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Design formulas for vibration control of sagged cables using passive MR dampers

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Abstract. In this paper, a method for analyzing the damping performance of stay cables incorporating magnetorheological (MR) dampers in the passive control mode is developed taking into account the cable sag and inclination, the damper coefficient, stiffness and mass, and the stiffness of damper support. Both numerical and asymptotic solutions are obtained from complex modal analysis. With the asymptotic solution, analytical formulas that evaluate the equivalent damping ratio of the sagged cable-damper system in consideration of all the above parameters are derived. The main thrust of the present study is to develop an general design formula and a universal curve for the optimal design of MR dampers for adjustable passive control of sagged cables. Two sag-affecting coefficients are derived to reflect the effects of cable sag on the maximum attainable damping ratio and the optimal damper coefficient. For the cable configurations commonly used in cable-stayed bridges, the sag-affecting coefficients are directly expressed in terms of the sag-extensibility parameter to facilitate the control design. A case study on adjustable passive vibration control of the longest cable (536 m) on Stonecutters Bridge is carried out to demonstrate the influence of the sag for the damper design, and to figure out the necessity of adjustability of damper coefficients for achieving maximum damping ratio for different vibration modes.

Keywords: sagged cable; vibration mitigation; MR damper; passive control; open-loop control

1. Introduction

Most of the previous research efforts on cable vibration control have been based on the assumption of taut strings. Adopting this assumption makes the analysis convenient and analytical solution possible. Kovacs (1982) was among the first to investigate cable vibration mitigation based on a taut string model. He studied the damper optimization by finding the intersection between the frequency response curves when the damper coefficient is zero and infinite, respectively. Making use of a taut string model, Yoneda and Maeda (1989) conducted complex eigenvalue analysis to evaluate the optimal damper coefficient and damping effect of cables attached with dampers. By judiciously grouping the cable and damper parameters into nondimensions, Pacheco *et al*. (1993) proposed a ‘universal curve’ of the normalized damping ratio versus the normalized damper coefficient for taut cables incorporating viscous dampers. Krenk (2000) and Main and Jones (2002a) investigated free vibrations of taut cables attached with viscous dampers via complex modal analysis method. Main and Jones (2002b) further extended their study to the free vibrations of taut cables attached with nonlinear dampers. Using complex modal analysis, Caracoglia and Jones (2007) explored the damping a taut cable using two viscous dampers on a single stay; Xu *et al*. (2007) proposed the method by using the adjustable fluid dampers to realize the reduction of cable vibration; Wu and Cai (2007) proposed a TMD-MR damper for vibration mitigation of a taut cable; Zhou *et al*. (2014a, 2015) investigated the free vibration of a taut cable with a spring, and with a damper and a concentrated mass; Zhou *et al*. (2014b) conducted the full-scale cable vibration mitigation experiment which shows the good capacity of viscous damper; Sun and Chen (2015) studied the free vibrations of a taut cable with a general viscoelastic damper modeled by fractional derivatives; and Cu *et al*. (2015, 2016) investigated the high damping rubber dampers and tuned mass-high damping rubber damper for a taut cable.

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Using the state-space method, Lu *et al*. (2017) investigated an inertial mass damper for mitigating taut cable vibration and its results implied that this kind of passive damper could provide a greater energy dissipation capacity than the traditional viscous damper. Duan *et al*. (2019) proposed the general design formulas for vibration control of taut cable using passive MR dampers. All these studies modelled the cable as a taut string.

On the other hand, advances in modern construction technology have resulted in increasing applications of long-span cables in cable-stayed bridges. For instance, as the world's first two cable-stayed bridges with their main span exceeding 1000 m, the Stonecutters Bridge has its longest stay cables of 536 m and the Sutong Bridge has its longest stay cables of 577 m. These long stay cables exhibit considerably large sags, and consequently, the effect of sag-extensibility on the cable dynamic characteristics is noticeable. A taut string model that neglects cable sag and inclination is valid for vibration control design of the cables with short to moderate lengths, but may result in inaccurate evaluation of damper parameters and damping performance in the vibration control design of long-span stay cables. Several researchers have examined this issue. Xu and Yu (1998) studied the mitigation of cable vibration using oil dampers by considering cable sag and inclination and showed that the sag significantly affected the maximum attainable damping ratio for long cables. Tabatabai and Mehrabi (2000) studied the influence of both cable sag and cable bending stiffness on the system damping ratios when using viscous dampers. Cremona (2001) generalized the Pacheco's 'universal curve' (Pacheco *et al*. 1993) for inclined cables by taking account of the sag-extensibility parameter. Krenk and Nielsen (2001) investigated vibrations of shallow cables attached with viscous dampers and derived an asymptotic solution accounting for the influence of cable sag. Krenk and Hogsberg (2005) studied the damping performance of shallow cables attached with visco-elastic dampers. Johnson *et al*. (2003) proposed a control-oriented model in consideration of cable sag and inclination and compared the damping performance of cables using passive, active and semiactive dampers, respectively. Wang *et al*. (2005) designed a new method for optimal design of viscous dampers to achieve multi-mode cable vibration control. Cheng *et al*. (2010) designed an optimal viscous damper for sagged cable using energy-based approach, without restriction on the damper location. Duan *et al*. (2018) investigated the damping effect of the viscous damper for a sagged cable using real-time hybrid simulation method.

Recent research interest on cable vibration control has been given to MR dampers and the first application of MR damping technology to bridge structures has been implemented on Dongting Lake Bridge for mitigation of cable vibration (Chen *et al*. 2004, Ko *et al*. 2002, Duan 2004, Duan *et al*. 2006). Johnson *et al*. (1999, 2003, 2007) formulated semi-active vibration control strategies for taut and sagged cables incorporating MR dampers, using state-space method. Duan *et al*. (2005) proposed the state-derivative feedback cable control using MR dampers. An experimental investigation of a taut cable with a MR damper was reported by Maslanka *et al*. (2007), which implied that an appropriately controlled MR damper provides a nearly constant damping in a wide range of cable vibration amplitudes. Or *et al*. (2008) developed the MR damper with embedded piezoelectric force sensors to facilitate the closed-loop cable vibration control. Zhou *et al*. (2006) investigated the semi-active control of three-dimensional vibrations of an inclined sag cable with magnetorheological dampers. Zhou *et al*. (2008) further studied the semi-active control performance of MR damper for shallow cable under harmonic axial support motion. Zhao and Zhu (2011) conducted the stochastic optimal semi-active control of stay cables by using magneto-rheological damper. Huang *et al*. (2012, 2015) used an optimally tuned MR damper to mitigate the vibration of stay cable and conducted a full-scale experimental verification for this optimally tuned MR damper. Chen *et al*. (2016) investigated the damping effect for stay cables using a newly developed self-sensing magnetorheological (MR) damper. From these researches, it is found that semi-active MR dampers with the aid of an appropriate real-time closed-loop control strategy are capable of offering much better damping performance than optimal passive dampers for cable vibration control.

In addition to the effectiveness in closed-loop control, MR dampers are superior to viscous dampers for cable vibration mitigation even when they are used in the passive mode (Duan 2004, Duan *et al*. 2016, Zhou and Sun 2013, Wang *et al*. 2018, Zhou *et al*. 2018). Viscous dampers of the same size can only afford optimal damping to one or a few cables on a cable-stayed bridge while the vibration in other cables attached with the same dampers may still fail to be suppressed due to insufficient damping. However, MR dampers of the same size can provide optimal or sub-optimal damping to each of the cables by tuning the damper voltage/current input to appropriate values. Even for a same cable, the dominant vibration mode can be different in different rain-wind-excitation events and can not be specified a priori. The passive (viscous) dampers designed for achieving optimal performance in one specific mode or appropriate (but not necessarily optimal) damping performance for several concerned modes are unalterable after they are installed and may provide insufficient damping even when the dominant mode is identified after the installation. Contrarily, MR dampers can provide the optimal (maximum) damping performance for the dominant mode after the dampers are installed, through adjusting the voltage/current input without changing the damper size. When being targeted to enhance the cable damping to suppress rain-wind-induced vibration, implementing MR dampers in an adjustable passive control mode is more practical and economical than in closed-loop control mode.

