Smart Structures and Systems, *Vol. 19, No. 3 (2017) 243-257*
DOI: https://doi.org/10.12989/sss.2017.19.3.243

Semi-analytical vibration analysis of functionally graded size-dependent nanobeams with various boundary conditions

Farzad Ebrahimi* and Erfan Salari

Department of Mechanical Engineering, Faculty of Engineering, Imam Khomeini International University, Qazvin, Iran

(Received April 2, 2015, Revised January 6, 2017, Accepted January 6, 2017)

Abstract. In this paper, free vibration of functionally graded (FG) size-dependent nanobeams is studied within the framework of nonlocal Timoshenko beam model. It is assumed that material properties of the FG nanobeam, vary continuously through the thickness according to a power-law form. The small scale effect is taken into consideration based on nonlocal elasticity theory of Eringen. The non-classical governing differential equations of motion are derived through Hamilton's principle and they are solved utilizing both Navier-based analytical method and an efficient and semi-analytical technique called differential transformation method (DTM). Various types of boundary conditions such as simply-supported, clamped-clamped, clamped-simply and clamped-free are assumed for edge supports. The good agreement between the presented DTM and analytical results of this article and those available in the literature validated the presented approach. It is demonstrated that the DTM has high precision and computational efficiency in the vibration analysis of FG nanobeams. The obtained results show the significance of the material graduation, nonlocal effect, slenderness ratio and boundary conditions on the vibration characteristics of FG nanobeams.

Keywords: free vibration; functionally graded nanobeam; timoshenko beam theory; differential transformation method; nonlocal elasticity theory

1. Introduction

Functionally graded materials (FGMs) are a new class of novel non-homogeneous materials which are generally composed of two different parts such as ceramic and metal in which the material properties changes smoothly between two surfaces. (Ebrahimi 2013). Possessing this structure, makes the FGMs receive wide application in modern industries including aerospace, mechanical, electronics, optics, chemical, biomedical, nuclear, and civil engineering. (Ebrahimi *et al*. 2009, Aghelinejad *et al*. 2011, Ebrahimi and Rastgoo 2008a, b, 2009, 2011). It is commonly believed that the nanotechnology will motivate a series of industrial revolutios in the following years. Therefore, Recently there has been growing interest for application of nonlocal continuum mechanics especially in the field of fracture mechanics, dislocation mechanics and micro/nano technologies. Structural elements such as beams, plates, and membranes in micro or nanolength scale are commonly used as components in micro/nano electromechanical systems (MEMS/NEMS). Therefore understanding the mechanical and physical properties of nanostructures is necessary for its practical applications. At nanolength scales, size effects often become prominent, which cause an increasing interest in the general area of nanotechnology. Applying the continuum elasticity theory to the analysis of the nanoscale structures, indicates that it is inadequate because of ignoring the small scale effects. Therefore, we need to consider the small length scales associated with nanostructures such as lattice spacing between individual atoms, surface properties, grain size, etc. Nonlocal elasticity theory introduced by Eringen accounts for the small-scale effects arising at the nanoscale level. It has been extensively applied to analyze the bending, buckling, vibration and wave propagation of beam-like elements in nanomechanical devices. Unlike the constitutive equation in classical elasticity, `, Reddy, and Levinson beam theories using the nonlocal differential constitutive relations of Eringen. In other scientific work, Wang and Liew (2007) carried out the static analysis of micro- and nano-structures based on nonlocal continuum mechanics using Euler-Bernoulli beam theory and Timoshenko beam theory. Aydogdu (2009) proposed a generalized nonlocal beam theory to study bending, buckling, and free vibration of nanobeams based on Eringen model using different beam theories. Phadikar and Pradhan (2010) reported finite element formulations for nonlocal elastic Euler-Bernoulli beam and Kirchoff plate. Pradhan and Murmu (2010) investigated the flapwise bending-vibration characteristics of a rotating nanocantilever by using Differential quadrature method (DQM). They noticed that small-scale effects play a significant role in the vibration response of a rotating nanocantilever. Civalek *et al*. (2010) presented a formulation of the equations of motion and bending of Euler–Bernoulli beam using the nonlocal elasticity theory for cantilever microtubules. Thai (2012) proposed a nonlocal shear deformation beam theory for bending, buckling, and vibration of nanobeams using the nonlocal differential constitutive relations of Eringen.

