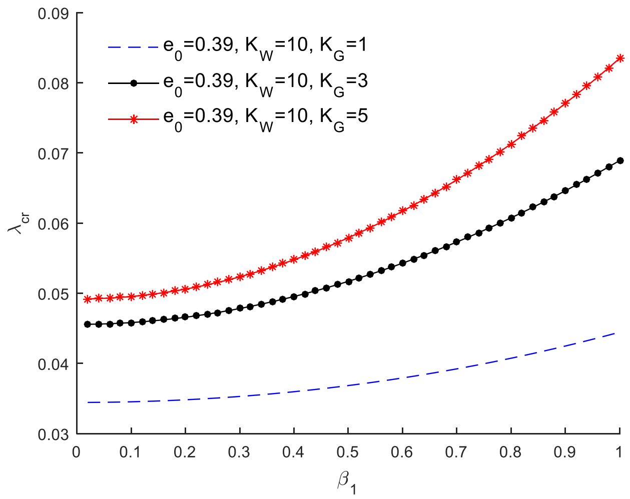


Fig. 8 Critical buckling load λ_{cr} versus the aspect ratio β_1 of DWCNTs for Winkler-type and Pasternak-type foundation models and the nonlocal parameter



(a) Effect of the nonlocal parameter e_0



(b) Effect of the parameter K_G

Fig. 9 The effects of the nonlocal parameter e_0 and the shear layer modulus K_G on the critical buckling load λ_{cr} of DWCNTs

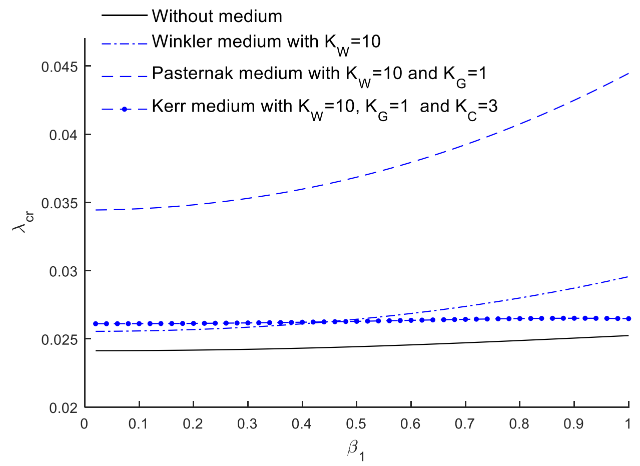


Fig. 10 Critical buckling load λ_{cr} versus the aspect ratio β_1 of DWCNTs for Winkler-type, Pasternak-type and Kerr-type foundation models and the nonlocal parameter $e_0 = 0.39$

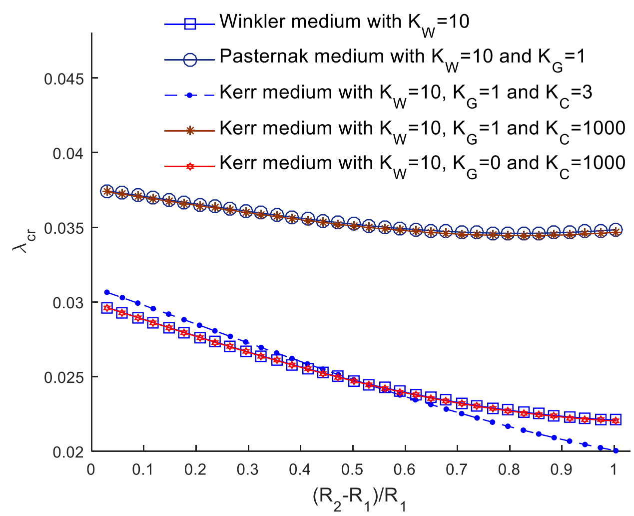


Fig. 11 Critical buckling load λ_{cr} versus the ratio $(R_2-R_1)/R_1$ of DWCNTs embedded in an elastic medium for the nonlocal parameter $e_0 = 0.39$ using different values of K_W, K_G and K_C