In this paper, a method is developed for analyzing the damping performance of inclined sagged cables incorporating MR dampers in the passive control mode, by using a three-element damper model (Powell 1994) and complex modal analysis (Krenk 2004). It takes into account the combined effect of the cable sag and inclination, damper coefficient, stiffness and mass, and stiffness of damper support. Then analytical formulas and a universal curve are derived from the asymptotic solution for designing the parameters of MR dampers to achieve the maximum damping ratio for a sagged cable. A comprehensive parametric study is conducted to evaluate the effects of cable and damper parameters on the system damping performance. Based on the parametric study, two sag-affecting coefficients are defined for quantifying the influence of cable sag and inclination on the system damping ratio and the optimal damper coefficient. A case study is carried out on the adjustable passive control of the longest cable (536 m) on Stonecutters Bridge.

2. Formulation

2.1 Equation of motion

An inclined sagged cable incorporating MR dampers is shown in Fig. 1. The distance between the supports is *l*, and the inclination angle (the angle between the chord *OO'* and the horizon) is θ. The MR damper subsystem, represented by the concentrated damper mass *M*, stiffness coefficient k_e, viscous coefficient c_e, frictional coefficient F_I and support stiffness k_s, is located at the distance x_d from the lower end. The cable has an axial elastic stiffness *EA*, and the ends of the cable are supported via springs with linear stiffness k_1 and k_2. The sag at the mid span is *f*. When $f/l \ll \frac{1}{8}$, the cable can be described using the shallow-cable theory that is based on the assumption of uniform mass distribution along the horizontal projection of the cable (Irvine 1981). The mass per unit length and the cable tension force along the chord *OO'* (x-axis) are denoted as m and T_0.

The governing equation for free oscillation of the cable-damper system can be expressed as Irvine (1981)

$$-T_0\frac{\partial^2}{\partial x^2}y(x,t)-T_0\frac{\partial^2}{\partial x^2}y_0(x,t)+m\frac{\partial^2 y(x,t)}{\partial t^2}+F_d\delta(x-x_d)=0 \qquad (2)$$

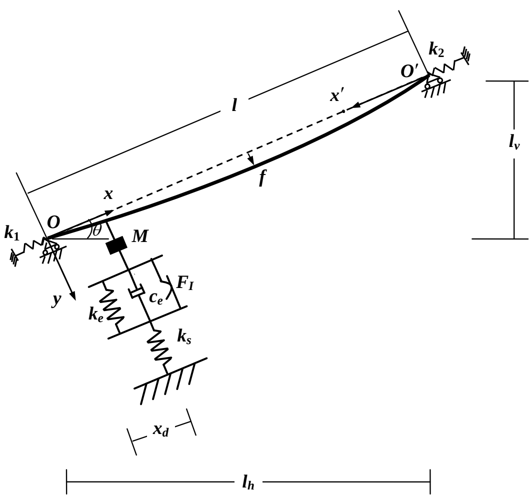


Fig. 1 Sagged cable-damper system

where y is the cable dynamic response around its static equilibrium configuration y_0.; T is the additional tension force due to cable vibration; and F_d is the MR damper force represented by Powell (1994)

$$F_d = My_d + k_e\Delta + c_e\dot{\Delta} + F_I\,\mathrm{sgn}(\dot{\Delta})$$
$$y_d = (1+\frac{k_e}{k_s})\Delta + \frac{c_e}{k_s}\dot{\Delta} + \frac{1}{k_s}F_I\,\mathrm{sgn}(\dot{\Delta}) \qquad (2a,b)$$

in which y_d is the cable movement at the damper location; and Δ the relative displacement between the damper piston and cover. The static equilibrium configuration y_0 and the sag at mid span *f* are expressed by

$$y_0(x) = 4f(1-\frac{x}{l})\frac{x}{l}\ ,\ f=\frac{mgl^2\cos\theta}{8T_0} \qquad (3a,b)$$

The additional tension force T stemming from cable vibration can be obtained by Krenk and Nielsen (2001)

$$T\left(\frac{L_e}{EA}+\frac{1}{k_1}+\frac{1}{k_2}\right)=\frac{8f}{l^2}\int_0^l y(x,t)dx \qquad (4)$$

where $L_e=\int_0^l\left(\frac{ds}{dx}\right)^3 dx \approx \left[1+8\left(\frac{f}{l}\right)^2\right]l$ is the effective length and can be approximated as *l* in practice.

2.2 Complex modal analysis

The cable vibration, additional cable tension force, damper force, and damper relative displacement are written in the following complex format

$$y(x,t)=\tilde{y}(x)\exp(i\omega t),\ T=\tilde{T}(x)\exp(i\omega t)$$
$$F_d=\tilde{F}_d(x)\exp(i\omega t),\ \Delta=\tilde{\Delta}\exp(i\omega t) \qquad (5a\text{-}d)$$

Following the similar procedure of complex modal analysis by Krenk and Nielsen (2001), the determinant equation that permits non-trivial solutions for the cable vibration amplitude at the damper location $\tilde{y}_d$ and the additional tension force amplitude $\tilde{T}$ is obtained as

$$\sin\left(\frac{\beta l}{2}\right)\left\{\sin\left(\frac{\beta l}{2}\right)-\cos\left(\frac{\beta l}{2}\right)\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]\right\}=$$
$$-2\left[\frac{1}{T_0\beta}\frac{\tilde{F}_d}{\tilde{y}_d}\right]\sin\left(\frac{\beta x_d}{2}\right)\sin\left(\frac{\beta x'_d}{2}\right)\times \qquad (6)$$
$$\left\{\sin\left(\frac{\beta l}{2}\right)-\cos\left(\frac{\beta x_d}{2}\right)\cos\left(\frac{\beta' x_d}{2}\right)\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]\right\}$$

where $x'_d = l - x_d$, β is the wavenumber defined as

$$\beta=\omega\sqrt{\frac{m}{T_0}} \qquad (7)$$

and λ^2 is the sag-extensibility parameter defined by Irvine and Caughey (1974)

$$\frac{1}{\lambda^2}=\left(\frac{L_e/l}{EA}+\frac{1}{k_1 l}+\frac{1}{k_2 l}\right)\frac{T_0^3}{(mgl\cos\theta)^2} \tag{8}$$

It is clear from Eq. (6) that the effect of the damper is reflected in the right-hand side of the equation and that the left-hand side of the equation will be equal to zero when the cable is undamped. It is also seen that the effect of cable sag on the damping ratio of the cable-damper system is uncoupled with all the damper parameters (damper mass, stiffness coefficient, viscous coefficient, frictional coefficient and support stiffness).

Using the equivalent energy method (Weber and Boston 2010, Huang and Jones 2011, Duan *et al*. 2019), an equivalent damping coefficient for frictional force can be obtained

$$W=\int_0^T F_I\,\mathrm{sgn}(\dot\Delta)\cdot\dot\Delta dt=4F_I\|\Delta\|,$$
$$W=\int_0^T c_I\dot\Delta\cdot\dot\Delta dt=c_I\pi\|\dot\Delta\|\|\Delta\|, c_I=\frac{4F_I}{\pi\|\dot\Delta\|} \tag{9a-c}$$

where, c_I is the equivalent viscous coefficient of frictional force; $\|\dot\Delta\|$ is the velocity amplitude of damper motion; $\|\Delta\|$ is the displacement amplitude of damper motion, is the energy dissipated within one cycle by dampers.

Substituting Eqs. (5(c)), (5(d)) into Eq. (2) and replacing $F_I\,\mathrm{sgn}(\dot\Delta)$ by $c_I\dot\Delta$ leads to

$$\tilde{y}_d=\left\{1+\frac{1}{k_s}\left[k_e+c_e(i\omega)+c_I(i\omega)\right]\right\}\tilde\Delta$$
$$\tilde{F}_d=-M\omega^2\tilde{y}_d+\left[k_e+c_e(i\omega)+c_I(i\omega)\right]\tilde\Delta \tag{10a,b}$$

For brevity, the following dimensionless parameters are defined

$$\eta_c=\frac{c_e}{\sqrt{T_0 m}}, \eta_I=\frac{c_I}{\sqrt{T_0 m}}, u_k=\frac{k_e l}{T_0}$$
$$u_s=\frac{k_s l}{T_0}, \gamma_M=\frac{M(\omega_n^0)^2 l}{T_0} \tag{11a-e}$$

where ω^2, n = 1, 2, 3… is the circular frequency of the undamped cables.