*Corresponding author, Professor
E-mail: febrahimy@eng.ikiu.ac.ir

http://www.techno-press.com/journals/sss&subpage=7
ISSN: 1738-1584 (Print), 1738-1991 (Online)

Recently several researches have been carried out on static and dynamic behavior of FGM structures. Asghari *et al*. (2010) studied the free vibration of the FGM Euler-Bernoulli microbeams, which has been extended to consider a size-dependent Timoshenko beam based on the modified couple stress. The dynamic characteristics of FG beam with power law material graduation in the axial or the transversal directions was examined by Alshorbagy *et al*. (2011). Also the analysis of mechanical characteristics of nanostructure is one of the interesting research topics. (Ebrahimi and Barati 2016f-n, Ebrahimi and Barati 2017). Ke and Wang (2011) exploited the size effect on dynamic stability of functionally graded Timoshenko microbeams. Eltaher *et al*. (2012, 2013a) presented a finite element formulation for free vibration analysis of FG nanobeams based on nonlocal Euler-Bernoulli beam theory. They also exploited the size-dependent static-buckling behavior of functionally graded nanobeams on the basis of the nonlocal continuum model. (Eltaher *et al*. 2013b). Using nonlocal Timoshenko and Euler-Bernoulli beam theory, Simsek and Yurtcu (2013) investigated analytically bending and buckling of FG nanobeam by analytical method. Thermal buckling and free vibration analysis of FG nanobeams subjected to temperature distribution have been exactly investigated by Ebrahimi and Salari (2015a-c) and Ebrahimi *et al*. (2015 a, b). Ebrahimi and Barati (2016o-q) investigated buckling behavior of smart piezoelectrically actuated higher-order size-dependent graded nanoscale beams and plates in thermal environment. As one may note, the most cited references deal with the modeling of micro/nano-beams are based on the assumptions that the material is homogeneous and a very limited literature is available for micro/nano-scale FGM structures. Motivated by this fact, in this study, differential transformation method is applied in analyzing vibration characteristics of FG size-dependent nanobeams. The superiority of the DTM is its simplicity and good precision and depends on Taylor series expansion while it takes less time to solve polynomial series. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. To the best knowledge of the authors, no research effort has been devoted so far to to find the solution of vibrational behavior of a FG nanobeams withbased on nonlocal timoshenko beam theroy employing DTM .

Motivated by these considerations, in this paper, the free vibration of FG nanobeams is investigated based on the nonlocal Timoshenko beam theory. It is assumed that material properties of the beam, such as Young's modulus and mass density, vary continuously through the beam thickness according to power-law form. The governing equations and the related boundary conditions are derived. These equations are solved using both analytical method and DTM and natural frequencies of FG nanobeam are obtained. The effects of nonlocal parameter, slenderness ratio, material graduations on the vibration responses of the FG nanobeam are discussed. Comparisons with the results from the existing literature are provided and the good agreement between the results of this article and those available in literature validated the presented approach. Numerical results are presented to serve as benchmarks for the application and the design of nanoelectronic and nano-drive devices, nano-oscillators, and nanosensors, in which nanobeams act as basic elements.

2. Theory and formulation

2.1 Power-law functionally graded material (P-FGM) beam

One of the most favorable models for FGMs is the power-law model, in which material properties of FGMs are assumed to vary according to a power law about spatial coordinates. The coordinate system for FG nano beam is shown in Fig. 1. The FG nanobeam is assumed to be composed of ceramic and metal and effective material properties (P_f) of the FG beam such as Young's modulus E_f, shear modulus G_f and mass density ρ_f are assumed to vary continuously in the thickness direction (z-axis direction) according to an power function of the volume fractions of the constituents while the Poisson's ratio is assumed to be constant in the thickness direction. According to the rule of mixture, the effective material properties, P, can be expressed as (Simsek and Yurtcu 2013)

$$P_f = P_c V_c + P_m V_m \tag{1}$$

where P_m, P_c, V_m and V_c are the material properties and the volume fractions of the metal and the ceramic constituents related by

$$V_c + V_m = 1 \tag{2.a}$$

The volume fraction of the ceramic constituent of the beam is assumed to be given by

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^P \tag{2.b}$$

Here p is the non-negative variable parameter (power-law exponent) which determines the material distribution through the thickness of the beam. Therefore, from Eqs. (1)-(2), the effective material properties of the FG nanobeam can be expressed as follows

$$P_f(z) = (P_c - P_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_m \tag{3}$$

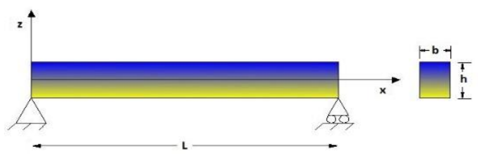


Fig. 1 Geometry and coordinates of functionally graded nanobeam

According to this material distribution, bottom and top surfaces of functionally graded beam are pure metal and pure ceramic repectively.