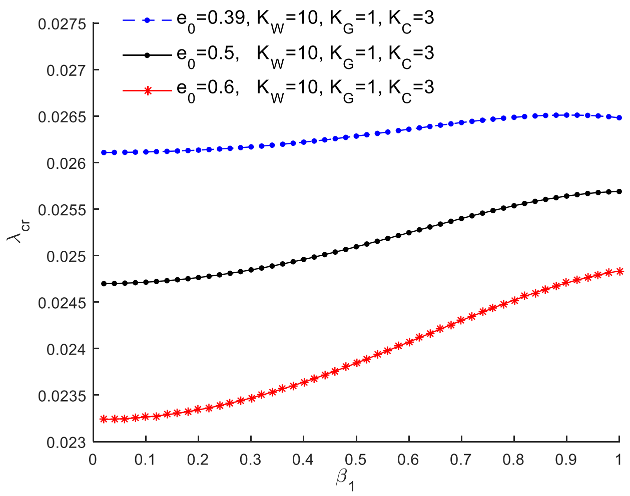
values of the parameter K_G of the Pasternak model (see Fig. 9(b)).

In the third numerical test, we are interested to study the nonlocal critical buckling load λ_{cr} of DWCNTs embedded in Kerr elastic medium. We consider the following three parameters of Kerr model: $K_W = 10\ nN/nm^3$, $K_G = 1\ nN/nm$ and $K_C = 3\ nN/nm^3$. With these parameters values, the Kerr medium makes the DWCNT less rigid compared to the Pasternak medium as shown in Fig. 10. But the comparison between Kerr and Winkler mediums depends on the value of K_C, where the critical load of the Kerr model varies between the critical load without medium and that of the Pastrenak medium as shown in Fig. 11. In this figure, we can note that the critical load decreases for increasing values of the ratio $(R_2-R_1)/R_1$ for the three models of Winkler, Pasternak and Kerr. We also observe that for great values of K_C the Kerr model tends to that of Pasternak; for great values of K_C with a Pasternak parameter K_G equals zero the Kerr model tends to that of Winkler.

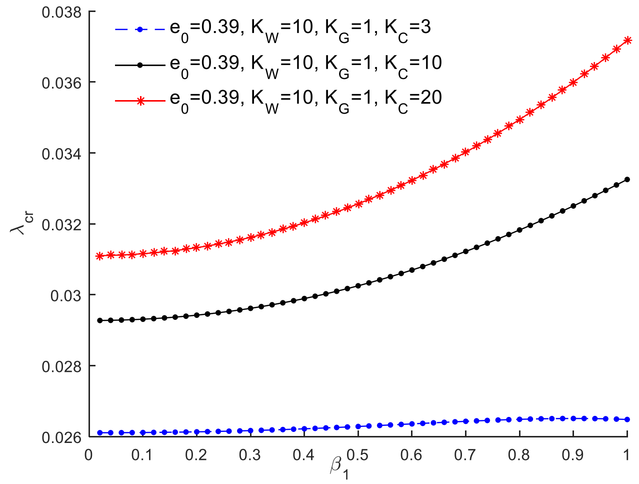
Regarding the effect of the nonlocal parameter e_0 and the modulus K_C of the upper spring, we observe the same remark as the other models (see Fig. 12).

7. Conclusions

In this paper, the buckling analysis of DWCNT embedded in an elastic foundation under axial compression is investigated by employing the nonlocal Donnell shell theory. The elastic foundation is based on Winkler, Pasternak and Kerr models which depend respectively to one, two and three parameters. Firstly, an analytical solution of the local critical buckling load of DWCNTs embedded in Winkler elastic medium under axial compression is



(a) Effect of the nonlocal parameter e_0



(b) Effect of the parameter K_C

Fig. 12 The effects of the nonlocal parameter e_0 and the upper spring modulus K_C on the critical load of DWCNTs

developed. Thereafter, several parametric studies have been performed to show the influence of elastic foundation on the critical buckling load of DWCNTs taking into account the effects of internal small length scale and the VdW interactions between the inner and outer nanotubes. The present study shows that the elastic foundation makes the DWCNT more rigid where the critical buckling load is greater than that without foundation under identical data and conditions.

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