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2D magneto-mechanical vibration analysis of a micro composite Timoshenko beam resting on orthotropic medium

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Abstract. In the present study, the free vibration analysis of a size-dependent micro composite Timoshenko beam model reinforced by various distributions of carbon nanotubes under temperature changes and two-dimensional magnetic field is investigated based on modified strain gradient theory. Also, the effects of environment are simulated by orthotropic elastic foundation and it is assumed that the material properties are temperature-dependent. Mathematical formulations are obtained using Hamilton's principle and the governing equations of motion are derived based on energy approach and variation method. These equations are solved using semi-analytical and numerical methods such as Navier's type solution, finite element method and generalized differential quadrature method for various boundary conditions. The obtained results of this study are compared with the other previous researches and there is a good agreement between them. The main purpose of this work is the comparison of various solution methods on the problem outputs. Thus, the results are compared together and the effects of solution approach on the dimensionless natural frequencies is developed. Moreover, the effects of length-to-thickness ratio, magnetic field, temperature changes, elastic foundation and carbon nanotubes volume fractions on the dimensionless natural frequencies are studied. The results of this article demonstrate that the micro composite Timoshenko beam reinforced by FG-O and FG-X CNTs have lowest and highest dimensionless natural frequency, respectively. It is investigated that the dimensionless natural frequency enhances by increasing the magnetic field in x and z-directions.

Keywords: 2D magnetic field; free vibration analysis of Timoshenko micro beam; reinforced by carbon nanotubes; temperature-dependent material properties; orthotropic medium

1. Introduction

Since the 18th century, free vibration analysis of beams has been considered using the Euler-Bernoulli beam theory (EBT) (Hsu 2016). Then, the researchers found that EBT predicted the results for the lower thickness-to-length ratio correctly in engineering problems (Banerjee 2001). Thus, later in the 19th century, introduced the reasonable theories such as Rayleigh (19th century) and Timoshenko (20th century) beam theories (TBT) by adding the effects of rotary inertia and transverse shear deformation on the vibratory of the beams (Hsu 2016), respectively. After the 20th century, with the spread the mechanical science, the higher order beam theories were known as first order shear deformation theory (FSDT), third order shear deformation theory (TSDT) and sinusoidal shear deformation theory (SSDT). By introducing these theories, many researchers in the world focused on the various structures such as bars (Canales and Mantari 2017, Sapountzakis *et al*. 2015 and Zhang *et al*. 2016a), beams (Domagalski 2018), plates (Soni *et al*. 2018, Qing *et al*. 2018 and Ozdemir 2018) and cylindrical shells (Plagianakos and Papadopoulos 2015, Yang *et al*. 2014 and Beni *et al*. 2015) and used different theories on these structures:

Pradhan and Mandal (2013) investigated vibration, buckling and bending of carbon nanotubes (CNTs) based on nonlocal TBT and sensitivity of thermal environment. Based on FSDT, Pandey and Pradyumna (2015) established a layer-wise finite element formulation for dynamic analysis of functionally graded material (FGM) sandwich shell in thermal and non-thermal environments. They concluded that this layer-wise formulation can be easily applied for any singly curved and doubly curved sandwich shells with various boundary conditions. Romanoff *et al*. (2016) developed non-classical sandwich beam theories and used them in micro and macro structural response. Kashani and Sani (2016) presented free vibration analysis of horizontal cylindrical shells using exact solution of governing equations and polar finite element method (FEM) based on Sander's thin-walled shell theory. They compared their results with experimental tests and showed that their methods have more agreement in comparison with the other theories. Ghasemi *et al*. (2017, 2018) presented a computational design methodology for topology optimization of multi-material based flexoelectric composites and based on a combination of isogeometric analysis. They investigated multi-phase vector level set (LS) model, efficiently satisfies multiple constraints and

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intrinsically avoids overlap or vacuum among different phases. Also, they showed various numerical examples demonstrating the significant enhancement in electromechanical coupling coefficient that can be obtained using topology optimization. Nanthakumar *et al*. (2016) examined an algorithm to solve the inverse problem of detecting inclusion interfaces in a piezoelectric structure is proposed using FEM. Their outputs demonstrated that iterative procedure proposed can determine the location and approximate shape of material sub-domains in the presence of higher noise levels. Areias *et al*. (2016) introduced a new staggered algorithm for elastic and elasto-plastic materials of fracture of finite-strain plates and shells based on a phase-field model of crack regularization. Using TSDT, Mohammadimehr *et al*. (2016a, b) studied static, buckling and free vibration analyses of micro composite Reddy plates reinforced by FG-CNTs based on strain gradient theory (MSGT). They assumed that the material properties are dependent on temperature changes and considered the effects of these changes on the maximum deflection, critical buckling load and natural frequencies. Ganesh *et al*. (2016) examined the free vibration analysis of delaminated composite plates based on the equivalent single layer (ESL) and FSDT theories using FEM for simply supported and clamp-free boundary conditions. AkhavanAlavi *et al*. (2019) presented active control of micro Reddy beam integrated with functionally graded nanocomposite sensor and actuator based on linear quadratic regulator method. Mohammadimehr *et al*. (2018c) illustrated bending, buckling, and free vibration analyses of carbon nanotube reinforced composite beams and experimental tensile test to obtain the mechanical properties of nanocomposite. Zhang *et al*. (2016b) described vibration analysis of CNTs reinforced FG-composite triangular plates subjected to in-plane stresses based on FSDT to study the effect of the transverse shear deformation of the plates using the element-free IMLS-Ritz method. They showed that the changes in CNTs-reinforced composite volume fraction and CNTs distribution had pronounced effects on the vibration behavior of various types of CNT-reinforced composite plates. For the uncertainty analysis, Vu-Bac *et al*. (2016) investigated a unified method for probabilistic sensitivity analysis for computationally expensive models with correlated inputs. It is shown that the spline regression model is more robust than polynomial regression model, i.e., the quadratic regression model, as the spline regression model can larger 80% exactly approximate the observed data while the quadratic regression without mixed terms fails to account for them. Also, Hamdia *et al*. (2018) investigated sensitivity and uncertainty analysis for flexoelectric nanostructures. Barquero-Cabrero *et al*. (2018) investigated a model of the elastic curve for rectangular Timoshenko beams with straight haunches under uniformly distributed load and moments in the ends considering the bending and shear deformations to obtain the deflections and rotations on the beam. Tabbakh and Nasihatgozar (2018) considered buckling response of polymeric plates reinforced with CNTs and coated by magneto strictive layer. Farzampour *et al*. (2019) illustrated the effect of flexural and shear stresses simultaneously for optimized design of butterfly-shaped dampers: Computational study.

Moreover, the vibration problems need to solve governing equations of motions using mathematical methods. These methods divide into three categories: exact analytical solutions, semi-analytical approach and numerical solution methods such as FEM, finite difference method (Hsu 2016, Lee and Schultz 2004) generalized differential quadrature method (GDQM) and mesh-free method. Usually, these methods determine natural frequencies and illustrate mode shapes to solve the eigenvalue and eigenvector problems by stiffness and mass matrices. In this article, we will talk about FEM and GDQM numerical solution methods. In the last years, FEM (Yang *et al*. 2016 and Zeinkiewicz and Taylor 2000) and GDQM has been extensively used in many fields of engineering (Paluszny *et al*. 2013) such as aerospace, civil and mechanical engineering, micro-electro-mechanical systems (MEMs) and nano-electro-mechanical systems (NEMs). Arani *et al*. (2012) presented the effect of CNT volume fraction on the magneto-thermo-electro-mechanical behavior of smart nanocomposite cylinder. Rajabi and Mohammadimehr (2019) investigated bending analysis of a micro sandwich skew plate using extended Kantorovich method based on Eshelby-Mori-Tanaka approach. Anitescu *et al*. (2019) presented a method for solving partial differential equations (PDEs) using artificial neural networks and an adaptive collocation strategy. They showed that this method increases the robustness of the neural network approximation. Guo *et al*. (2019) investigated bending problems of thin Kirchhoff plates using a deep collocation method (DCM). Their method took advantage of computational graphs and backpropagation algorithms involved in deep learning. They demonstrated that DCM can proved to be suitable to the bending analysis of Kirchhoff plate of various geometries. A novel nonlocal operator theory based on the variational principle proposed for solution of PDEs by Rabczuk *et al*. (2019). They concluded that their formulation allows the assembling of the tangent stiffness matrix with ease and simplicity, which is necessary for the eigenvalue analysis such as the waveguide problem. Moreover, they solved the differential electromagnetic vector wave equations based on electric fields using this operator. Mohammadimehr and Mehrabi (2017, 2018) discussed about stability and free vibration analyses of double-bonded micro composite sandwich micro tubes conveying fluid flow based on EBT and Reddy cylindrical shell theories using GDQM. They demonstrated that flow velocity has a special role on the stability of micro structures and increasing velocity lead to decrease dimensionless natural frequencies. Yue *et al*. (2016) predicted surface effects and flexoelectricity of Timoshenko beam using Navier's analytical solution approach based on Hamilton's principle. Setoodeh *et al*. (2018) considered free vibration analysis of three-layered FG-CNTs reinforced composite spherical panels based on higher order shear deformation thick-moderately panels theories and using transformed differential quadrature (TDQ) method. Soleimani and Beni (2018) studied size dependent axisymmetric shell element formulation by using the modified couple stress theory (MCST) in place of classical continuum theory. They showed that the rigidity of the nano-shell in the MCST is greater than that in classical continuum theory, which led to increase in natural frequencies. Furtak and Rodacki (2018) discussed about the load-bearing capacity of composite timber-glass I-beams, which were interesting alternative beams of ceilings and roofs. Shahedi and Mohammadimehr (2019) presented vibration analysis of rotating fully-bonded and delaminated sandwich beam with CNTRC face sheets and AL-foam flexible core in thermal and moisture environments.