By combining Eqs. (2), (5), (10) and (11), Eq. (6) can be expressed as

$$\sin(\frac{\beta l}{2})\left\{\sin(\frac{\beta l}{2})-\cos(\frac{\beta l}{2})\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}(\frac{\beta l}{2})^3\right]\right\}=$$
$$-2\frac{1}{\beta l}\left\{\frac{\left[u_k+i\beta l(\eta_c+\eta_I)\right]}{1+\frac{1}{u_s}\left[u_k+i\beta l(\eta_c+\eta_I)\right]}-\gamma_M(\frac{\omega_n}{\omega_n^0})^2\right\}\times$$
$$\sin\left(\frac{\beta x_d}{2}\right)\sin\left(\frac{\beta x_d'}{2}\right)\times$$
$$\left\{\sin\left(\frac{\beta l}{2}\right)-\cos\left(\frac{\beta x_d}{2}\right)\cos\left(\frac{\beta' x_d}{2}\right)\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]\right\} \tag{12}$$

which will be used for solution of the wavenumbers for the cable-damper system taking into account the (dimensionless) damper mass γ_M, stiffness coefficient u_k, viscous coefficient η_c, frictional coefficient η_I, support stiffness u_s, as well as the cable sag and inclination characterized by λ^2.

3. Solution of free vibrations

The free oscillation of a damped cable with sag and inclination occurs in two types of vibration modes: nearly antisymmetric modes and nearly symmetric modes (Krenk and Nielsen 2001). In the nearly antisymmetric modes, the cable is lifted at one side to the mid-span while being lowered at the opposite side with respect to the static profile; in the nearly symmetric modes, the cable is lifted simultaneously at both sides (for small λ^2), or the side parts of the cable near the supports are lowered when the middle part of the cable is lifted with respect to the static profile (for large λ^2). Both numerical and asymptotic solutions for the nearly antisymmetric and nearly symmetric vibrations are obtained.

3.1 Nearly antisymmetric vibrations

3.1.1 Numerical solution

By dividing both sides by $\cos(\frac{\beta l}{2})$, Eq. (12) can be expressed as

$$\tan(\frac{\beta l}{2})=\frac{2\Phi\sin^2(\frac{\beta x_d}{2})}{\left\{\Omega+2\Phi\sin(\frac{\beta x_d}{2})\cos(\frac{\beta x_d}{2})\right\}} \tag{13a}$$

where Φ, which reflects the effect of the damper force, is expressed by

$$\Phi=\frac{1}{T_0\beta}\frac{\tilde{F}_d}{\tilde{y}_d}=\frac{1}{\beta l}\left\{\frac{\left[u_k+i\beta l(\eta_c+\eta_I)\right]}{1+\frac{1}{u_s}\left[u_k+i\beta l(\eta_c+\eta_I)\right]}-\gamma_M(\frac{\omega_n}{\omega_n^0})^2\right\} \tag{13b}$$

and Ω, which reflects the effect of the sag-extensibility, is expressed by

$$\Omega=\frac{\sin(\frac{\beta l}{2})-\cos(\frac{\beta l}{2})\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]}{\sin(\frac{\beta l}{2})-\cos(\frac{\beta x_d}{2})\cos(\frac{\beta x_d'}{2})\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]} \tag{13c}$$

The numerical solution is obtained by the fixed-point iteration of Eq. (13), i.e., substituting the current estimate of βl into the right-hand side and then using the equation to calculate a new estimate (Duan 2004). The iteration is started from the wavenumber of undamped vibrations

$$\beta_n^0 l = n\pi, n = 2,4,\ldots. \quad (14)$$

The solution is found on a branch of the tangent function where

$$2\tan(\frac{\beta_n l}{2}) = 2\tan(\frac{\beta_n l}{2} - k\pi) = \tan(\frac{\beta_n l - n\pi}{2}) \approx \beta_n l - n\pi = \beta_n l - \beta_n^0 l \quad (15)$$

where n=2k and k = 1, 2, 3, ...

The numerical solution converges rapidly when $\frac{nx_d}{l}$ is less than 0.5, i.e., when the damper is located between the cable end and the nearest antinode. After solving $\beta_n l$, the equivalent modal damping ratio of the cable-damper system for nearly antisymmetric vibrations is obtained as (Duan 2004)

$$\xi = \frac{\mathrm{Im}(\beta_n l)}{\mathrm{Re}(\beta_n l)} \quad (16)$$

3.1.2 Asymptotic solution

The asymptotic solution is obtained by substituting the undamped wavenumbers $\beta_n^0 = \frac{n\pi}{l}$ into the right-hand side of Eq. (13(a)), which leads to

$$\tan(\frac{\beta_n l}{2}) = \frac{\frac{1}{2}\Phi\sin^2(\beta_n^0 x_d)}{1+\Phi\sin(\beta_n^0 x_d)\cos^2(\frac{\beta_n^0 x_d}{2})} \quad (17)$$

Using the approximation given in Eq. (15) and substituting Eq. (13b) into Eq. (17) leads to

$$\beta_n l = \beta_n^0 l \times \left\{1+\frac{\left[B+i\beta_n^0 l(\eta_c+\eta_I)\left(1-\frac{\gamma_M}{u_s}\right)\right]S^2\frac{x_d}{l}}{\left(1+BSC'\frac{x_d}{l}+\frac{u_k}{u_s}\right)+i\beta_n^0 l(\eta_c+\eta_I)\left[\frac{1}{u_s}+\left(1-\frac{\gamma_M}{u_s}\right)SC'\frac{x_d}{l}\right]}\frac{x_d}{l}\right\} \quad (18)$$

where

$$B = \left[u_k(1-\frac{\gamma_M}{u_s}) - \gamma_M\right],$$
$$S = \frac{\sin(\beta_n^0 x_d)}{\beta_n^0 x_d}, \quad (19a\text{-}c)$$
$$C' = \cos^2(\frac{\beta_n^0 x_d}{2})$$

From Eqs. (18) and (16), the normalized damping ratio of the cable-damper system for nearly antisymmetric vibrations is obtained as

$$\frac{\xi_n}{\frac{x_d}{l}} = \frac{\beta_n^0 l(\eta_c+\eta_I)S^2\frac{x_d}{l}}{[1+U_{k,M,s}+U_{k,M}]^2+\left[\beta_n^0 l(\eta_c+\eta_I)V_{M,s}C'S\frac{x_d}{l}\right]^2} \quad (20)$$

where

$$U_{k,M,s} = \frac{u_k}{u_s}(1-\gamma_M C'S\frac{x_d}{l}),$$
$$U_{k,M} = (u_k-\gamma_M)C'S, \quad (21a\text{-}c)$$
$$V_{M,s} = \left(\frac{1}{u_s}(1-\gamma_M C'S\frac{x_d}{l})+C'S\frac{x_d}{l}\right)/\left(C'S\frac{x_d}{l}\right)$$

In the above, $U_{k,M,s}$ is a factor accounting for the coupled effect of the damper stiffness u_k, damper mass γ_M and support stiffness u_s; $U_{k,M}$ is a factor accounting for the coupled effect of the damper stiffness and mass; and $V_{M,s}$ is a factor accounting for the coupled effect of the damper mass and support stiffness. When $\frac{nx_d}{l} \le 0.15$, the solution is almost unaffected when approximating S and C' to 1(Duan 2004). For nearly antisymmetric vibrations the value of β_n^0 is equal to $\frac{n\pi}{l}$.

Fig. 2 illustrates the numerical and asymptotic solutions for nearly antisymmetric vibrations of a sagged cable attached with an ideal viscous damper ($u_k = 0$, $\gamma_M = 0$) at the location $\frac{x_d}{l} = 0.02$ with ideal support ($1/u_s = 0$). It is seen that the asymptotic solution provides a very good approximation of the numerical solution and the cable sag has a negligible influence on the curve of the normalized system damping ratio $\xi/(x_d/l)$ versus the normalized damper coefficient κ which is defined as

$$\kappa = \frac{1}{\pi}(\eta_c+\eta_I)n\frac{x_d}{l} \quad (22)$$

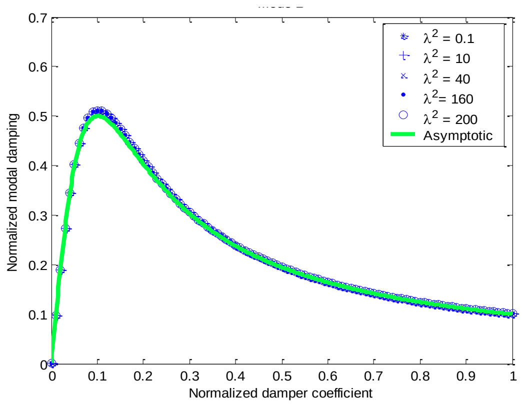
3.2 Nearly symmetric vibrations

3.2.1 Numerical solution

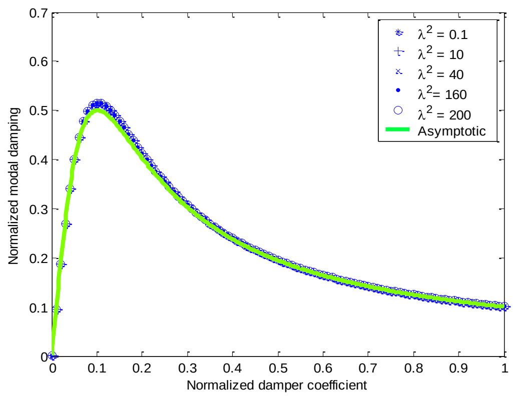
By dividing both sides of Eq. (12) by $\sin(\frac{\beta l}{2})$ and using a trigonometric relation, one obtains