2.2 Kinematic relations

It is well-known that the Euler-Bernoulli beam theory always overestimates the natural frequency of free vibration due to the fact that the effects of rotary inertia and shear deformation are neglected in this theory and Timoshenko beam theory presents a more realistic model of the beam. Based on Timoshenko beam theory, the displacements of an arbitrary point along the x- and z-axes, denoted by $u_x(x,z,t)$ and $u_z(x,z,t)$ respectively, are

$$u_x\left(x,z,t\right)=u\left(x,t\right)+z\,\varphi(x,t) \tag{4.a}$$

$$u_z\left(x,z,t\right)=w\left(x,t\right) \tag{4.b}$$

where $u(x,t)$ and $w(x,t)$ are displacement components in the midplane, φ is the rotation of the beam cross-section and t is the time. Based on the displacement field in Eq. (4), the normal strain ε_{xx} and shear strain γ_{xz} can be expressed as follows

$$\varepsilon_{xx}=\varepsilon_{xx}^{0}+zk^{0},\quad \varepsilon_{xx}^{0}=\frac{\partial u(x,t)}{\partial x},\quad k^{0}=\frac{\partial\varphi(x,t)}{\partial x} \tag{5}$$

$$\gamma_{xz}=\frac{\partial w}{\partial x}+\varphi \tag{6}$$

where ε_{xx}^{0} and k^{0} are the extensional strain and bending strain respectively. Based on the Hamilton's principle, which states that the motion of an elastic structure during the time interval $t_1<t<t_2$ is such that the time integral of the total dynamics potential is extremum (Tauchert 1974)

$$\int_0^t \delta(U-T+W_{ext})dt=0 \tag{7}$$

where U is strain energy, T is kinetic energy and W_{ext} is work done by external forces. The first variation of the strain energy can be calculated as

$$\delta U=\int_v \sigma_{ij}\delta\varepsilon_{ij}dV=\int_v(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xz}\delta\gamma_{xz})dV \tag{8}$$

Substituting Eqs. (5) and (6) into Eq.(8) yields

$$\delta U=\int_0^L(N(\delta\varepsilon_{xx}^{0})+M(\delta k^{0})+Q(\delta\gamma_{xz}))dx \tag{9}$$

where N, M are the axial force and bending moment, respectively and Q is the shear force. These stress resultants used in Eq. (9) are defined as

$$N=\int_A\sigma_{xx}dA,\quad M=\int_A\sigma_{xx}z\,dA,\quad Q=\int_A K_s\sigma_{xz}dA \tag{10}$$

where K_s is the shear correction factor. The kinetic energy for Timoshenko beam can be written as

$$T=\frac{1}{2}\int_0^L\int_A\rho(z)\left(\left(\frac{\partial u_x}{\partial t}\right)^2+\left(\frac{\partial u_z}{\partial t}\right)^2\right)dA\,dx \tag{11}$$

And the first variation of the Eq. (11) can be obtained as

$$\delta T=\int_0^L\left[I_0\left(\frac{\partial u}{\partial t}\frac{\partial\delta u}{\partial t}+\frac{\partial w}{\partial t}\frac{\partial\delta w}{\partial t}\right)+I_1\left(\frac{\partial\varphi}{\partial t}\frac{\partial\delta u}{\partial t}+\frac{\partial u}{\partial t}\frac{\partial\delta\varphi}{\partial t}\right)+I_2\frac{\partial\varphi}{\partial t}\frac{\partial\delta\varphi}{\partial t}\right]dx \tag{12}$$

where (I_0, I_1, I_2) are the mass moment of inertias, defined as follows

$$(I_0,I_1,I_2)=\int_A\rho(z)(1,z,z^2)\,dA \tag{13}$$

The first variation of external forces work of the beam can be written as

$$\delta W_{ext}=\int_0^L\left(f(x)\delta u+q(x)\delta w\right)dx \tag{14}$$

In which $f(x)$ and $q(x)$ are external axial and transverse loads distribution along length of beam, respectively. Substituting Eqs. (9), (12) and (14) into Eq. (7) and setting the coefficients of δu, δw and $\delta\varphi$ to zero, the governing equations of motion for vibration analysis of FGM beams can be written as

$$\frac{\partial N}{\partial x}+f=I_0\frac{\partial^2 u}{\partial t^2}+I_1\frac{\partial^2\varphi}{\partial t^2} \tag{15.a}$$

$$\frac{\partial Q}{\partial x}+q=I_0\frac{\partial^2 w}{\partial t^2} \tag{15.b}$$

$$\frac{\partial M}{\partial x}-Q=I_1\frac{\partial^2 u}{\partial t^2}+I_2\frac{\partial^2\varphi}{\partial t^2} \tag{15.c}$$