In this work, a comparison between different solving methods in eigenvalue problems is presented based on Navier's type solution (as a semi-analytical approach), FEM and GDQM (as numerical solutions) for simply-simply supported (S-S) and clamped-clamped supported (C-C) boundary conditions, respectively. In fact, the main novelty of the present study is the comparison of FE and GDQ methods for two-dimensional (2D) magneto-thermo-mechanical vibration analysis of size-dependent micro composite Timoshenko beam model that there is not in the previous researches. For this purpose, a micro composite Timoshenko beam model reinforced by various distribution of CNTs has been chosen based on size-dependent MSGT in micro/ nano scale. It is assumed that the material properties of matrix and reinforcement are temperature-dependent and changes with temperature. Also, the effects of environment are simulated by orthotropic elastic foundation versus Winkler spring constant and Pasternak shear modulus in two-directional local axis. Moreover, 2D magnetic fields in length and thickness of micro beam are applied and the effects of these fields on the first dimensionless natural frequencies are investigated.

2. Simulation

2.1 Geometry

According to Figs. 1 and 2, a micro composite Timoshenko beam model reinforced by uniform (UD) and functionally graded single layered carbon nanotubes (FG-SWCNTs (10,10)) in presence of 2D magnetic fields in x and z-directions is considered with length L, width b and thickness h. This micro beam rested in an orthotropic foundation with Winkler spring coefficient k_w and Pasternak shear modulus $k_{g\xi}$ and $k_{g\eta}$ in ξ and η local direction.

The limitations and assumption of this model are stated as follows:

(1) The displacement fields based on first-order shear deformation theory for a micro sandwich beam are considered.
(2) In this study, 2D magneto-mechanical loads are considered.
(3) To compare, the Navier's method as semi-analytical solution as well as numerical methods including finite element and differential quadrature methods are illustrated.
(4) The delamination between core and face sheets is not considered.
(5) The agglomeration effect in this work is ignored.

In this article, the system is simulated as a composite structure consist of Poly {(mphenylenevinylene)-co-[(2,5-dioctoxy-p-phenylene) vinylene]}[1] as a matrix and CNTs as reinforcement that properties of matrix and reinforcement

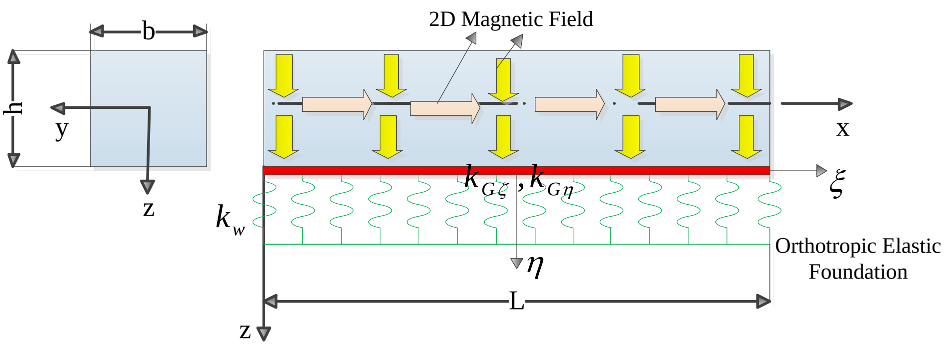


Fig. 1 Schematic of micro composite Timoshenko beam in an orthotropic elastic medium in the presence of 2D magnetic field

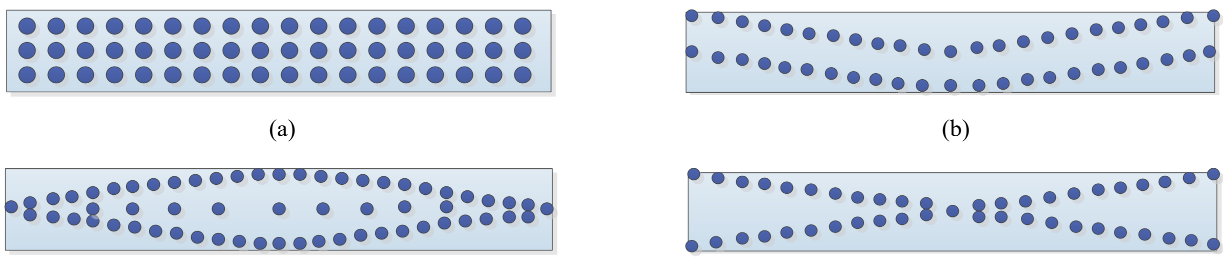


Fig. 2 Schematic of various distribution of CNTs as a reinforcement (a) UD; (b) FG-V; (c) FG-O; (d) FG-X

[1] - PmPv

Table 1 The temperature-dependent material properties of PmPv composite matrix (Mohammadimehr and Mehrabi 2017)

Material properties	Temperature (K)				
	300	400	500	600	700
E_m (MPa)	2100	1630	1160	690	220
G_m (MPa)	783.6	608.3	432.3	257.5	82.04
α_m $(10^{-6}/K)$	45.00	47.25	49.50	51.75	54.00

Table 2 The temperature-dependent material properties of CNTs (Lei *et al*. 2013b, 2015)

Material properties	Temperature (K)				
	300	400	500	600	700
E_{11} (GPa)	5646.6	5581.3	5530.8	5495.2	5474.4
E_{22} (GPa)	7080.0	6998.1	6934.8	6890.2	6864.1
G_{12} (GPa)	1944.5	1956.9	1964.3	1966.8	1964.4
α_{11} $(10^{-6}/K)$	3.458	4.115	4.536	4.720	4.667

are assumed temperature-dependent and are shown in Tables 1 and 2.

In the present study, uniform distribution (UD), FG-V, FG-O and FG-X distributions of CNTs are assumed to reinforce the micro composite Timoshenko beam that Volume fraction for these distributions is defined as follows (Ardestani *et al*. 2017 and Song *et al*. 2016)

$$V_{CNT}(z)=\begin{cases}V_{CNT}^{*}(UD)\\(1+\dfrac{2z}{h})V_{CNT}^{*}(FG-V)\\2(1-\dfrac{|2z|}{h})V_{CNT}^{*}(FG-O)\\2\dfrac{|2z|}{h}V_{CNT}^{*}(FG-X)\end{cases}\tag{1}$$

where Mohammadimehr *et al*. (2018a)

$$V_{CNT}^{*}=\frac{w_{CNT}}{w_{CNT}+(\frac{\rho_{CNT}}{\rho_m})(1-w_{CNT})}\tag{2}$$

where w_{CNT}, ρ_m and ρ_{CNT} are CNTs mass fraction, matrix density and carbon nanotubes density, respectively.

2.2 Generalized rule of mixture

The generalized rule of mixture for estimation of magneto-thermo-mechanical properties of micro composite Timoshenko beam reinforced by UD and FG-CNTs are expressed as follows:

Mechanical properties (Mohammadimehr and Mehrabi 2017)

$$\begin{cases}E_{11}=\eta_1 V_{CNT}E_{11}^{CNT}+V_m E_m\\\dfrac{\eta_2}{E_{22}}=\dfrac{V_{CNT}}{E_{22}^{CNT}}+\dfrac{V_m}{E_m}\\\dfrac{\eta_3}{G_{12}}=\dfrac{V_{CNT}}{G_{12}^{CNT}}+\dfrac{V_m}{G_m}\\\upsilon_{12}=V_{CNT}\upsilon_{12}^{CNT}+V_m\upsilon_m\end{cases}\tag{3}$$

Table 3 η_i force transformation between SWCNTs and polymeric matrix (Zhu *et al*. 2012)

V_{CNT}^{*}	η_1	η_2	η_3
0.11	0.149	0.934	0.934
0.14	0.150	0.941	0.941
0.17	0.149	1.384	1.384

where E_{11}^{CNT}, E_{22}^{CNT}, $G_{12}^{CNT}$and υ_{12}^{CNT} are Young's modulus, shear modulus and Poisson's ratio of SWCNTs in longitudinal and transverse directions, respectively. Also, E_m, G_m and υ_m denote Young's modulus, shear modulus and Poisson's ratio of the isotropic matrix. Moreover, η_i $(i=1,2,3)$ introduces force transformation between SWCNTs and polymeric matrix that the values of η_i are shown in Table 3.

In the Eq. (3) V_m is the matrix volume fraction and it can be described as follows (Mohammadimehr *et al*. 2020)

$$V_m+V_{CNT}=1\tag{4}$$

Thermal properties (Mohammadimehr *et al*. 2016a)

$$\begin{cases}\alpha_{11}(z)=V_{CNT}\alpha_{11}^{CNT}+V_m\alpha^m\\\alpha_{22}(z)=(1+\upsilon_{12}^{CNT})V_{CNT}\alpha_{22}^{CNT}\\\qquad+(1+\upsilon^m)V_m\alpha^m-\upsilon_{12}\alpha_{11}\end{cases}\tag{5}$$

where, $\alpha_{11}(z)$ and $\alpha_{22}(z)$ are the equal thermal expansion coefficients of micro structure in longitudinal and transverse directions, respectively. Moreover, α_{11}^{CNT}, α_{22}^{CNT} and α^m are the thermal expansion coefficients of CNTs and matrix, respectively.