$$\tan(\frac{\beta l}{2}) = \left[\frac{\beta l}{2} - \frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right] + \frac{2\Phi\Sigma^2\left[\frac{\beta l}{2} - \frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]^2}{1+2\Phi\Sigma\left\{1-\Sigma\left[\frac{\beta l}{2} - \frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right]\right\}} \quad (23)$$

where Φ is defined in Eq. (13(b)) and



(a) 2nd mode



(b) 4th mode

Fig. 2 Numerical and asymptotic solutions for nearly antisymmetric vibrations of a damped cable under different sag (x_d/l= 0.02): (a) 2nd mode and (b) 4th mode

$$\Sigma = \frac{\sin(\frac{\beta x_d}{2})\sin(\frac{\beta x'_d}{2})}{\sin(\frac{\beta l}{2})} \tag{24}$$

The numerical solution can be obtained by the fixed-point iteration of Eq. (23), i.e., substituting the current estimate of βl into the right-hand side and then using the equation to calculate a new estimate. The initial guess for iteration is taken as the wavenumber of undamped vibrations. After solving $\beta_n l$, the equivalent modal damping ratio of the cable-damper system for nearly symmetric vibrations is calculated using Eq. (16).

3.2.2 Asymptotic solution

The asymptotic solution for nearly symmetric vibrations of the cable-damper system can be derived upon some approximations.

$$\tan(\frac{\beta l}{2})-\left[\frac{\beta l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3\right] \approx \tan(\frac{\beta_n^0 l}{2})-\left[\frac{\beta_n^0 l}{2}-\frac{4}{\lambda^2}\left(\frac{\beta_n^0 l}{2}\right)^3\right] + \left[\tan^2(\frac{\beta_n^0 l}{2})+\frac{12}{\lambda^2}\left(\frac{\beta_n^0 l}{2}\right)^2\right]\frac{\beta_n l-\beta_n^0 l}{2} \tag{25}$$

The left-hand side of Eq. (25) is linearized around the undamped value. By introducing Eq. (25) into Eq. (23), one obtains

$$\frac{\beta_n l-\beta_n^0 l}{2} = \frac{\Phi\sin^2(\beta_n^0 x_d)\Xi\tan(\frac{\beta_n^0 l}{2})}{2\left[1+\Phi\sin(\beta_n^0 x_d)\Xi\right]\left[\tan^2(\frac{\beta_n^0 l}{2})+\frac{12}{\lambda^2}\left(\frac{\beta_n^0 l}{2}\right)^2\right]} \tag{26}$$

where

$$\Xi = \left[1-\frac{\tan\left(\frac{\beta_n^0 x_d}{2}\right)}{\tan\left(\frac{\beta_n^0 l}{2}\right)}\right] \tag{27}$$

The approximation given in Eq. (27) is valid for $\frac{x_d}{l} << 1$ and $\tan(\frac{\beta_n l}{2}) \neq 0$. Using this approximation and substituting the expression of Φ into Eq. (26) leads to

$$\beta_n l = \beta_n^0 l \times \left(1+\frac{\left[B+i\beta_n^0 l(\eta_c+\eta_l)\left(1-\frac{\gamma_M}{u_s}\right)\right]S^2\frac{x_d}{l}}{\left(1+BSC''\frac{x_d}{l}+\frac{u_k}{u_s}\right)+i\beta_n^0 l(\eta_c+\eta_l)\left[\frac{1}{u_s}+\left(1-\frac{\gamma_M}{u_s}\right)SC''\frac{x_d}{l}\right]}\frac{x_d}{l}\frac{1}{W_{\xi,\lambda^2}}\right) \tag{28}$$

where

$$B=\left[u_k(1-\frac{\gamma_M}{u_s})-\gamma_M\right], \quad S=\frac{\sin(\beta_n^0 x_d)}{\beta_n^0 x_d}, \quad C''=1,$$
$$W_{\xi,\lambda^2}=1+\frac{12}{\lambda^2}\left(\frac{\beta_n^0 l}{2}\right)^2 \Big/ \tan^2(\frac{\beta_n^0 l}{2}) \tag{29a-d}$$

From Eqs. (28) and (16), the normalized damping ratio of the cable-damper system for nearly symmetric vibrations is obtained as

$$\frac{\xi_n}{\frac{x_d}{l}} = \frac{\beta_n^0 l(\eta_c+\eta_l)S^2\frac{x_d}{l}}{[1+U_{k,M,s}+U_{k,M}]^2+\left[\beta_n^0 l(\eta_c+\eta_l)V_{M,s}C''S\frac{x_d}{l}\right]^2}\frac{1}{W_{\xi,\lambda^2}} \tag{30}$$

where

$$U_{k,M,s}=\frac{u_k}{u_s}(1-\gamma_M C''S\frac{x_d}{l}),$$
$$U_{k,M}=(u_k-\gamma_M)C''S,$$
$$V_{M,s}=\left(\frac{1}{u_s}(1-\gamma_M C''S\frac{x_d}{l})+C''S\frac{x_d}{l}\right)\Big/\left(C''S\frac{x_d}{l}\right) \tag{31a-c}$$

In the above, $U_{k,M,s}$, $U_{k,M}$ and $V_{M,s}$ are factors accounting for the coupled effect of the damper stiffness u_k, damper mass γ_M and support stiffness u_s; and W_{ξ,λ^2} is the sag-affecting coefficient of damping ratio characterizing the effect of cable sag on the damping ratio, which is equal to 1 when $\lambda^2 = 0$, and larger than 1 when $\lambda^2 > 0$. When $\frac{x_d}{l} << 1$, the solution is almost unaffected when approximating S and C' to 1 (Duan 2004).

For nearly symmetric vibrations the value of β_n^0 is different from $\frac{n\pi}{l}$. The sag-affecting coefficient of damper coefficient is introduced here to reflect this difference and for convenience of expression

$$W_{\eta,\lambda^2} = \frac{\beta_n^0 l}{n\pi} \tag{32}$$

which is equal to 1 when $\lambda^2 = 0$ and larger than 1 when $\lambda^2 > 0$, for $n = 1, 3, 5, \ldots$.

Examining Eq. (20) for nearly antisymmetric vibrations (modes of even orders) and Eq. (30) nearly symmetric vibrations (modes of odd orders) and generalizing the sag-affecting coefficient of damping ratio W_{ξ,λ^2} and sag-affecting coefficient of damper coefficient W_{η,λ^2} as

$$W_{\xi,\lambda^2} = \begin{cases} 1 + \dfrac{12}{\lambda^2}\left(\dfrac{\beta_n^0 l}{2}\right)^2 \Big/ \tan^2\left(\dfrac{\beta_n^0 l}{2}\right) & n = 2k-1 \\ 1 & n = 2k \end{cases} \quad k = 1, 2, 3, \ldots \tag{33}$$

$$W_{\eta,\lambda^2} = \begin{cases} \beta_n^0 l / n\pi & n = 2k-1 \\ 1 & n = 2k \end{cases} \quad k = 1, 2, 3, \ldots \tag{34}$$

one obtains a general formula for evaluating the equivalent modal damping ratio of the sagged cable-damper system as

$$\frac{\xi_n}{\frac{x_d}{l}} = \frac{n\pi W_{\eta,\lambda^2}(\eta_c + \eta_I)S^2 \frac{x_d}{l}}{[1 + U_{k,M,s} + U_{k,M}]^2 + \left[n\pi W_{\eta,\lambda^2}(\eta_c + \eta_I)V_{M,s}CS\frac{x_d}{l}\right]^2} \frac{1}{W_{\xi,\lambda^2}} \tag{35}$$

where C and S are approximated as 1 when $\frac{x_d}{l} < 0.15$ (Duan 2004); or in the expression

$$\frac{\xi_n}{\frac{x_d}{l}} = \frac{\pi^2 \kappa W_{\eta,\lambda^2} S^2}{[1 + U_{k,M,s} + U_{k,M}]^2 + \left[\pi^2 \kappa W_{\eta,\lambda^2} V_{M,s} CS \frac{x_d}{l}\right]^2} \frac{1}{W_{\xi,\lambda^2}} \tag{36}$$

where κ is the normalized damper coefficient defined in Eq. (22). By defining a generalized system damping ratio and a generalized damper coefficient as

$$\frac{\xi^*}{\frac{x_d}{l}} = \frac{\xi}{\frac{x_d}{l}} W_{\xi,\lambda^2}, \quad \kappa^* = \kappa W_{\eta,\lambda^2} = \frac{1}{\pi} W_{\eta,\lambda^2}(\eta_c + \eta_I) n \frac{x_d}{l} \tag{37a,b}$$