Also the corresponding boundry conditions are given as

$$N=0 \text{ or } u=0 \quad \text{at } x=0 \text{ and } x=L \tag{16.a}$$

$$Q=0 \text{ or } w=0 \quad \text{at } x=0 \text{ and } x=L \tag{16.b}$$

$$M=0 \text{ or } \varphi=0 \quad \text{at } x=0 \text{ and } x=L \tag{16.c}$$

2.3 The Nonlocal Timoshenko beam equations and boundary conditions

Based on Eringen's nonlocal elasticity model (Eringen and Edelen 1972), the stress at a reference point x in a body is considered as a function of strains of all points in the near region. This assumption is agreement with experimental observations of atomic theory and lattice dynamics in phonon scattering in which for a homogeneous and isotropic elastic solid the nonlocal stress-tensor components σ_{ij} at any point x in the body can be expressed as

$$\sigma_{ij}(x) = \int_{\Omega} \alpha(|x' - x|, \tau)\, t_{ij}(x')\, d\Omega(x') \tag{17}$$

where $t_{ij}(x')$ are the components of the classical local stress tensor at point x which are related to the components of the linear strain tensor ε_{kl} by the conventional constitutive relations for a Hookean material, i.e.

$$t_{ij} = C_{ijkl}\varepsilon_{kl} \tag{18}$$

The meaning of Eq. (17) is that the nonlocal stress at point x is the weighted average of the local stress of all points in the neighborhood of x, the size of which is related to the nonlocal kernel $\alpha(|x'-x|,\tau)$. Here $|x'-x|$ is the Euclidean distance and τ is a constant given by

$$\tau = \frac{e_0 a}{l} \tag{19}$$

which represents the ratio between a characteristic internal length, a (such as lattice parameter, C-C bond length and granular distance) and a characteristic external one, l (e.g., crack length, wavelength) trough an adjusting constant, e_0, dependent on each material. The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. According to for a class of physically admissible kernel $\alpha(|x'-x|,\tau)$ it is possible to represent the integral constitutive relations given by Eq. (17) in an equivalent differential form as

$$(1-(e_0 a)\nabla^2)\sigma_{kl} = t_{kl} \tag{20}$$

where ∇^2 is the Laplacian operator. Thus, the scale length $e_0 a$ takes into account the size effect on the response of nanostructures. For an elastic material in the one dimensional case, the nonlocal constitutive relations may be simplified as

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\,\varepsilon_{xx} \tag{21}$$

$$\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\,\gamma_{xz} \tag{22}$$

where σ and ε are the nonlocal stress and strain, respectively. E is the Young's modulus, $G=E/2(1+v)$ is the shear modulus (where v is the poisson's ratio). For nonlocal Timoshenko FG beam, Eqs. (21) and (22) can be written as (Ebrahimi *et al.* 2016)

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z)\,\varepsilon_{xx} \tag{23}$$

$$\sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G(z)\,\gamma_{xz} \tag{24}$$

where ($\mu=(e_0 a)^2$). Integrating Eqs. (23) and (24) over the beam's cross-section area, we obtain the force-strain and the moment-strain of the nonlocal Timoshenko FG beam theory can be obtained as follows

$$N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx}\frac{\partial u}{\partial x} + B_{xx}\frac{\partial \varphi}{\partial x} \tag{25}$$

$$M - \mu \frac{\partial^2 M}{\partial x^2} = B_{xx}\frac{\partial u}{\partial x} + D_{xx}\frac{\partial \varphi}{\partial x} \tag{26}$$

$$Q - \mu \frac{\partial^2 Q}{\partial x^2} = C_{xz}\left(\frac{\partial w}{\partial x} + \varphi\right) \tag{27}$$

In the above equations the following cross-sectional rigidities are defined

$$(A_{xx}, B_{xx}, D_{xx}) = \int_A E(z)(1, z, z^2)\, dA \tag{28}$$

$$C_{xz} = K_s \int_A G(z)\, dA \tag{29}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (15.a) into Eq. (25) as follows

$$N = A_{xx}\frac{\partial u}{\partial x} + B_{xx}\frac{\partial \varphi}{\partial x} + \mu\left(I_0 \frac{\partial^3 u}{\partial x\,\partial t^2} + I_1 \frac{\partial^3 \varphi}{\partial x\,\partial t^2} - \frac{\partial f}{\partial x}\right) \tag{30}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of M from Eq. (15.c) into Eq. (26) as follows

$$\begin{aligned} M &= B_{xx}\frac{\partial u}{\partial x} + D_{xx}\frac{\partial \varphi}{\partial x} + \\ &\mu\left(I_1 \frac{\partial^3 u}{\partial x\,\partial t^2} + I_2 \frac{\partial^3 \varphi}{\partial x\,\partial t^2} + I_0 \frac{\partial^2 w}{\partial t^2} - q\right) \end{aligned} \tag{31}$$