3. Formulation

3.1 Displacement fields of Timoshenko beam model

Displacement field of the micro composite Timoshenko beam reinforced by CNTs can be expressed as follows

$$\begin{cases} u(x,z,t) = u_0(x,t) - z(c_1 w_0(x,t) - c_2\varphi_0) \\ v(x,z,t) = 0 \\ w(x,z,t) = w_0(x,t) \end{cases} \tag{6}$$

where u, v and w are the displacements of micro Timoshenko beam in x, y and z direction, respectively. Also, u_0, v_0, w_0, c_1 and c_2 denote middle surface at $z = 0$ and constant coefficients that equal to 1 for Timoshenko micro composite beams, respectively. Moreover, φ_0 is a function of deflection and rotation of cross-section of micro composite Timoshenko beam that in this article, it is defined as follows

$$\varphi_0(x,t) = w_{0,x}(x,t) - \theta_x(x,t) \tag{7}$$

where θ_x is angle of rotation of the cross-section of any point on the beam mid-plane. It is noted that, in the previous researches u_0 is assumed equal to zero, but in the present study, u_0 is considered in the displacement fields equation and it will show that u_0 has no important effect on the obtained results, then, this parameter will ignore in the FEM, GDQM and Navier's solution methods.

Strain-displacement relations can be expressed as follows (Mohammadimehr *et al*. 2020)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{8}$$

Substituting Eq. (3) into the Eq. (4) lead to derive the strain-displacement components as follows

$$\begin{cases} \varepsilon_x = u_{0,x}(x,t) - c_1 z w_{0,xx}(x,t) + c_2 z\varphi_{0,x}(x,t) \\ \gamma_{xz} = (1-c_1)w_{0,x}(x,t) + c_2\varphi_0(x,t) \\ \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{yz} = 0 \end{cases} \tag{9}$$

Using Hook's Law, stress-strain relations can be stated as follows (Chau-Dinh *et al*. 2012)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{Bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha_{11}(z)\Delta T \\ \varepsilon_y - \alpha_{22}(z)\Delta T \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{10}$$

where, σ_{xx} and σ_{yy} are normal stresses while σ_{yz}, σ_{xz}, and σ_{xy} denote the shear stresses. ΔT is the temperature changes and Q_{ij} is defined as follows (Mohammadimehr and Mehrabi 2017)

$$\begin{cases} Q_{11} = \dfrac{E_{11}}{1-\upsilon_{12}\upsilon_{21}}; \quad Q_{22} = \dfrac{E_{22}}{1-\upsilon_{12}\upsilon_{21}}; \quad Q_{12} = \dfrac{\upsilon_{21}E_{11}}{1-\upsilon_{12}\upsilon_{21}} \\ Q_{44}(z) = G_{23}(z) = \dfrac{E_{22}(z)}{2(1+\upsilon_{23})}; \quad Q_{55}(z) = Q_{66}(z) = G_{12}(z) = \dfrac{E_{11}(z)}{2(1+\upsilon_{12})} \end{cases} \tag{11}$$

Therefore

$$\begin{cases} \sigma_x = \dfrac{E_{11}(z)}{1-\upsilon_{12}\upsilon_{21}}\left(-zc_1\dfrac{\partial^2 w_0}{\partial x^2} - \alpha_{11}(z)\Delta T\right) \\ \tau_{xz} = \dfrac{E_{11}(z)}{2(1+\upsilon_{12})}\left((1-c_1)\dfrac{\partial w_0}{\partial x} + c_2\phi_0\right) \end{cases} \tag{12}$$

3.2 Strain potential energy - The modified strain gradient theory (MSGT)

In contrast to the classical gradient elasticity theory including length scale parameters (Kultu and Omurtag 2012) there are various theories such as MCST, MSGT and most general strain gradient theory (MGST). In 2002, Yang *et al*. (2002) first presented MCST (Ke and Wang 2011). Then, MSGT proposed by Lam *et al*. (2003) and MGST (Ansari *et al*. 2013 and Chu *et al*. 2018) extended two previous theories by introducing new constant of length scale parameters. In this work, MSGT is chosen to derive strain potential energy contain classical strain ε, dilatation gradient vector γ, symmetric rotation $\chi^{(s)}$and deviatoric stretch gradient $\eta^{(1)}$ tensors (Mohammadimehr and Mehrabi 2017). This theory can be written as follows (Thai *et al*. 2017)

$$U = \frac{1}{2}b\int_A (\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + m_{ij}^{(s)}\chi_{ij}^{(s)} + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)})dA \tag{13}$$

where U, p_i, $m_{ij}^{(s)}$ and $\tau_{ijk}^{(1)}$ are strain potential energy and higher-order stresses which are described as follows (Ma *et al*. 2008 and Mirsalehi *et al*. 2017)

$$\begin{cases} P_i = 2Gl_0^2\gamma_i \\ \tau_{ijk}^{(1)} = 2Gl_1^2\eta_{ijk}^{(1)} \\ m_{ij}^{(s)} = 2Gl_2^2\chi_{ij}^{(s)} \end{cases} \tag{14}$$

in which (Ma *et al*. 2008 and Mirsalehi *et al*. 2017)

$$\begin{cases} \gamma_i = \varepsilon_{mm,i} \\ \eta_{ijk}^{(1)} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) - \frac{1}{15}\begin{bmatrix} \delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mj,m}) \\ +\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) \\ +\delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}) \end{bmatrix} \\ \chi_{ij}^{(s)} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \end{cases} \tag{15}$$

It is noted that in the Eq. (14), l_0, l_1 and l_2 are additional material length scale parameters related to dilatation, deviatoric stretch and rotation gradients, respectively. Also, in the Eq. (15), θ_i represents the components of rotation vector θ and written as follows (Ma *et al*. 2008 and Mirsalehi *et al*. 2017)

$$\boldsymbol{\theta_i} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{curl}(\boldsymbol{u}))_{\boldsymbol{i}} \tag{16}$$

where u_i refers to the components of displacement vector u.

Substituting Eqs. (9) and (15) into the Eq. (16) the components of dilatation gradient vector, symmetric rotation and deviatoric stretch gradient tensors investigated as follows

$$\begin{cases} \gamma_x = u_{0,xx}(x,t) - c_1 z u_{0,xxx}(x,t) + c_2 z \varphi_{0,xx}(x,t) \\ \gamma_z = -c_1 w_{0,xx}(x,t) + c_2 \varphi_{0,x}(x,t) \end{cases} \tag{17}$$

$$\begin{cases} (A_{11}, D_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(1, z^2)\, dz \\ A_{55} = \int_{-\frac{h}{2}}^{\frac{h}{2}} k_s Q_{55} dz \end{cases}$$

$$\begin{cases} (P^{(0)}, P^{(2)}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} 2 G l_0^2 (1, z^2)\, dz \\ (T^{(0)}, T^{(2)}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{4}{5} G l_1^2 (1, z^2)\, dz \\ M^{(0)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{4} G l_2^2 dz \end{cases} \tag{21}$$

In the Eq. (21), k_s is the shear coefficient that it is considered equal to *5/6* for Timoshenko micro beams.

$$\begin{cases} \eta_{xxx}^{(1)} = \frac{2}{5}\left[u_{0,xx}(x,t) - \alpha_1 z w_{0,xxx}(x,t) + \alpha_2 z \varphi_{0,xx}(x,t)\right] \\ \eta_{xxz}^{(1)} = \eta_{xzx}^{(1)} = \eta_{zxx}^{(1)} = \frac{4}{15}\left[(1-2c_1) w_{0,xx}(x,t) + 2c_2 \varphi_{0,x}(x,t)\right] \\ \eta_{xyy}^{(1)} = \eta_{yxy}^{(1)} = \eta_{yyx}^{(1)} = \eta_{xzz}^{(1)} = \eta_{zxz}^{(1)} = \eta_{zzx}^{(1)} = -\frac{1}{5}\begin{bmatrix} u_{0,xx}(x,t) - c_1 z w_{0,xxx}(x,t) \\ +c_2 z \varphi_{0,xx}(x,t) \end{bmatrix} \\ \eta_{yyz}^{(1)} = \eta_{yzy}^{(1)} = \eta_{zyy}^{(1)} = -\frac{1}{15}\left[(1-2c_1) w_{0,xx}(x,t) + 2c_2 \varphi_{0,x}(x,t)\right] \\ \eta_{zzz}^{(1)} = -\frac{1}{5}\left[(1-2c_1) w_{0,xx}(x,t) + 2c_2 \varphi_{0,x}(x,t)\right] \end{cases} \tag{18}$$

$$\chi_{yx}^{(s)} = \chi_{xy}^{(s)} = -\frac{1}{4}\left[(1+c_1) w_{0,xx}(x,t) - c_2 \varphi_{0,x}(x,t)\right] \tag{19}$$