Eq. (36) becomes

$$\frac{\xi_n^*}{\frac{x_d}{l}} = \frac{\kappa^* \pi^2 S^2}{[1 + U_{k,M,s} + U_{k,M}]^2 + \left[\kappa^* \pi^2 V_{M,s} CS \frac{x_d}{l}\right]^2} \tag{38}$$

Eqs. (35), (36) and (38) are general formulas that can be used to evaluate the attainable damping ratio for a cable-damper system, taking into account all the damper parameters as well as the cable sag and inclination. It is seen from Eq. (36) that a ‘universal curve’ of the normalized system damping ratio versus the normalized damper coefficient similar to the Pacheco’s curve (Pacheco *et al*. 1993) cannot be obtained for sagged cables because this relation also depends on the sag-extensibility parameter λ^2. However, after introducing the definition given in Eq. (37), the relation between the generalized system damping ratio and the generalized damper coefficient is independent of λ^2 and therefore a ‘generalized universal curve’ of the generalized system damping ratio versus the generalized damper coefficient can be generated using Eq. (38). This ‘generalized universal curve’ can be easily used in designing MR dampers to achieve desired cable vibration control effectiveness (Duan 2004, Duan *et al*. 2005). It is known from Eq. (38) that the effects of damper parameters on the system damping performance for sagged cables are the same as those for taut cables because the sag-extensibility parameter λ^2 does not explicitly appear in the equation.

4. Analysis of damping performance

4.1 Effect of damper parameters

The maximum attainable damping ratio and the optimal damper coefficient for sagged cables can be obtained from

$$\frac{\partial \xi_n}{\partial(\eta_c + \eta_I)} = 0 \tag{39}$$

as

$$\xi_{n,opt} = \xi_{n,opt}\Big|_{(\frac{1}{u_s}, u_k, \gamma_M, \lambda^2=0)} \Gamma_\xi^* \tag{40}$$

and

$$(\eta_c + \eta_I)_{opt} = (\eta_c + \eta_I)_{opt}\Big|_{(\frac{1}{u_s}, u_k, \gamma_M, \lambda^2=0)} \Gamma_\eta^* \quad \text{or}$$
$$\kappa_{opt} = \kappa_{opt}\Big|_{\left(\frac{1}{u_s}, u_k, \gamma_M, \lambda^2=0\right)} \Gamma_\eta^* \tag{41a,b}$$

where the correction coefficient of damping ratio Γ_ξ^* and the correction coefficient of damper coefficient Γ_η^* are defined as

$$\Gamma_\xi^* = \frac{1}{W_{\xi,\lambda^2}} \cdot \frac{1}{V_{M,s}(1 + U_{k,M,s} + U_{k,M})}, \tag{42a,b}$$

$$\Gamma_\eta^* = \frac{1}{W_{\eta,\ \lambda^2}} \cdot \frac{(1+U_{k,M,s}+U_{k,M})}{V_{M,s}}$$

The expressions of $\xi_{n,opt}$, $(\eta_c+\eta_I)_{opt}$ and κ_{opt} when $\frac{1}{u_s}$, u_k, and γ_M are all zero are given as

$$\frac{\xi_{n,opt}\big|_{(\frac{1}{u_s},u_k,\gamma_M=0)}}{\frac{x_d}{l}} = \frac{S^2}{2CS},\quad (\eta_c+\eta_I)_{opt}\big|_{(\frac{1}{u_s},u_k,\gamma_M=0)} = \frac{1}{n\pi CS\frac{x_d}{l}},$$
$$\kappa_{opt}\big|_{(\frac{1}{u_s},u_k,\gamma_M=0)} = \frac{1}{\pi^2 CS} \quad (43a\text{-}c)$$

where C and S are approximated as 1 when $\frac{x_d}{l} < 0.15$ (Duan 2004).

The effects on the optimal damping ratio and damper coefficient of individual damper parameters can also explicitly expressed by the two sag-affecting factors Γ_ξ^* and Γ_η^*. When there is no damper mass and the support is ideal, i.e. $\frac{1}{u_s} = \gamma_M = 0$, the effect of damper stiffness is expressed as

$$\Gamma^*_{\xi,u_k\big|\frac{1}{u_s}=\gamma_M=0} = \frac{1}{W_{\xi,\lambda^2}(1+u_kCS\frac{x_d}{l})},$$
$$\Gamma^*_{\eta,u_k\big|\frac{1}{u_s}=\gamma_M=0} = \frac{(1+u_kCS\frac{x_d}{l})}{W_{\eta,\lambda^2}} \quad (44)$$

When there is no damper stiffness and the support is ideal, i.e. $\frac{1}{u_s} = u_k = 0$, the effect of damper mass is given as

$$\Gamma^*_{\xi}\big|_{\frac{1}{u_s}=u_k=0} = \frac{1}{W_{\xi,\lambda^2}(1-\gamma_M CS\frac{x_d}{l})},\ \Gamma^*_{\eta}\big|_{\frac{1}{u_s}=u_k=0} = \frac{(1-\gamma_M CS\frac{x_d}{l})}{W_{\eta,\lambda^2}} \quad (45)$$

When there is no damper mass and stiffness, i.e., $\gamma_M = u_k = 0$, the effect of support stiffness is derived as

$$\Gamma^*_{\xi,u_s|\gamma_M=u_k=0} = \frac{1}{W_{\xi,\lambda^2}\left[\frac{1}{u_sCS\frac{x_d}{l}}+1\right]},$$
$$\Gamma^*_{\eta,u_s|\gamma_M=u_k=0} = \frac{1}{W_{\eta,\lambda^2}\left[\frac{1}{u_sCS\frac{x_d}{l}}+1\right]} \quad (46)$$

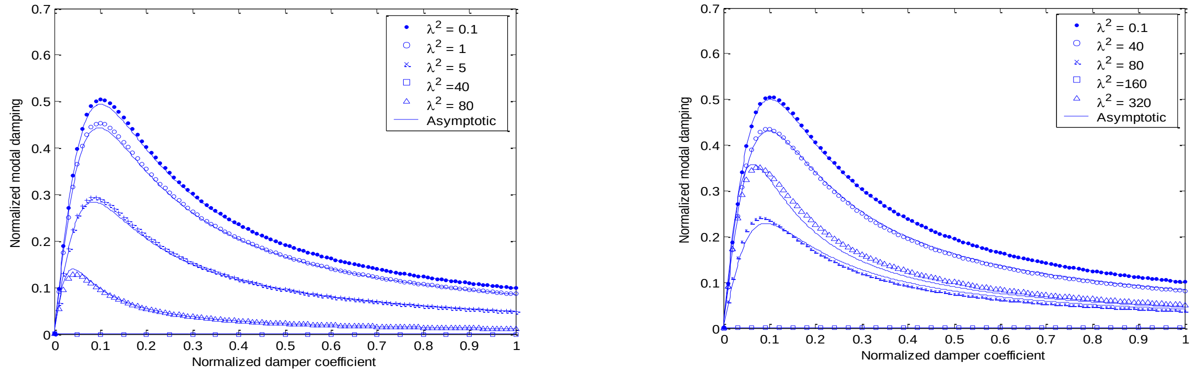
It is known from Eqs. (44)-(46) that apart from the effect of cable sag reflected by W_{ξ,λ^2} and W_{η,λ^2}, the effects of damper parameters including the damper stiffness u_k, damper mass γ_M, and stiffness of damper support u_s on the system damping performance for sagged cables are the same as those for taut cables (Duan *et al*. 2019).