By substituting for the second derivative of Q from Eq. (15.b) into Eq. (27), the following expression for the nonlocal shear force is derived

$$Q = C_{xz}\left(\frac{\partial w}{\partial x} + \varphi\right) + \mu\left(I_0 \frac{\partial^3 w}{\partial x\,\partial t^2} - \frac{\partial q}{\partial x}\right) \tag{32}$$

The nonlocal governing equations of Timoshenko FG nanobeam in terms of the displacement can be derived by substituting for N, M and Q from Eqs. (30)-(32), into Eq. (15) as follows

$$\begin{aligned} &A_{xx}\frac{\partial^2 u}{\partial x^2} + B_{xx}\frac{\partial^2 \varphi}{\partial x^2} + \\ &\mu\left(I_0 \frac{\partial^4 u}{\partial t^2 \partial x^2} + I_1 \frac{\partial^4 \varphi}{\partial t^2 \partial x^2} - \frac{\partial^2 f}{\partial x^2}\right) \\ &-I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 \varphi}{\partial t^2} + f = 0 \end{aligned} \tag{33.a}$$

$$C_{xz}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial \varphi}{\partial x}\right)+\mu\left(I_0\frac{\partial^4 w}{\partial t^2\partial x^2}-\frac{\partial^2 q}{\partial x^2}\right) -I_0\frac{\partial^2 w}{\partial t^2}+q=0 \tag{33.b}$$

$$B_{xx}\frac{\partial^2 u}{\partial x^2}+D_{xx}\frac{\partial^2 \varphi}{\partial x^2}-C_{xz}\left(\frac{\partial w}{\partial x}+\varphi\right)+ \mu\left(I_1\frac{\partial^4 u}{\partial t^2\partial x^2}+I_2\frac{\partial^4 \varphi}{\partial t^2\partial x^2}\right)-I_1\frac{\partial^2 u}{\partial t^2}-I_2\frac{\partial^2 \varphi}{\partial t^2}=0 \tag{33.c}$$

3. Solution method

3.1 Implementation of differential transformation method

The differential transforms method provides an analytical solution procedure in the form of polynomials to solve ordinary and partial differential equations with small calculation errors and ability to solve nonlinear equations with boundary conditions value problems. Using DTM technique, the ordinary and partial differential equations can be transformed into algebraic equations, from which a closed-form series solution can be obtained easily. In this method, certain transformation rules are applied to both the governing differential equations of motion and the boundary conditions of the system in order to transform them into a set of algebraic equations. The solution of these algebraic equations gives the desired results of the problem. In this method, differential transformation of kth derivative function y(x) and differential inverse transformation of Y(k) are respectively defined as follows (Abdel-Halim Hassan 2002)

$$\mathrm{Y(k)} = \frac{1}{\mathrm{k!}}\left[\frac{\mathrm{d^k}}{\mathrm{dx^k}}\mathrm{y(x)}\right]_{\mathrm{x=0}} \tag{34}$$

$$\mathrm{y(x)} = \sum_{0}^{\infty}\mathrm{x^k Y(k)} \tag{35}$$

In which y(x) is the original function and Y(k) is the transformed function. Consequently from Eqs. (34), (35) we obtain

$$\mathrm{y(x)} = \sum_{\mathrm{k=0}}^{\infty}\frac{\mathrm{x^k}}{\mathrm{k!}}\left[\frac{\mathrm{d^k}}{\mathrm{dx^k}}\mathrm{y(x)}\right]_{\mathrm{x=0}} \tag{36}$$

Eq. (36) reveals that the concept of the differential transformation is derived from Taylor's series expansion. In real applications the function y(x) in Eq. (36) can be written in a finite form as

$$\mathrm{y(x)} = \sum_{\mathrm{k=0}}^{\mathrm{N}}\mathrm{x^k Y(k)} \tag{37}$$

In this calculations $\mathrm{y(x)} = \sum_{\mathrm{n+1}}^{\infty}\mathrm{x^k Y(k)}$ is small enough to be neglected, and N is determined by the convergence of the eigenvalues. From the definitions of DTM in Eqs. (34)-(36), the fundamental theorems of differential transforms method can be performed that are listed in Table 1 while Table 2 presents the differential transformation of conventional boundary conditions. According to the basic transformation operations introduced in Table 1, the transformed form of the governing Eq. (33) around x_0=0 may be obtained as