Finally, the strain potential energy is obtained by substituting Eq's. (14) to (19) into the Eq. (13) as follows

$$U = \frac{1}{2} b \int_0^L \begin{Bmatrix} (P^{(0)} + T^{(0)}) u_{0,xx}^2 + A_{11} u_{0,x}^2 + c_1^2 (P^{(2)} + T^{(2)}) w_{0,xxx}^2 \\ + \left[c_1^2 (D_{11} + P^{(0)}) + (1+c_1)^2 M^{(0)} + \frac{2}{3}(1-2c_1)^2 T^{(0)}\right] w_{0,xx}^2 \\ + c_2^2 (P^{(2)} + T^{(2)}) \varphi_{0,xx}^2 + c_2^2 (D_{11} + M^{(0)} + P^{(0)} + \frac{8}{3} T^{(0)}) \varphi_{0,x}^2 \\ + c_2^2 A_{55} \varphi_0^2 - 2 c_1 c_2 (P^{(2)} + T^{(2)}) w_{0,xxx} \varphi_{0,xx} \\ -2\left[c_1 c_2 (D_{11} + P^{(0)}) + c_2 (1+c_1) M^{(0)} - \frac{4}{3} c_2 (1-2c_1) T^{(0)}\right] w_{0,xx} \varphi_{0,x} \end{Bmatrix} dx \tag{20}$$

where A_{ij}, D_{ij}, $P^{(i)}$, $M^{(i)}$ and $T^{(i)}$ are the mechanical constants and described as the following form

3.3 Kinetic energy

The kinetic energy equation of micro composite Timoshenko beams reinforced by CNTs can be defined as follows (Lenci *et al*. 2017)

$$T = \frac{1}{2} \int_A b\left[u_{,t}^2(x,z,t) + w_{,t}^2(x,z,t)\right] dA \tag{22}$$

Substituting Eq. (6) into the Eq. (22), the following relation can be derived as follows

$$T = \frac{b}{2}\int_0^L \left[I^{(0)}\left(u_{0,t}^2(x,t) + w_{0,t}^2(x,t)\right) + I^{(2)}\begin{pmatrix} c_1^2 w_{0,xt}^2(x,t) + c_2^2\varphi_{0,t}^2(x,t) \\ -2c_1c_2 w_{0,xt}(x,t)\varphi_{0,t}(x,t) \end{pmatrix}\right] dx \tag{23}$$

In which $I^{(0)}$ and $I^{(2)}$ are the principle mass inertia and the rotation inertia, respectively, that these parameters are defined as

$$(I^{(0)}, I^{(2)}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)(1, z^2)\, dz \tag{24}$$

3.4 The work done by external forces

In the present work, the micro composite Timoshenko beam reinforced by CNTs is considered resting on an orthotropic elastic medium and two-dimensional magnetic field is applied in length and thickness direction of micro beam. Thus, the work done by external forces is described as the following form

$$V = V_{Orthotropic}^{ElasticMedium} + V_{2D}^{MagneticField} \tag{25}$$

In which $V_{Orthotropic}^{ElasticMedium}$ and $V_{2D}^{MagneticField}$ are the work done by orthotropic elastic foundation and 2D magnetic field, respectively. these works, can be investigated as follows:

$$V = \frac{b}{2}\int_0^{L\int} \begin{bmatrix} k_w w_0^2(x,t) - (k_{g\xi}\cos^2\theta + k_{g\eta}\sin^2\theta - Y^{(0)}H_x^2) w_{0,xx}(x,t) w_0(x,t) \\ +Y^{(0)}H_xH_z(w_{0,xx}(x,t)u_0(x,t) - u_{0,xx}(x,t)w_0(x,t)) - Y^{(2)}H_z^2 \\ \times (\frac{Y^{(0)}}{Y^{(2)}} u_{0,xx}(x,t)u_0(x,t) + c_1^2 w_{0,xxx}(x,t)w_{0,x}(x,t) - c_1c_2\varphi_{0,xx}(x,t) \\ \times w_{0,x}(x,t) - c_1c_2\varphi_0(x,t)w \quad {}^2_2{}_{0,xx}{}_{0}{}_{0,xxx} \end{bmatrix} \tag{33}$$

3.4.1 Orthotropic elastic foundation

$$V_{Orthotropic}^{ElasticMedium} = \frac{b}{2}\int_0^L F^e\, w_0(x,t)\, dx \tag{26}$$

where, F^e is the elastic medium forces based on Winkler spring constant and orthotropic Pasternak shear modulus that it is presented as follows (Soni *et al.* 2018)

$$F^e = k_w w_0(x,t) - k_{g\zeta} w_{0,xx}(x,t)\cos^2(\theta) - k_{g\eta} w_{0,xx}(x,t)\sin^2(\theta) \tag{27}$$

3.4.2 2D magnetic field

Based on the Maxwell relations between mechanical equations and 2D magnetic field the external work done can be defined as follows (Soni *et al.* 2018)

$$V_{2D}^{MagneticField} = \frac{b}{2}\int_0^L [f_{xl}u(x,z,t) + f_{zl}w(x,z,t)]dx \tag{28}$$

where f_{xl}, f_{zl}, u and w are the components of Lorentz forces in the x and z-directions and displacement vectors that these forces are investigated as the following form (Jelodari and Nilseresht 2018)

$$\vec{f_l} = (f_{xl}, f_{yl}, f_{zl}) = \eta(\vec{J}\times\vec{H}) \tag{29}$$

in which η, $\vec{J}$ and $\vec{H}$ are the magnetic permeability, magnetic current density vector and 2D magnetic intensity vector, respectively (Mohammadimehr *et al.* 2018a)

$$\vec{H} = (H_x, 0, H_z) \tag{30}$$

$$\vec{J} = \vec{\nabla}\times\vec{h} \tag{31}$$

In the Eq. (31), $\vec{h}$ is the perturbation of magnetic field vector and is derived based on the Eq. (32) as follows (Soni *et al.* 2018)

$$\vec{h} = \vec{\nabla}\times(\overrightarrow{(u,w)}\times\vec{H}) \tag{32}$$

3.4.3 External works

The work done by external elastic medium and magnetic fields forces is examined by substituting Eqs. (29) to (32) and Eq. (27) into the Eq. (28) and Eq. (26), respectively and substituting them into Eq. (25) yields the following equation

where $Y^{(0)}$ and $Y^{(2)}$ are the magnetic field constant and described as the following form

$$(Y^{(0)}, Y^{(2)}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \eta(1, z^2)\, dz \tag{34}$$

3.5 Equilibrium and governing equations of motion

In the present work, Hamilton's principle based on energy approach is employed to derive the equilibrium and governing equations of motions of micro composite Timoshenko beam reinforced by CNTs under mechanical and magnetic fields. According to the Hamilton's principle the total energy of micro system can be written as follows (Ansari *et al.* 2011 and Nguyen-Thanh *et al.* 2015)

$$\prod = T - (U + V) \tag{35}$$

Substituting Eqs. (20), (23) and (33) into the Eq. (35) lead to derived equilibrium equations of motions as follows

$$\Pi=-\frac{b}{2}\int_0^L\left\{\begin{array}{l} A_{11}u_{0,x}^2+(\mathrm{P}^{(0)}+T^{(0)})u_{0,xx}^2+c_1^2(\mathrm{P}^{(2)}+T^{(2)})\,\mathrm{w}_{0,\mathrm{xxx}}^2-2c_1c_2(\mathrm{P}^{(2)}+T^{(2)}) \\ \times\mathrm{w}_{0,\mathrm{xxx}}\,\varphi_{0,xx}+\left[c_1^2(\mathrm{D}_{11}+\mathrm{P}^{(0)})+(1+c_1)^2M^{(0)}+\frac{2}{3}(1-2c_1)^2T^{(0)}\right]\mathrm{w}_{0,xx}^2 \\ +c_2^2A_{55}\varphi_0^2+c_2^2(\mathrm{P}^{(2)}+T^{(2)})\varphi_{0,xx}^2+c_2^2(\mathrm{D}_{11}+M^{(0)}+\mathrm{P}^{(0)}+\frac{8}{3}T^{(0)})\varphi_{0,x}^2 \\ -2\left[c_1c_2(\mathrm{D}_{11}+\mathrm{P}^{(0)})+c_2(1+c_1)M^{(0)}-\frac{4}{3}c_2(1-2c_1)T^{(0)}\right]\mathrm{w}_{0,xx}\,\varphi_{0,x} \\ +k_w\,w_0^2-(\mathrm{k}_{g\xi}\cos^2\theta+\mathrm{k}_{g\eta}\sin^2\theta-Y_0^{(0)}H_x^2)\,w_{0,xx}\,w_0+Y_0^{(2)}H_xH_z \\ \times(w_{0,xx}\,u_0-u_{0,xx}\,w_0)-I^{(0)}\left(u_{0,t}^2+w_{0,t}^2\right)-I^{(2)}\begin{pmatrix} c_1^2\,w_{0,xt}^2+c_2^2\,\varphi_{0,t}^2 \\ -2\,c_1\,c_2\,w_{0,xt}\,\varphi_{0,t}\end{pmatrix} \\ -H_z^2\begin{pmatrix} Y_0^{(0)}u_{0,xx}\,u_0+c_1^2Y_0^{(2)}w_{0,xxx}\,w_{0,x}-c_1c_2Y_0^{(2)}\varphi_{0,xx}\,w_{0,x} \\ -c_1c_2Y_0^{(2)}\varphi_0\,w_{0,\mathrm{xxx}}+c_2^2Y_0^{(2)}\varphi_{0,xx}\,\varphi_0 \end{pmatrix} \end{array}\right\}dx \tag{36}$$

Also, the governing equations of motions of micro composite Timoshenko beam can be established by applying the variation operator on the equilibrium equations of motions and calculating the first variation of total energy by substituting Eqs. (7) and (36) into the Eq. (37) as follows

$$\delta\Pi=\int_0^{t_1}(\delta T-\delta U-\delta V)\,dt=0 \tag{37}$$

$$\begin{cases} \delta u_0: & (A_{11}+Y^{(0)}H_z^2)u_{0,xx}-(P^{(0)}+T^{(0)})u_{0,xxxx}=I^{(0)}u_{0,tt} \\ \delta w_0: & -k_w w_0+(A_{55}+k_{g\xi}\cos^2\theta+k_{g\eta}\sin^2\theta-2Y^{(0)}H_x^2)w_{0,xx}-A_{55}\theta_{x,x} \\ \delta\theta_x: & -A_{55}\theta+\left(D_{11}+M^{(0)}+P^{(0)}+\frac{8}{3}T^{(0)}+2Y^{(2)}H_z^2\right)\theta_{x,xx}-(P^{(2)}+T^{(2)})\theta_{x,xxxx} \\ & +A_{55}w_{0,x}+\left(M^{(0)}-\frac{4}{3}T^{(0)}-2Y^{(2)}H_z^2\right)w_{0,xxx}=I^{(2)}\theta_{x,tt} \end{cases} \tag{38}$$

As seen in the Eq. (38) δu_0 is independent δw_0 and $\delta\theta_x$. Thus, it can be said that this equation has no effect on the natural frequencies and the other parameters. In fact, u_0 can be neglected in the all of equations and numerical results can be solved based on w_0 , θ_x.