It turns out that the damper stiffness decreases the maximum attainable damping ratio and increases the optimal damper coefficient; the damper mass counteracts the effect of the damper stiffness and may increase the maximum attainable damping ratio when the damper mass is appropriate; and softening of the support stiffness decreases both the maximum attainable damping ratio and the optimal damper coefficient. As a result, one should endeavor to eliminate damper stiffness, make damper support stiff enough, and utilize the benefit of damper mass in designing MR dampers for cable vibration control. To attain the maximum damping ratio for all cables on a cable-stayed bridge, voltage/current input to the MR dampers should be adjusted with the optimal damper coefficient specific for each cable and vibration mode.

4.2 Effect of cable sag

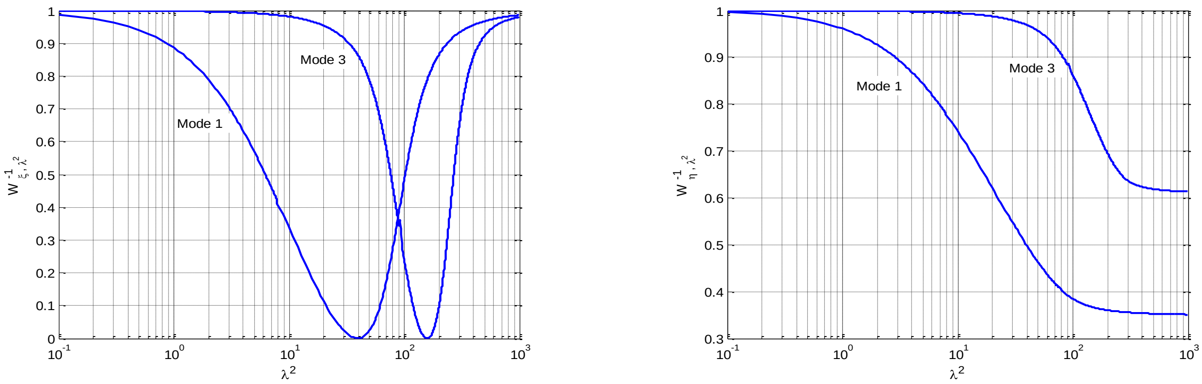
The frequency of a symmetric vibration mode increases with an increase in the sag-extensibility parameter λ^2, and can increase beyond the frequency of the antisymmetric mode immediately above: the so-called frequency crossover phenomenon (Irvine and Caughey 1974). For example, the 1st modal frequency can be larger than the 2nd modal frequency. For an inclined cable, the frequency avoidance may replace the frequency crossover, because the curvature varying along the cable leads to modes that are hybrid in shape; i.e., neither symmetric nor antisymmetric (Triantafyllou 1984). For example, the 1st modal shape has zones of inverse motion near the supports; i.e. the side parts of the cable near the supports are lowered when the middle part of the cable is lifted with respect to the static profile. With sag-extensibility parameter λ^2 around the frequency crossover or avoidance value, the motion will be greatly reduced near the cable ends, and therefore render a damper in this region ineffective. For the undamped cable, the frequency avoidance occurs for the nearly symmetric vibration modes (n=1,3,5…), when the sag-extensibility factor parameter $\lambda^2 = (n+1)^2\pi^2$ (Krenk and Nielsen 2001, Duan 2004).

Fig. 3 shows the numerical and asymptotic solutions for nearly symmetric vibrations of a sagged cable attached with an ideal viscous damper at the location $\frac{x_d}{l} = 0.02$ with ideal support. The diagrams of the normalized system damping ratio $\frac{\xi}{\frac{x_d}{l}}$ versus the normalized damper coefficient κ are here plotted to illustrate the influence of the sag-extensibility parameter λ^2. The values of λ^2 = 0.1, 1, 5, 40, 80 for the first mode and λ^2 = 0.1, 40, 80, 160,



(a) 1st mode (b) 3rd mode

Fig. 3 Numerical and asymptotic solutions for nearly symmetric vibrations of a damped cable under different sag (x_d/l = 0.02): (a) 1st mode and (b) 3rd mode



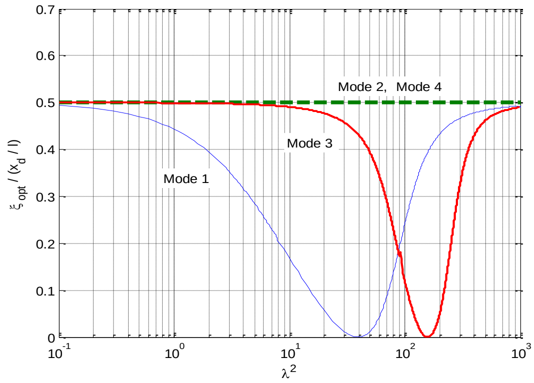
(a) $\frac{1}{W_{\xi,\lambda^2}}$ (b) $\frac{1}{W_{\eta,\lambda^2}}$

Fig. 4 Sag-affecting coefficients W_{ξ,λ^2} for system damping ratio and W_{η,λ^2} for damper coefficient versus sag-extensibility parameter λ^2 : (a) $\frac{1}{W_{\xi,\lambda^2}}$ and (b): $\frac{1}{W_{\eta,\lambda^2}}$

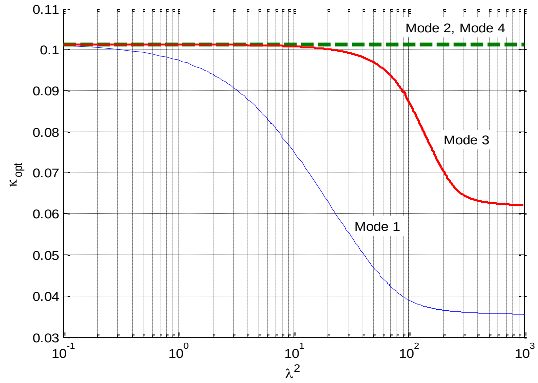
320 for the third mode are selected such that the first value represents the taut cable, the second and third values are smaller than the frequency avoidance value (Krenk and Nielsen 2001, Duan 2004) of $(n+1)^2\pi^2$, the fourth is close to the frequency avoidance value, and the fifth is larger than the frequency avoidance value. It is seen that the asymptotic solution agrees well with the numerical solution for all the λ^2 values. It is observed that the system maximum attainable damping ratio decreases with the increase of λ^2 at onset and becomes almost zero when λ^2 approaches to the frequency avoidance value. After λ^2 is beyond the frequency avoidance value, the maximum damping ratio increases with increasing λ^2, but never exceeds the value 0.52 corresponding to λ^2 = 0 (taut cables). It is also seen that the optimal damper coefficient which achieves the maximum damping ratio always decreases with the increase of λ^2, regardless of whether λ^2 is smaller or larger than the frequency avoidance value.

Fig. 4 illustrates the sag-affecting coefficients for the system damping ratio and the damper coefficient versus the sag-extensibility parameter λ^2. It is observed that when λ^2 is equal to 1, the maximum damping ratio for the first mode is decreased by about 10% in comparison with that when λ^2 = 0, and the value of the corresponding damper coefficient is reduced by about 5%. When λ^2 is up to 10, the decrease will be about 65% and 25%, respectively. Therefore, for the first mode vibration, the effect of cable sag can be ignored in engineering application when $\lambda^2 < 1$, but is significant and not ignorable when $\lambda^2 > 1$. However, the effect of cable sag on the third mode is negligible (less than 1%) when λ^2 is not larger than 10. For the higher modes, the effect of cable sag becomes more trivial in the range of 0< λ^2 <10.

Fig. 5 where the curves of the maximum damping ratio and the optimal damper coefficient versus the sag-

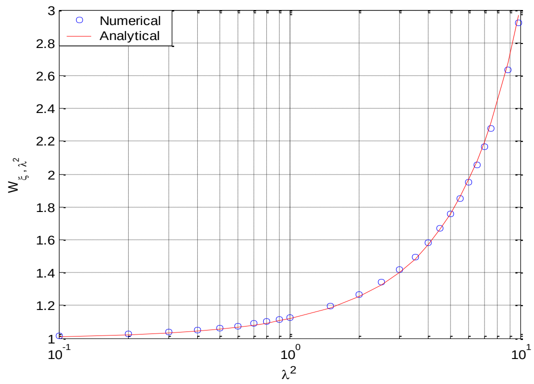


(a) maximum normalized damping ratio

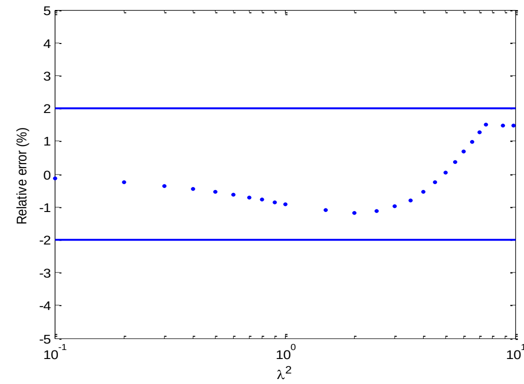


(b) optimal normalized damper coefficient

Fig. 5 Effect of sag-extensibility parameter λ^2 on system damping performance



(a) numerical and analytical results



(b) relative error

Fig. 6 Validation of empirical formula for sag-affecting coefficient of damping ratio W_{ξ,λ^2} ($\lambda^2 \le 10$)

extensibility parameter λ^2 are plotted for the first four modes. When all the four modes are concerned, the maximum system damping ratio is controlled by the first mode in the range of 0< λ^2 <10 because the first mode has the lowest attainable damping ratio.