$$A_{xx}(k+1)(k+2)U[k+2]+B_{xx}(k+1)(k+2)\phi[k+2] -\mu(I_0\omega^2(k+1)(k+2)U[k+2]+I_1\omega^2(k+1)(k+2)\phi[k+2]) +I_0\omega^2U[k]+I_1\omega^2\phi[k]=0 \tag{38}$$

$$C_{xz}((k+1)(k+2)W[k+2]+(k+1)\phi[k+1]) -\mu I_0\omega^2(k+1)(k+2)W[k+2]+I_0\omega^2W[k]=0 \tag{39}$$

$$B_{xx}(k+1)(k+2)U[k+2]+D_{xx}(k+1)(k+2)\phi[k+2]-C_{xz}((k+1)W[k+1]+\phi[k]) -\mu(I_1\omega^2(k+1)(k+2)U[k+2]+I_2\omega^2 (k+1)(k+2)\phi[k+2])+I_1\omega^2U[k]+I_2\omega^2\phi[k]=0 \tag{40}$$

Table 1 Some of the transformation rules of the one-dimensional DTM (Chen and Ju 2004)

Original function	Transformed function
$y(x) = \lambda\varphi(x)$	$Y(k) = \lambda\Phi(k)$
$y(x) = \varphi(x) \pm \theta(x)$	$Y(k) = \Phi(k) \pm \Theta(k)$
$y(x) = \frac{d\varphi}{dx}$	$Y(k) = (k+1)\Phi(k+1)$
$y(x) = \frac{d^2\varphi}{dx^2}$	$Y(k) = (k+1)(k+2)\Phi(k+1)$
$y(x) = \varphi(x)\theta(x)$	$Y(k) = \sum_{l=0}^{k}\Phi(l)\Theta(k-l)$
$y(x) = x^m$	$Y(k) = \delta(k-m) = \begin{cases}1 & k=m\\ 0 & k\neq 0\end{cases}$

Table 2 Transformed boundary conditions (B.C.) based on DTM (Chen and Ju 2004)

X=0		X=1	
Original BC	Transformed BC	Original BC	Transformed BC
f(0)=0	F[0]=0	f(1)=0	$\sum_{k=0}^{\infty}F[k]=0$
$\frac{df}{dx}(0)=0$	F[1]=0	$\frac{df}{dx}(1)=0$	$\sum_{k=0}^{\infty}kF[k]=0$
$\frac{d^2f}{dx^2}(0)=0$	F[2]=0	$\frac{d^2f}{dx^2}(1)=0$	$\sum_{k=0}^{\infty}k(k-1)F[k]=0$
$\frac{d^3f}{dx^3}(0)=0$	F[3]=0	$\frac{d^3f}{dx^3}(1)=0$	$\sum_{k=0}^{\infty}k(k-1)(k-2)F[k]=0$

where $U[k]$, $W[k]$ and $\phi[k]$ are the transformed functions of u, w and φ respectively. Additionally, the differential transform method is applied to various boundary conditions by using the theorems introduced in Table 2 and the following transformed boundary conditions are obtained.

- Simply supported-Simply supported

$$W[0]=0,\ \phi[1]=0,\ U[0]=0$$
$$\sum_{k=0}^{\infty} W[k]=0,\ \sum_{k=0}^{\infty} k\ \phi[k]=0,\ \sum_{k=0}^{\infty} k\ U[k]=0 \qquad (41.a)$$

- Clamped-Clamped

$$W[0]=0,\ \phi[0]=0,\ U[0]=0$$
$$\sum_{k=0}^{\infty} W[k]=0,\ \sum_{k=0}^{\infty} \phi[k]=0,\ \sum_{k=0}^{\infty} U[k]=0 \qquad (41.b)$$

- Clamped-Simply supported

$$W[0]=0,\ \phi[0]=0,\ U[0]=0$$
$$\sum_{k=0}^{\infty} W[k]=0,\ \sum_{k=0}^{\infty} k\ \phi[k]=0,\ \sum_{k=0}^{\infty} k\ U[k]=0 \qquad (41.c)$$

- Clamped-Free

$$W[0]=0,\ \phi[0]=0,\ U[0]=0$$
$$\sum_{k=0}^{\infty} k\ \phi[k]=0,\ \sum_{k=0}^{\infty} k\ (k-1)(k-2)W[k]=0,\ \sum_{k=0}^{\infty} k\ U[k]=0 \qquad (41.d)$$