4. Numerical and Semi-analytical solution methods

In this study, Navier's solution method, GDQM and FEM are applied in order to solve the equilibrium and governing equations of micro composite Timoshenko beam reinforced by CNTs. In this section these semi-analytical and numerical methods are introduced:

4.1 Navier's type solution method

One of the semi-analytical approach that can employ to obtain the free vibration of the micro composite Timoshenko beam reinforced by CNTs rested in an orthotropic elastic medium is the Navier's type solution method for two edges simply supported boundary conditions. This method predicted the principle variables based on harmonic functions, so that the boundary conditions of the problem are satisfied. But what is important in this method is that the boundary conditions must be simply supported at the ends of structure. These S-S boundary conditions of Timoshenko micro beams at $x=0,L$ are given as follows (Lei *et al.* 2013a)

$$\begin{cases} w_0(x=0,L)=0 \\ w_{0,xx}(x=0,L)=0 \\ \theta_{x,x}(x=0,L)=0 \end{cases} \tag{39}$$

The following expansion of the displacement field can be written as follows

$$\begin{cases} w_0(x,t)=\sum_{m=1}^{\infty}W_m\sin(\frac{m\pi}{L}x)\,e^{i\omega t} \\ \theta_x(x,t)=\sum_{m=1}^{\infty}\theta_m\cos(\frac{m\pi}{L}x)\,e^{i\omega t} \end{cases} \tag{40}$$

where W_m, θ_m,ω and m are the undetermined Fourier coefficients, vibration frequency and transverse wave number, respectively. Substituting Eq. (40) into the Eq. (38) yields

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}-\omega^2\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}=0 \tag{41}$$

where

$$\begin{cases} K_{11} = (\frac{m\pi}{L})^4 \, (M^{(0)} + \frac{2}{3}T^{(0)}) + A_{55}(\frac{m\pi}{L})^2 \\ K_{12} = K_{21} = (\frac{m\pi}{L})^3 \, (M^{(0)} - \frac{4}{3}T^{(0)}) - A_{55}(\frac{m\pi}{L}) \\ K_{22} = (\frac{m\pi}{L})^4 \, (P^{(2)} + T^{(2)}) + (\frac{m\pi}{L})^2 \, (D_{11} + P^{(0)} + M^{(0)} + \frac{8}{3}T^{(0)}) + A_{55} \end{cases} \quad (42)$$

$$\begin{cases} M_{11} = I^{(0)} \\ M_{12} = M_{21} = 0 \\ M_{22} = I^{(2)} \end{cases} \quad (43)$$

In fact, for other boundary conditions, the Navier's method is incapable of solving the problem and does not provide accurate outputs. Therefore, it is necessary to use numerical methods to solve other boundary conditions. In the present work, authors examine the finite element method (FEM) and generalized differential quadrature method (GDQM).

4.2 Generalized differential quadrature method (GDQM)

In this present paper, GDQ and FE numerical methods are applied to solve the equilibrium and governing equations of motions of micro composite Timoshenko beam.

In this section, the Chebyshev-Gauss-Lobatto grid points are employed to perform GDQ method because this solution domain can predict nodes according to the critical conditions of the problem and it considers the number of nodes closer to the structural boundaries of the structure and decreases the number of nodes as it moves away from the boundaries.

Thus, the generate grid points in *x*-direction can be defined as follows (Ansari *et al.* 2014)

$$\psi_i = \frac{b-a}{2}\left(1 - cos\left(\frac{i-1}{N-1}\right)\pi\right) + a; \quad i = 1,2,\dots,N \quad (44)$$

where N, a and b are the grid point and two ends of the micro beam along the *x*-direction, respectively. Fig. 3 shows the effect of the Chebyshev-Gauss-Lobatto grid points on the length of micro composite Timoshenko beam model.

Moreover, in order to separate displacement field variable (Arani and Amir 2013)

$$\begin{cases} W_0(x,t) = W_0(x)e^{\omega t} \\ \theta_x(x,t) = \theta_x(x)e^{\omega t} \end{cases} \quad (45)$$

Thus, based on GDQM, the displacement functions are approximated as the follows (Arani and Amir 2013)

$$\psi_i = \frac{b-a}{2}\left(1 - cos(\frac{i-1}{N-1}\right)\pi) + a; \quad i = 1,2,\dots,N \quad (46)$$

where $C_{jm}^{(i)}$ refers to weighting coefficients corresponding to the first and higher order derivatives can be established as follows (Ansari *et al.* 2014 and Mohammadimehr *et al.* 2019)

$$[C_x^{(r)}]_{ij} = \begin{cases} I_x, r = 0 \\ \dfrac{\prod_{j=1;j\neq i}^{N}(x_i - x_j)}{(x_i - x_j)\prod_{i=1;i\neq j}^{N}(x_j - x_i)}, \begin{cases} r = 1 \\ i,j = 1,2,\dots,N \\ i \neq j \end{cases} \\ \lambda\left[C_{ij}^{(1)}C_{ii}^{(r-1)} - \dfrac{C_{ij}^{(r-1)}}{x_i - x_j}\right], \begin{cases} r = 2,3,\dots,N-1 \\ i,j = 1,2,\dots,N \\ i \neq j \end{cases} \\ -\sum_{j=1;j\neq i}^{N} C_{ij}^{(r)}, \begin{cases} r = 1,2,\dots,N-1 \\ i,j = 1,2,\dots,N \\ i = j \end{cases} \end{cases} \quad (47)$$

where I_x is $N \times N$ identity matrix. Applying the first and

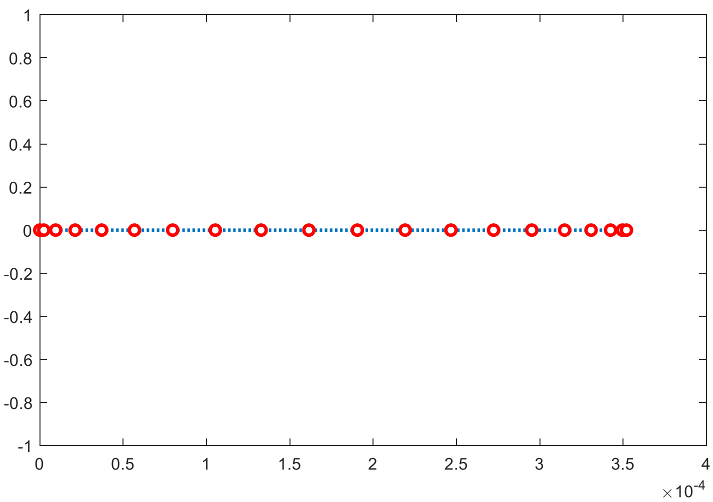


Fig. 3 The Chebyshev-Gauss-Lobatto grid point in *x*- direction of micro Timoshenko beam

higher order derivatives and boundary conditions lead to discretize governing equations of motions using the GDQ technique based on

$$[M_d]X_{d,tt}+[K_d]X_d=0,\quad X_d==[W_{01},\dots W_{0N},\dots,\theta_{x1},\dots,\theta_{xN}]^T \tag{48}$$

where, M_d, K_d and X_d denote the domain mass and stiffness matrices and displacement vector, respectively.

4.3 Finite element method (FEM)

In this study, the micro composite Timoshenko beam element has two nodes, in which two degrees of freedom associated with the degree of freedom of transverse displacement and bending rotation considered for each node.

It is noted that FEM cannot solve the simply supported boundary conditions, because this boundary condition is defined for a micro Timoshenko composite beam as follows

$$\begin{cases} W=0 \\ \dfrac{\partial^2 W}{\partial x^2}(x=0,L)=0 \end{cases} \tag{49}$$

In the above relation, it is need W and $\frac{\partial^2 W}{\partial x^2}$ apply in the stiffness and mass matrices, but FEM cannot predicted $\frac{\partial^2 W}{\partial x^2}$ in the stiffness matrix because stiffness matrix can define θ_x and W only. However, for clamped-clamped boundary condition are defined as follows

$$\begin{cases} W(x=0,L)=0 \\ \dfrac{\partial W}{\partial x}(x=0,L)=0 \end{cases} \tag{50}$$

In this B.C., stiffness matrix in the FEM is describes based on θ_x and W. Thus, FE method can predict the stiffness matrix components very well for C-C boundary conditions.