The sag-extensibility parameter λ^2 for stay cables in actual cable-stayed bridges is usually within the range of 0< λ^2 <10. Only the damping performance of the first mode is significantly affected by the sag in this range. To facilitate engineering application, it is preferable to provide analytical expressions of the sag-affecting coefficients W_{ξ,λ^2} and W_{η,λ^2} in terms of λ^2. Such analytical expressions can be obtained by introducing an approximate solution of $\beta_n^0 l$ into Eqs. (33) and (34) as

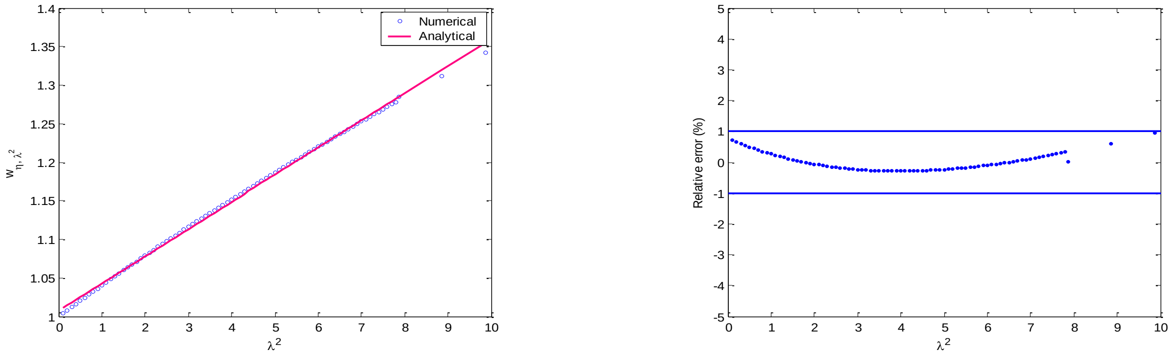
$$W_{\xi,\lambda^2} = \begin{cases} 1+0.11\lambda^2(1+0.035\lambda^2)^2 & n=1,\ \lambda^2 \le 10 \\ 1 & n>1,\ \lambda^2 \le 10 \end{cases} \tag{47}$$

$$W_{\eta,\lambda^2} = \begin{cases} 1+0.035\lambda^2 & n=1,\ \lambda^2 \le 10 \\ 1 & n>1,\ \lambda^2 \le 10 \end{cases} \tag{48}$$

Fig. 6 provides a comparison between the numerical solution of Eq. (33) and the analytical solution obtained by Eq. (47). It is seen that the relative error of the empirical formula Eq. (47) for W_{ξ,λ^2} is less than 2%.

Similarly, Fig. 7 shows the comparison between the numerical solution of Eq. (34) and the analytical solution obtained by Eq. (48). The relative error of the empirical formula Eq. (48) for W_{η,λ^2} is less than 1% in the range of 0< λ^2 <10. Both of the empirical formulae are accurate enough for engineering application. After so doing, the damper design for sagged cables has become quite convenient with the use of the general formulas or the 'generalized universal curve' together with Eqs. (47) and (48).

Fig. 8 illustrates the diagrams of the normalized system damping ratio $\xi/(x_d/l)$ versus the normalized damper coefficient κ and the generalized system damping ratio $\xi/(x_d/l)$ versus the generalized damper coefficient κ^* for the first vibration mode of sagged cables with $\lambda^2 < 10$. The asymptotic solution obtained using the general formulae Eqs. (36) and (38) agrees well with the numerical solution from Eq. (23). It is observed that the plot of the normalized system damping ratio versus the normalized



(a) numerical and analytical results (b) relative error

Fig. 7 Validation of empirical formula for sag-affecting coefficient of damping ratio W_{η,λ^2} ($\lambda^2 \le 10$)

damper coefficient relies on the sag-extensibility parameter λ^2, and therefore is not a universal curve for sagged cables. However, the plots of the generalized system damping ratio versus the generalized damper coefficient for different λ^2 are merging to form a universal curve. It should be noted that the universal curve plotted in Fig 8(b) is also applicable to other vibration modes as long as $\frac{nx_d}{l} \le 0.15$ (Duan 2004). Therefore, this universal curve makes the damper design for sagged cables as easily as for taut cables.

5. Case study

The longest cable (536 m) on the cable-stayed Stonecutters Bridge in Hong Kong is studied to demonstrate the influence of the sag for the damper design, and to clarify the necessity of the adjustability of damper coefficients for achieving maximum damping ratio for different vibration modes. The main parameters of this cable are shown in Table 1, where *l* is the cable length, is the inclination angle, *EA* the elastic stiffness, T_0 the cable tension force along the cable chord, m the mass per unit length, and *D* the diameter of the cable.

Among many kinds of cable vibrations, the rain-wind-induced vibration has the largest amplitude and is the most dangerous. Although the mechanism of rain-wind excitation is still a conundrum, the rain-wind-induced cable vibration can be mitigated if the cable damping is as high as making the Scruton number greater than 10 (Irwin 1997, Yamada 1997, Tanaka 2003). This is the so-called Irwin's criterion

$$S_e = \frac{m\xi}{\rho_{air} D^2} > 10 \tag{49}$$

where S_e is the Scruton number, *D* is the cable diameter, *m* is the cable mass per unit of length; ξ is damping ratio, and ρ_{air} is the mass density of the air, 1.225 kg/m^3. According to this relationship, the minimum damping ratio to prevent rain-wind-induced vibration should 0.44% for this cable.

Table 1 Parameters of the longest cable on Stonecutters Bridge

Parameter	l (m)	θ (°)	EA (10^6N)	T_0 (kN)	m (kg/m)	D (m)
Value	536	19	2080	6167	110.6	0.2

The sag f and sag-extensibility parameter λ^2 for this cable can be obtained from Eqs. (3) and (8) as

$$f = 5.97\, m \;\; and \;\; \lambda^2 = 2.97 \tag{50}$$

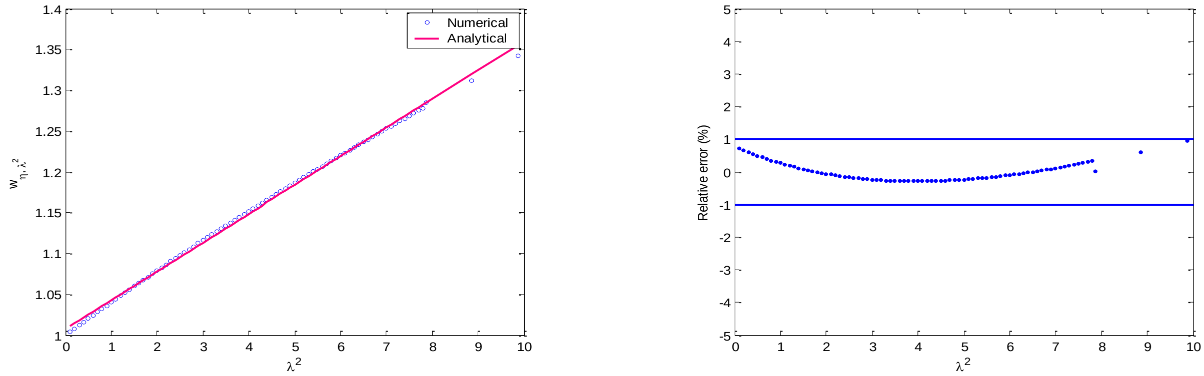
It is obvious that the sag is very large, even though it is only about 1% of the cable length; and that the sag-extensibility λ^2 is also not negligible because it is larger than 1. By substituting the value of λ^2 into Eqs. (47) and (48), the following modification factors due to the sag are obtained as

$$W_{\xi,\lambda^2} = \begin{cases} 1.352 & n=1 \\ 1 & n>1 \end{cases} \text{ and } W_{\eta,\lambda^2} = \begin{cases} 1.094 & n=1 \\ 1 & n>1 \end{cases} \tag{51a,b}$$