By using Eqs. (38)-(40) together with the transformed boundary conditions one arrives at the following eigenvalue problem

$$\begin{bmatrix} M_{11}(\omega) & M_{12}(\omega) & M_{13}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) & M_{23}(\omega) \\ M_{31}(\omega) & M_{32}(\omega) & M_{33}(\omega) \end{bmatrix} [C]=0 \qquad (42.a)$$

where $[C]$ correspond to the missing boundary conditions at x=0. For the non-trivial solutions of Eq. (42.a), it is necessary that the determinant of the coefficient matrix set equal to zero

$$\begin{vmatrix} M_{11}(\omega) & M_{12}(\omega) & M_{13}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) & M_{23}(\omega) \\ M_{31}(\omega) & M_{32}(\omega) & M_{33}(\omega) \end{vmatrix} =0 \qquad (42.b)$$

Solution of Eq. (42.b) is simply a polynomial root finding problem. In the present study, the Newton-Raphson method is used to solve the governing equation of the non-dimensional natural frequencies. Solving Eq. (42.b), the i^{th} estimated eigenvalue for n^{th} iteration ($\omega=\omega_i^{(n)}$) may be obtained and the total number of iterations is related to the accuracy of calculations which can be determined by the following equation

$$\left|\omega_i^{(n)}-\omega_i^{(n-1)}\right|<\varepsilon \qquad (43)$$

In this study ε=0.0001 considered in procedure of finding eigenvalues which results in 4 digit precision in estimated eigenvalues. Further a Matlab program has been developed according to DTM rule stated above. As mentioned before, DTM implies an iterative procedure to obtain the high-order Taylor series solution of differential equations. The Taylor series method requires a long computational time for large orders, whereas one advantage of employing DTM in solving differential equations is a fast convergence rate and a small calculation error.

3.2 Analytical solution

Here, based on the Navier type method, an analytical solution of the governing equations for free vibration of a simply supported FG nanobeam is presented. The displacement functions are expressed as product of undetermined coefficients and known trigonometric functions to satisfy the governing equations and the conditions at x=0, L. The following displacement fields are assumed to be of the form

$$u(x,t)=\sum_{n=1}^{\infty} U_n \cos(\frac{n\pi}{L}x)\, e^{i\omega_n t} \qquad (44)$$

$$w(x,t)=\sum_{n=1}^{\infty} W_n \sin(\frac{n\pi}{L}x)\, e^{i\omega_n t} \qquad (45)$$

$$\varphi(x,t)=\sum_{n=1}^{\infty} \phi_n \cos(\frac{n\pi}{L}x)\, e^{i\omega_n t} \qquad (46)$$

where (U_n, W_n, ϕ_n) are the unknown Fourier coefficients to be determined for each n value. Boundary conditions for simply supported beam are as Eq. (47)

$$u(0)=0,\ \frac{\partial u}{\partial x}(L)=0$$
$$w(0)=w(L)=0,\ \frac{\partial\varphi}{\partial x}(0)=\frac{\partial\varphi}{\partial x}(L)=0 \qquad (47)$$

Substituting Eqs. (44)-(46) into Eqs. (33.a)-(33.c) respectively, leads to Eqs. (48)-(50)

$$(-A_{xx}(\frac{n\pi}{L})^2+I_0(1+\mu(\frac{n\pi}{L})^2)\omega_n^{\ 2})U_n$$
$$+(-B_{xx}(\frac{n\pi}{L})^2+I_1(1+\mu(\frac{n\pi}{L})^2)\omega_n^{\ 2})\phi_n=0 \qquad (48)$$

$$(-C_{xz}(\frac{n\pi}{L})^2+I_0(1+\mu(\frac{n\pi}{L})^2)\omega_n^{\ 2})W_n-C_{xz}(\frac{n\pi}{L})\phi_n=0 \qquad (49)$$

$$(-B_{xx}(\frac{n\pi}{L})^2 + I_1(1+\mu(\frac{n\pi}{L})^2)\omega_n^{\ 2})U_n$$
$$+(-D_{xx}(\frac{n\pi}{L})^2 - C_{xz} + I_2(1+\mu(\frac{n\pi}{L})^2)\omega_n^{\ 2}))\phi_n - C_{xz}(\frac{n\pi}{L})W_n = 0 \tag{50}$$

By setting the determinant of the coefficient matrix of the above equations, we obtain a quadratic polynomial for $\omega_n^{\ 2}$. By setting this polynomial to zero, we can find ω_n.