Thus, a cubic polynomial interpolation function is considered for displacement field as follow

$$w^e=a_0+a_1x+a_2x^2+a_3x^3 \tag{51}$$

in which a_0 , a_1, a_2 and a_3 are unknown constants. In Timoshenko beam theory, the shear deformation is equal to $\frac{\partial w}{\partial x}-\theta_x$, where θ_x is the bending rotation. It is clear displacement derivation with respect to *x* in the shear strain formula, reduced order of polynomial in Eq. (51), therefore a polynomial with same power should be consider for the bending rotation as follow

$$\theta^e=b_0+b_1x+b_2x^2 \tag{52}$$

in which b_0, b_1 and b_2 are unknown constants. The constant value assumption for the shear strain considered as follow

$$\frac{\partial w}{\partial x}-\theta=\varphi_0 \tag{53}$$

The unknown constants in Eqs. (51) and (52) determined from nodal variables at two ends of the element and using Eq. (53) as follows

$$\begin{cases} a_0=w_i \\ a_1=\theta_i+\varphi_0 \\ a_2=-\dfrac{3w_i+2l\theta_i+3l\phi_0-3w_j+l\theta_j}{l^2} \\ a_3=\dfrac{2w_i+l\theta_i+2l\phi_0-2w_j+l\theta_j}{l^3} \end{cases} \tag{54}$$

$$\begin{cases} b_0=\theta_i \\ b_1=-\dfrac{2\left(3w_i+2l\theta_i+3l\phi_0-3w_j+l\theta_j\right)}{l^2} \\ b_2=\dfrac{3\left(2w_i+l\theta_i+2l\phi_0-2w_j+l\theta_j\right)}{l^3} \end{cases} \tag{55}$$

Now, only one constant remains unknown in formulation which it is φ_0. Using the condition of minimum strain energy in element, the φ_0 value could be determined. The strain energy in Timoshenko beam element without foundation is the sum of bending and shear strain energies and calculated as following

$$\frac{\partial U^{(e)}}{\partial\varphi_0}=\frac{b}{2}\frac{\partial}{\partial\varphi_0}\left(\int_0^L\left\{\begin{array}{l} c_1^2(P^{(2)}+T^{(2)})\,w_{0,xxx}^{(e)2}+c_2^2(P^{(2)}+T^{(2)})\varphi_{0,xx}^{(e)2} \\ +\begin{bmatrix} c_1^2(D_{11}+P^{(0)})+(1+c_1)^2\,M^{(0)} \\ +\frac{2}{3}(1-2c_1)^2\,T^{(0)} \end{bmatrix}w_{0,xx}^{(e)2} \\ +c_2^2(D_{11}+M^{(0)}+P^{(0)}+\frac{8}{3}T^{(0)})\varphi_{0,x}^{(e)2} \\ +c_2^2A_{55}\varphi_0^{(e)2}-2c_1c_2(P^{(2)}+T^{(2)})\,w_{0,xxx}^{(e)}\,\varphi_{0,xx}^{(e)} \\ -2\begin{bmatrix} c_1c_2(D_{11}+P^{(0)}) \\ +c_2(1+c_1)M^{(0)} \\ -\frac{4}{3}c_2(1-2c_1)T^{(0)} \end{bmatrix}w_{0,xx}^{(e)}\,\varphi_{0,x}^{(e)} \end{array}\right\}dx\right) \tag{56}$$

which yields the following expression for the constant value of shear strain

$$\varphi_0=\Lambda\frac{2\left(w_j-w_i\right)-l\left(\theta_i+\theta_j\right)}{l} \tag{57}$$

where Λ is the constants value versus mechanical and magnetic property of micro composite Timoshenko beam. Substituting Eqs. (54) and (57) into Eqs. (51) and (52) yields the shape functions for Timoshenko beam element as following

$$\begin{cases} w^e = [N_w^1 \quad N_w^2 \quad N_w^3 \quad N_w^4][D] = [N_w][D] \\ {\theta_x}^e = [N_{\theta_x}^1 \quad N_{\theta_x}^2 \quad N_{\theta_x}^3 \quad N_{\theta_x}^4][D] = [N_{\theta_x}][D] \\ \varphi_0 = [N_{\varphi_0}^1 \quad N_{\varphi_0}^2 \quad N_{\varphi_0}^3 \quad N_{\varphi_0}^4][D] = [N_{\varphi_0}][D] \end{cases} \tag{58}$$

$$[D] = [w_i \quad \theta_i \quad w_j \quad \theta_j]^T \tag{59}$$

$$\begin{cases} N_w^1 = \dfrac{(x-l)(2x^2 + 2\Lambda xl - 4\Lambda x^2 - xl - l^2)}{l^3} \\ N_w^2 = \dfrac{x(x-l)(\Lambda l - 2\Lambda x + x - l)}{l^2} \\ N_w^3 = \dfrac{x(3xl + 4\Lambda x^2 + 2\Lambda l^2 - 2x^2 - 6\Lambda lx)}{l^3} \\ N_w^4 = \dfrac{x(x-l)(\Lambda l - 2\Lambda x + x)}{l^2} \end{cases} \tag{60}$$

$$\begin{cases} N_\theta^1 = \dfrac{6x(l-x)(2\Lambda - 1)}{l^3} \\ N_\theta^2 = \dfrac{(l-x)(l + 6\Lambda x - 3x)}{l^2} \\ N_\theta^3 = -\dfrac{6x(l-x)(2\Lambda - 1)}{l^3} \\ N_\theta^4 = \dfrac{x(3x - 6\Lambda x + 6\Lambda l - 2l)}{l^2} \end{cases} \tag{61}$$

The strain energy of the Timoshenko beam element with elastic foundation effect is written as following

$$U = \frac{b}{2}\int_0^{L_E} \left\{ \begin{aligned} &[D]^T \frac{\partial^3}{\partial x^3}[N_w]^T c_1^2 (P^{(2)} + T^{(2)}) \frac{\partial^3}{\partial x^3}[N_w][D] \\ &+[D]^T \frac{\partial^2}{\partial x^2}[N_w]^T \begin{bmatrix} c_1^2(D_{11} + P^{(0)}) + (1+c_1)^2 M^{(0)} \\ +\frac{2}{3}(1-2c_1)^2 T^{(0)} \end{bmatrix} \frac{\partial^2}{\partial x^2}[N_w][D] \\ &+[D]^T \frac{\partial^2}{\partial x^2}[N_{\varphi_0}]^T c_2^2 (P^{(2)} + T^{(2)}) \frac{\partial^2}{\partial x^2}[N_{\varphi_0}][D] \\ &+[D]^T \frac{\partial}{\partial x}[N_{\varphi_0}]^T c_2^2 (D_{11} + M^{(0)} + P^{(0)} + \frac{8}{3} T^{(0)}) \frac{\partial}{\partial x}[N_{\varphi_0}][D] \\ &-[D]^T \frac{\partial^3}{\partial x^3}[N_w]^T 2c_1 c_2 (P^{(2)} + T^{(2)}) \frac{\partial^2}{\partial x^2}[N_{\varphi_0}][D] \\ &-2[D]^T \frac{\partial^2}{\partial x^2}[N_w]^T \begin{bmatrix} c_1 c_2 (D_{11} + P^{(0)}) + c_2(1+c_1) M^{(0)} \\ -\frac{4}{3} c_2 (1-2c_1) T^{(0)} \end{bmatrix} \frac{\partial}{\partial x}[N_{\varphi_0}][D] \\ &+[D]^T [N_{\varphi_0}]^T c_2^2 A_{55} [N_{\varphi_0}][D] \end{aligned} \right\} dx \tag{62}$$

The kinetic energy of the Timoshenko beam element with inclusion of the rotary inertia effect is given by

$$T = \frac{b}{2}\int_0^{L} \begin{bmatrix} [\dot{D}]^T [N_w]^T I^{(0)} [N_w][\dot{D}] \\ +I^{(2)} \begin{pmatrix} [\dot{D}]^T \frac{\partial}{\partial x}[N_w]^T c_1^2 \frac{\partial}{\partial x}[N_w][\dot{D}] + [\dot{D}]^T [N_{\varphi_0}]^T c_2^2 [N_{\varphi_0}][\dot{D}] \\ -2[\dot{D}]^T \frac{\partial}{\partial x}[N_w]^T c_1 c_2 [N_{\varphi_0}][\dot{D}] \end{pmatrix} \end{bmatrix} dx \tag{63}$$

The external work done by can be written as follows

$$V=\frac{b}{2}\int_0^L\begin{bmatrix}[D]^T[N_w]^T k_w[N_w][D]-[D]^T\frac{\partial}{\partial x}[N_w]^T\begin{pmatrix}k_{g\xi}\cos^2(\theta)\\+k_{g\eta}\sin^2(\theta)\\-Y^{(0)}H_x^2\end{pmatrix}[N_w][D]\\ -Y^{(2)}H_z^2\begin{pmatrix}[D]^T\frac{\partial^3}{\partial x^3}[N_w]^T c_1^2\frac{\partial}{\partial x}[N_w][D]\\ -[D]^T\frac{\partial^2}{\partial x^2}[N_{\varphi_0}]^T c_1c_2\frac{\partial}{\partial x}[N_w][D]\\ -[D]^T[N_{\varphi_0}]^T c_1c_2\frac{\partial^3}{\partial x^3}[N_w][D]\\ +[D]^T\frac{\partial^2}{\partial x^2}[N_{\varphi_0}]^T c_2^2[N_{\varphi_0}][D]\end{pmatrix}\end{bmatrix}dx \quad (64)$$

To derive equation of motion, Lagrangian function is defined as follows

$$L=\sum\left(U^{(e)}-T^{(e)}+W^{(e)}\right) \quad (65)$$

and by using

$$\frac{d}{dt}\left(\frac{\partial L}{\partial[\dot{D}]}\right)-\frac{\partial L}{\partial[D]}=0 \quad (66)$$

leads to the governing equation of motion in matrix form as follows

$$[M][\ddot{D}]+[K][D]=[0] \quad (67)$$

By harmonic motion assumption with circular frequency, equilibrium equation of motion in Eq. (67) is changed to

$$([M]\omega^2+[K])[D]=0 \quad (68)$$

where ω, $[M]$ and $[K]$ are the natural frequencies, global consistent mass and stiffness matrices.