Considering the aesthetics of the bridge and the convenience of installation, it is practical to choose the damper location as $x_d/l \le 0.01$ (i.e., $x_d = 5.36$ m away from the cable end, or 1.74 m in height from the level of the deck). Then, supposing that the damper is an MR damper without damper stiffness and mass and installed with an ideal support (i.e., $u_k = \gamma_M = \frac{1}{u_s} = 0$), the maximum damping ratio and the optimal damper coefficient for the first vibration mode from Eqs. (42) and (43) are obtained as

$$\xi_{opt} = \frac{1}{W_{\xi,\lambda^2}} \frac{1}{2} \frac{x_d}{l} = 0.37\%$$

$$(\eta_c + \eta_I)_{opt} = \frac{1}{W_{\eta,\lambda^2}} \frac{1}{n\pi \frac{x_d}{l}} = 29.09 \tag{52a,b}$$



(a) traditional curve (b) universal curve

Fig. 8 Traditional and universal curves for sagged cables ($\lambda^2 \le 10$, 1st mode)

Table 2 Optimal adjustable passive control of the longest cable on Stonecutters Bridge (x_d/l = 1.0%, x_d = 5.36 m, h = 1.74 m)

Mode Number n	Taut cable ($\lambda^2 = 0$)			Sagged cable ($\lambda^2 = 2.68$)		
	Frequency (1/s)	c_{opt} (10^6Ns/m)	ξ_{opt} (%)	Frequency (1/s)	c_{opt} (10^6Ns/m)	ξ_{opt} (%)
1	0.22	0.83	0.50	0.24	0.76	0.37
2	0.44	0.41	0.50	0.44	0.41	0.50
3	0.66	0.28	0.50	0.66	0.28	0.50

Meanwhile, a total equivalent damper coefficient is obtained as

$$c = c_e + c_I = (\eta_e + \eta_I)\sqrt{T_0 m} \tag{53}$$

Therefore, corresponding to the nondimensional optimal damper coefficient $(\eta_c + \eta_I)_{opt}$, one obtains the dimensional optimal damper coefficient $c_{opt} = (c_e + c_I)_{opt}$.

The dimensional optimal damper coefficient for the first mode is obtained as

$$c_{opt} = \sqrt{T_0 m}(\eta_c + \eta_I)_{opt} = 0.76 \times 10^6 \, Ns/m \tag{54}$$

Similarly, we can obtain the results for other modes, as shown in Table 2, as well as corresponding results by ignoring the effect of sag. It is seen that due to the effect of the sag, the maximum damping ratio decreases by about 30%, from 0.50% to 0.37%; and the optimal damper coefficient decreases by about 10%. When installed with the 'optimal damper' according to the result of the taut cable, the achievable damping ratio can be obtained from Eq. (35) as

$$\xi_1 = \frac{x_d}{l} \frac{n\pi W_{\eta,\lambda^2}(\eta_c + \eta_I)\dfrac{x_d}{l}}{1 + \left[n\pi W_{\eta,\lambda^2}(\eta_c + \eta_I)\dfrac{x_d}{l}\right]^2} \frac{1}{W_{\xi,\lambda^2}} = 0.368\% \tag{55}$$

Comparing ξ_1 and ξ_{opt} in Eq. (52), it is found that the overestimation of the optimal damper coefficient will not induce a large decrease in the maximum damping ratio. However, the overestimation of the maximum damping ratio is noticeable. The minimum damping ratio for the control of rain-wind-induced vibration is 0.44% according to Eq. (49). Therefore, installing an ideal passive damper at this location cannot offer enough damping of the first mode. But without considering the effect of the sag-extensibility, the damping ratio is very much overestimated, which would lead to a failure in suppressing the rain-wind-induced cable vibration.

As for other modes, the maximum attainable damping ratio (0.50%) is larger than the requirement (0.44%). But, as discussed earlier, the damper stiffness and support stiffness may decrease the actual damping ratio. Assuming that 80% of the maximum damping ratio can be achieved, this will be hardly enough to meet the requirement. In this situation, the damper has to be installed at a higher position.

The values of the maximum damping ratio and optimal damper coefficient are shown in Table 3 for the first three modes with the installation heights of 2 m, 2.25 m, and 2.5 m. It is seen that the optimal damper coefficient decrease with the increase of the installation height and the increase of the modal number. For this longest cable, the damper installation height should be 2.5 m considering the unfavorable effects of damper stiffness and support stiffness to assure the maximum attainable damping ratio is larger than 0.44%, and equivalent damper coefficient range should cover but be not limited to 190 ~ 530 kNs/m. To achieve the optimal adjustable passive control, the MR damper should

Table 3 Damper design for the longest cable on Stonecutters Bridge

Items		Installation height (m)		
		2.0	2.25	2.5
x_d (m)		6.14	6.91	7.68
x_d / l		0.011	0.013	0.014
ξ_{opt} (%)	Mode 1	0.42	0.48	0.53
	Mode 2	0.57	0.65	0.72
	Mode 3	0.57	0.65	0.72
$(\eta_c + \eta_I)_{opt}$	Mode 1	25.39	22.57	20.31
	Mode 2	13.89	12.34	11.11
	Mode 3	9.258	8.229	7.406
$(c_e + c_I)_{opt}$ (10^6Ns/m)	Mode 1	0.66	0.59	0.53
	Mode 2	0.36	0.32	0.29
	Mode 3	0.24	0.21	0.19

be properly tuned to offer the calculated optimal damper coefficients for each of the concerned vibration modes.

The rain-wind-induced cable vibrations are usually dominated by one of the first few modes. However, the specific dominant mode in need to be controlled is a *priori* unknown for a given cable. As a result, the design of dampers is usually conducted by considering several possible modes and determining the damper coefficient to achieve desired damping ratio for all the concerned modes. As shown in Fig. 5, the first mode will have the lowest maximum attainable damping due to the effect of sag. So the damper size (damper coefficient) is determined equal to the optimal damper coefficient corresponding to the first mode. With such a damper size, the cable will achieve the maximum damping ratio in the first mode. However, the damping ratios achieved in the other three modes are not their maximum attainable damping ratios. When passive dampers are designed in this way, these damping ratios cannot be altered even when a dominant mode different from the first mode is identified later. When smart MR dampers are used, the system damping ratio for the dominant mode can be enhanced to its maximum damping ratio through adjusting the voltage/current input to alter the damper coefficient, even when the dampers have been installed on the bridge. This is a salient advantage of MR dampers over passive dampers for cable vibration control.

6. Conclusions

In this paper, a method for analyzing the damping performance of inclined sagged cables incorporating MR dampers in the passive control mode is developed. It takes into account the cable sag and inclination, damper coefficient, stiffness and mass, and stiffness of damper support. Based on the asymptotic solution, analytical formulae relating the system damping ratio with the damper coefficient are obtained, from which the maximum attainable damping ratio and the corresponding optimal damper coefficient can be easily calculated. Two sag-affecting coefficients have been derived to analytically evaluate the effect of cable sag on the system damping performance. By defining the generalized system damping ratio and the generalized damper coefficient, a 'generalized universal curve' is configured which makes the damper design for sagged cables as easily as for taut cables.

The analysis and parametric studies show that the nearly symmetric vibrations may be significantly affected by cable sag whereas the nearly antisymmetric vibrations are hardly affected by cable sag. The effect of cable sag is to decrease the system maximum attainable damping ratio and the optimal damper coefficient. However, the effect of cable sag on the system damping performance is independent of the effects of damper parameters (damper stiffness, damper mass, damping coefficient, and stiffness of damper support). Therefore, the observations and conclusions regarding the effects of the damper parameters obtained for taut cables are also valid for sagged cables. For the stay cables with the sag-extensibility parameter $\lambda^2 < 10$ usually used on cable-stayed bridges, only the first mode can be affected by the sag for the damping performance. A case study has been carried out to demonstrate the influence of the sag for the damper design, as well as the necessity of adjustability of damper coefficients for achieving maximum damping ratio for each of the vibration modes.

Even through, this study is initiated for MR dampers, but it is applicable for general external dampers for cable vibration mitigation, such as the electro-rheological dampers (Powell 1994), and inertial mass dampers (Lu *et al*. 2017, Wang *et al*. 2019), and so on.

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