5. Validation

In the present study, in order to display the efficiency of the present model, the obtained results are compared with the other results of the previously papers. Table 4 shows the comparison of the first natural frequencies of the isotropic homogenous Timoshenko micro beam wit $E = 1440\ MPa$, $\rho = 1220\ Kg/m^3$, $\nu = 0.38$, $L = 10h$, $b = h$ and $l = 17.6\ \mu m$ for S-S boundary conditions based on MCST ($l_0 = l_1 = 0$, $l_2 = l$). As seen that there is a good agreement between the present obtained results with Navier's and GDQ methods and those of analytical solutions. Also, in order to validate the FEM, results of this method are compared with GDQM for C-C boundary conditions based on MSGT ($l_0 = l_1 = l_2 = l$).

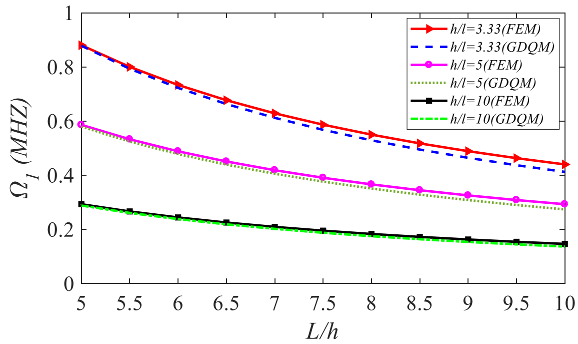


Fig. 4 Comparison of the GDQM and FEM first natural frequencies of Timoshenko micro composite beam for C-C boundary conditions based on MSGT

Fig. 4 depicted the comparison of the first natural frequencies of isotropic homogenous Timoshenko micro beam obtained from FE and GDQ methods based on MSGT. It is shown that there is a very small difference between the two methods. Thus, it is concluded that the FEM has a good agreement with the GDQM and it can use in the present work correctly.

Table 4 Comparison of the obtained first natural frequencies (MHZ) of homogenous Timoshenko micro beam with the other previous studies

h/l	First natural Frequencies (MHZ) for S-S				
	Yang *et al.* (2002)	Ma *et al.* (2008)	Ansari *et al.* (2011)	Navier	GDQM
3.33	0.1180	0.1229	0.1227	0.1229	0.1229
5	0.07636	0.07780	0.07782	0.07782	0.07782
10	0.03746	0.03760	0.03780	0.03765	0.03765

6. Results and discussion

In this work, the effects of various boundary conditions, semi-analytical and numerical solution methods, thickness-to-material length scale parameter and length-to-thickness ratios, 2D magnetic field, temperature changes, CNTs volume fractions, various UD and FG distribution of CNTs, Winkler and Pasternak constants are presented on the free vibration analysis of micro composite Timoshenko beam model reinforced by CNTs embedded in an orthotropic elastic foundation based on MSGT. It is assumed that the material length scale parameter of micro beam to be $l = 17.6\,\mu m$. Also, the used parameters in this section are considered as the following form

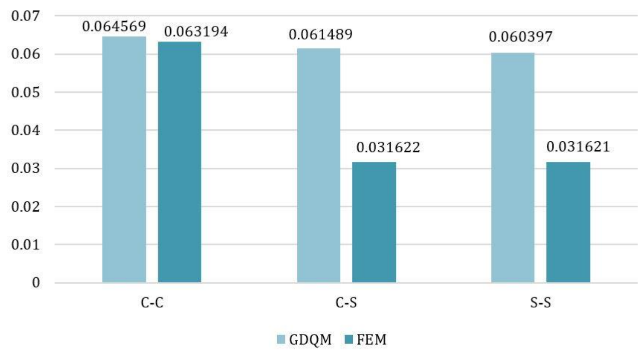


Fig. 5 Comparison of FEM and GDQM obtained results for C-C, C-S and S-S boundary conditions

$$\begin{cases} h = 4l; b = h; L = 8h; k_w = 2(GN/m^3); k_{g\zeta} = k_{g\eta} = 2000(N/m) \\ H_z = H_x = 2(A/\mu m); \Delta T = 50; \Omega_1 = \omega h\sqrt{\dfrac{I_{10}}{A_{110}}} \end{cases} \tag{69}$$

where

$$\begin{cases} A_{110} = \int_{-\frac{h}{2}}^{\frac{h}{2}} E_m dz \\ I_{10} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_m dz \end{cases} \tag{70}$$

Table 5 depicted the first natural frequencies of micro composite Timoshenko micro beam reinforced by CNTs with varying total number of grid point and number of elements in the GDQM and FEM, respectively. It is seen that the results were converged when $N_E \geq 110$ in FEM, while in GDQM, first natural frequencies were converged for $N \geq 11$. It is showed that C-C boundary conditions have the greatest consistency.

Fig. 5 compared the results of FEM and GDQM on the first natural frequencies for C-C, C-S and S-S boundary conditions. It has been shown that the C-C boundary conditions in the FEM are more consistent than the C-S and S-S boundary conditions. In fact, since the FEM cannot define $\frac{\partial w}{\partial x}$ in the stiffness matrix therefore, an error occurs in simply supported boundary conditions. thus, dimension-it can be said that FEM has the best performance in C-C boundary conditions while, GDQM has the best results for all boundary conditions. Therefore, in this section, the obtained results present based on Navier's and GDQ methods for simply-simply boundary conditions and based on FEM and GDQ methods for clamped-clamped boundary conditions.

Fig. 6 illustrates the effect of temperature changes on the first dimensionless natural frequencies of micro composite Timoshenko beam reinforced by CNTs resting on an orthotropic elastic foundation under 2D magnetic fields for S-S and C-C boundary conditions based on Navier's, FE and GDQ solution methods. It is showed that due to reducing the stiffness of micro structures the first less natural frequencies decrease when the temperature of system increases. Also, it is observed that the dimensionless natural frequencies in C-C boundary condition is more than S-S boundary condition because when the system supported is clamped the freedom of system is lower than simply boundary conditions.

Figs. 7 and 8 indicate effects of 2D magnetic field on the dimensionless natural frequencies of MSGT Timoshenko micro composite beam for S-S and C-C

Table 5 Convergence of elements and grid points for various boundary conditions based on FE and GDQ methods

Ω_{FEM}				Ω_{GDQM}			
Number of elements (N_E)	C-C	C-S	S-S	Number of grids (N)	C-C	C-S	S-S
10	0.063396	0.031625	0.031605	7	0.064561	0.061486	0.060396
30	0.063218	0.031623	0.031621	9	0.064569	0.061489	0.060397
50	0.063202	0.031622	0.031621	11	0.064569	0.061489	0.060396
70	0.063198	0.031622	0.031622	13	0.064569	0.061489	0.060396
90	0.063196	0.031622	0.031621	15	0.064569	0.061489	0.060396
110	0.063194	0.031622	0.031621	17	0.064569	0.061489	0.060396
130	0.063194	0.031622	0.031621	19	0.064569	0.061489	0.060396
150	0.063194	0.031622	0.031621	21	0.064569	0.061489	0.060396

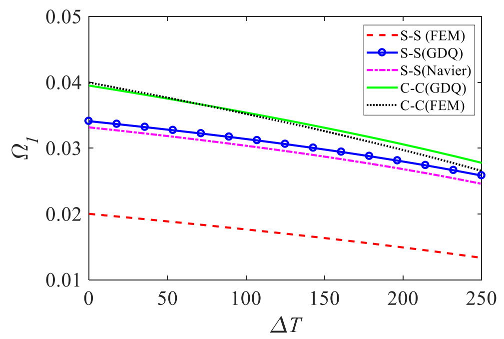
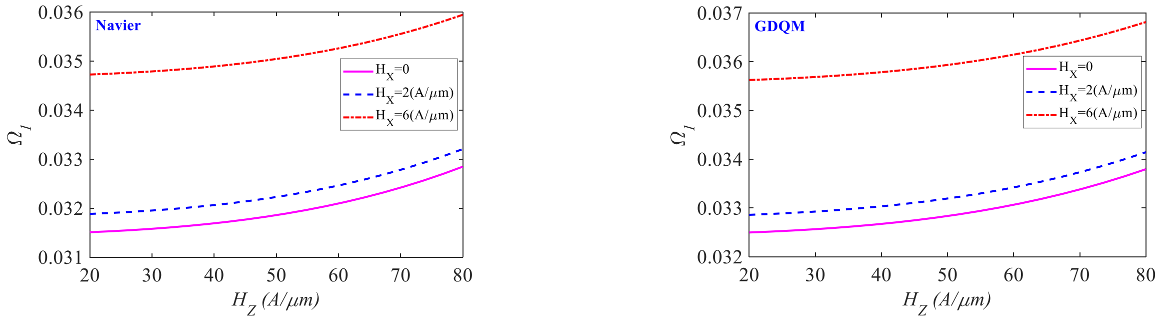


Fig. 6 Comparison of the first dimensionless natural frequencies of Timoshenko micro composite beam for various boundary conditions versus temperature changes

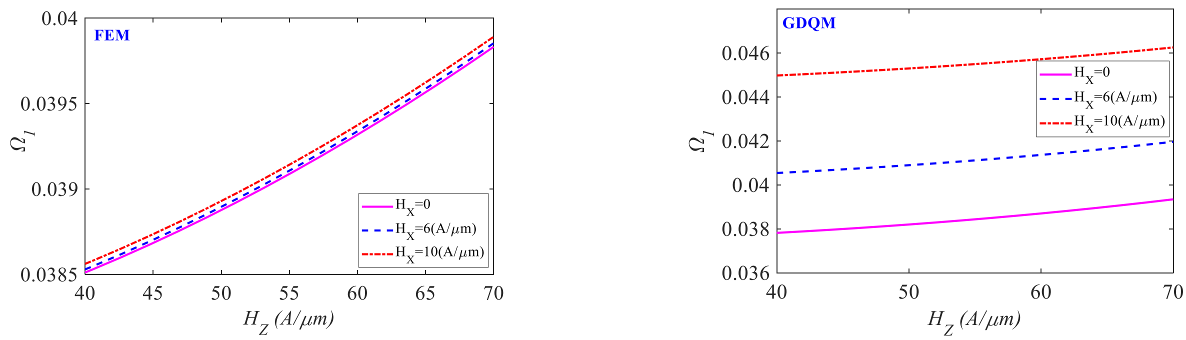
boundary conditions, respectively. It is investigated that the dimensionless natural frequency enhances by increasing the magnetic field in x and *z*-directions. Fig. 8 compared GDQ and Navier's type solution on the dimensionless natural frequencies S-S micro beam. This figure illustrated that there are a few differences between GDQ and Navier's methods (Table 6) and they almost the same behavior on the obtained results.

The dimensionless natural frequencies of the micro composite Timoshenko beam reinforced by CNTs for various distribution types of CNTs versus *L*/*h* is shown in Figs. 9 and 10 based on S-S and C-C boundary conditions, respectively. It is predictable that the micro composite Timoshenko beam reinforced by FG-*O* and FG-*X* CNTs have lowest and highest dimensionless natural frequency,



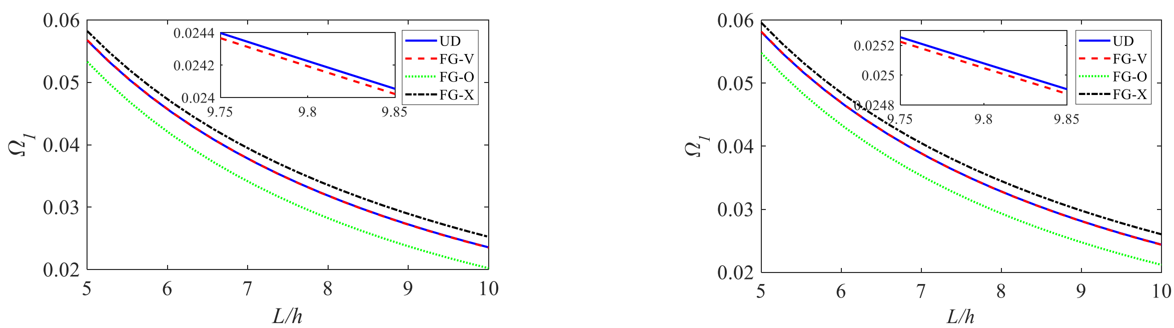
(a) (b)

Fig. 7 Comparison of 2D magnetic field on the first dimensionless natural frequencies for S-S boundary condition using: (a) Navier's; and (b) GDQ methods



(a) (b)

Fig. 8 Comparison of 2D magnetic field on the first dimensionless natural frequencies for C-C boundary condition using: (a) FE; and (b) GDQ methods



(a) (b)

Fig. 9 Comparison of (a) Navier's; and (b) GDQ methods on the first dimensionless natural frequencies for S-S boundary condition versus various type of CNTs distributions

Table 6 Comparison of Navier and GDQ output dimensionless natural frequencies based on 2D magnetic field

		Navier	GDQM	Error
$H_x = 0$	$H_z = 20$	0.03149	0.03249	3.078%
	$H_z = 50$	0.03184	0.03283	3.016%
	$H_z = 80$	0.03282	0.03379	2.871%
$H_x = 2(A/\mu m)$	$H_z = 20$	0.03187	0.03286	3.013%
	$H_z = 50$	0.03219	0.03319	3.013%
	$H_z = 80$	0.03319	0.03414	2.783%
$H_x = 6(A/\mu m)$	$H_z = 20$	0.03473	0.03562	2.499%
	$H_z = 50$	0.03503	0.03595	2.560%
	$H_z = 80$	0.03595	0.03682	2.363%

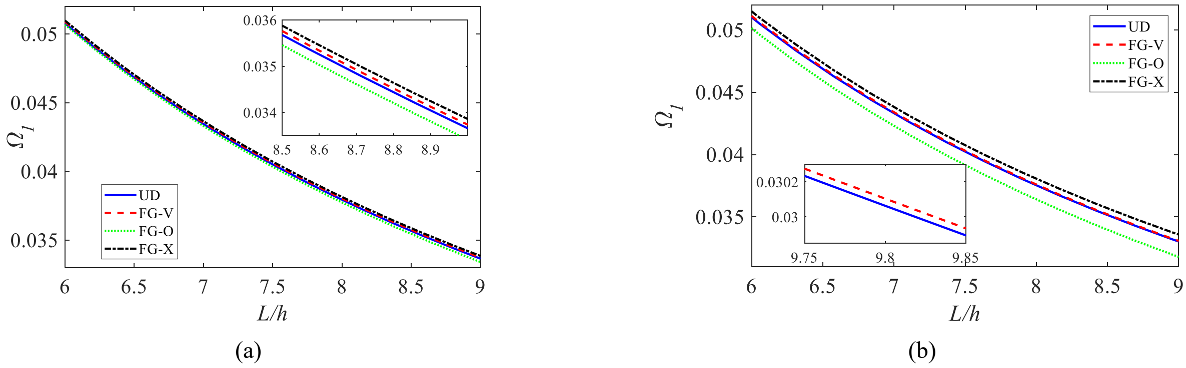


Fig. 10 Comparison of (a) FE; and (b) GDQ methods on the first dimensionless natural frequencies for C-C boundary condition versus various type of CNTs distributions

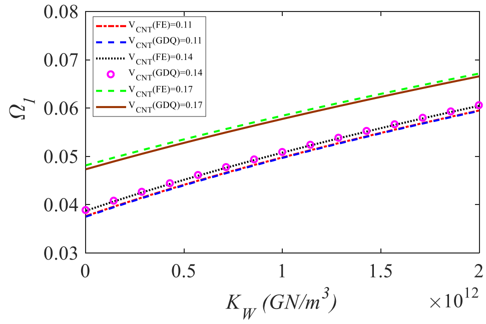


Fig. 11 Effects of Winkler constant and CNTs volume fraction on the first dimensionless natural frequencies for C-C boundary condition based on MSGT versus FEM and GDQM

respectively. Because of linearly distribution of CNTs lead to tolerate different loadings easily, thus in FG-*X* reinforcements, CNTs have the highest natural frequencies in comparison of the other UD and FG distributions. Also, these figures demonstrate that the micro beam stiffness has inverse relation with length of Timoshenko micro beams because the first dimensionless natural frequency reduces with increasing of the length to thickness ratio.

Fig. 11 compares the effects of Winkler spring constant and CNTs volume fraction on the dimensionless natural frequencies of Timoshenko micro composite beam reinforced by CNTs. This figure show that orthotropic elastic foundation and CNTs volume fraction have special effect on the stiffness of micro beams. Therefore, dimension-less natural frequencies increase with increasing of the Winkler constant and CNTs volume fraction. In fact, it can be said that the CNTs reinforcements lead to enhance the micro structures stiffness.

7. Conclusions

In this paper, free vibration analysis of a micro composite Timoshenko beam reinforced by various distribution of CNTs under temperature changes and 2D magnetic field is investigated based on MSGT. Moreover, a comparison between different solving methods in eigenvalue problems presented based on Navier's type solution, FEM and GDQM for S-S and C-C boundary conditions. The obtained results compared together and the effects of solution approach on the dimensionless natural frequencies developed. The effects of various parameter considered and showed that:

(1) In the present study, applying external works (2D magnetic field) had a good effect on the natural frequency.
(2) When the micro structure reinforced by carbon nanotubes and applied the magnetic field increases dimensionless natural frequencies.
(3) FG-*O* and FG-*X* CNTs had lowest and highest

dimensionless natural frequency, respectively.

(4) With increasing of temperature changes, the dimensionless natural frequencies of micro composite Timoshenko beam decreases.

(5) When length-to-thickness ratio of Timoshenko micro composite beam enhances, the dimensionless natural frequencies decreases.

(6) Carbon nanotubes volume fraction led to increase the stiffness of micro composite Timoshenko beam and dimensionless natural frequencies.

(7) The dimensionless natural frequencies of micro composite Timoshenko beam are enhanced by increasing the orthotropic elastic medium constants.

(8) The generalized differential quadrature method was the most consistent with analytical results. This method was responsive to all boundary conditions and the least error was observed in the obtained results.

(9) The finite element method has a good agreement with GDQM for clamped-clamped boundary condition.

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