4. Numerical results and discussions

Through this section, a numerical testing of the procedure as well as parametric studies are performed in order to establish the validity and usefulness of the DTM approach. The effect of FG distribution, nonlocality effect and thickness ratios on the natural frequencies of the FG nanobeam will be figured out. The functionally graded nanobeam is composed of steel and alumina (Al_2O_3) where its properties are given in Table 3. The bottom surface of the beam is pure steel, whereas the top surface of the beam is pure alumina. The beam geometry has the following dimensions: L (length)=10,000 nm, b (width)=1000 nm and h (thickness)=100 nm. Relation described in Eq. (51) are performed in order to calculate the non-dimensional natural frequencies

$$\bar{\omega} = \omega L^2 \sqrt{\rho A / EI} \tag{51}$$

Table 3 Material properties of FGM constituents

Properties	Steel	Alumina (Al_2O_3)
E	210 (Gpa)	390 (GPa)
ρ	7800 (kg/m^3)	3960 (kg/m^3)
ν	0.30	0.24

Table 4 Convergence study for the first two natural frequencies for simply supported FG nanobeam (L/h=100, μ=2*10^{-12})

Method	k	p=0		p=0.5		P=2		P=10	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
DTM	14	9.0181	27.2771	7.0780	21.4091	5.8693	17.7528	5.1871	15.6890
	15	9.0180	27.9031	7.0780	21.9005	5.8693	18.1603	5.1871	16.0491
	16	9.0180	30.4655	7.0780	23.9117	5.8692	19.8280	5.1870	17.5228
	17	9.0180	29.9819	7.0780	23.5321	5.8692	19.5132	5.1870	17.2447
	18	9.0180	29.4141	7.0780	23.0864	5.8692	19.1437	5.1870	16.9181
	19	9.0180	29.4441	7.0780	23.1100	5.8692	19.1632	5.1870	16.9353
	20	9.0180	29.4995	7.0780	23.1535	5.8692	19.1993	5.1870	16.9672
	21	9.0180	29.4965	7.0780	23.1512	5.8692	19.1973	5.1870	16.9655
	22	9.0180	29.4912	7.0780	23.1470	5.8692	19.1939	5.1870	16.9625
	23	9.0180	29.4915	7.0780	23.1472	5.8692	19.1940	5.1870	16.9626
	24	9.0180	29.4919	7.0780	23.1475	5.8692	19.1943	5.1870	16.9628
	25	9.0180	29.4919	7.0780	23.1475	5.8692	19.1943	5.1870	16.9628
	26	9.0180	29.4918	7.0780	23.1475	5.8692	19.1943	5.1870	16.9628
	27	9.0180	29.4918	7.0780	23.1475	5.8692	19.1943	5.1870	16.9628
	28	9.0180	29.4918	7.0780	23.1475	5.8692	19.1943	5.1870	16.9628
Analytical		9.0180	29.4918	7.0779	23.1473	5.8692	19.1940	5.1870	16.9628

where I=bh^3/12 is the moment of inertia of the cross section of the beam. Table 4 shows the convergence study of DTM method for first two frequencies of nanobeam with various gradient indexes. It is found that in DTM method after a certain number of iterations eigenvalues converged to a value with good precision, so the number of iterations is important in DTM method convergence. From results of Table 4, high convergence rate of the method may be easily observed and it may be deduced that k=26 leads to accurate results. As seen in Table 4, second natural frequency converged after 26 iterations with 4 digit precision while the first natural frequency converged after 16 iterations.

After looking into the satisfactory results for the convergence of frequencies, one may compare the nondimensional frequencies of FG nanobeam associated with different slenderness ratios and constituent volume fraction indexes. To evaluate accuracy of the natural frequencies predicted by the present method, natural frequencies of simply supported FG nanobeam with various volume fraction index and L/h ratios previously analyzed by finite element method are reexamined. Tables 5-12 compares the results of the present study (both analytical and DTM-based solution) and the results presented by Eltaher *et al*. (2012) which has been obtained by finite element method for FG nanobeam with different FG distribution indexes, length-to-thickness ratios and nonlocal parameters. The non-dimensional fundamental frequencies of simply-supported FG nanobeam is presented in Tables 5-12, which figures out the effect of nonlocal parameter (varying from 0 to 5), material distribution index (varying from 0 to 10) and length-to-thickness ratios (varying from 20 to 100) on the natural frequency characteristics of FG nanobeam. One may clearly notice here that the Non-dimensional fundamental frequency parameters obtained in the present investigation are in excellent agreement to the results presented by analytical solution and the results provided by finite element method (Eltaher *et al*. 2012) for all cases that are used for comparison and validates the proposed method of solution. First of all, when the two parameters vanish (μ*10^{-12}=0 and p=0) the classical isotropic beam theory is rendered. Furthermore, the effects of slenderness ratios on the dimensionless frequency are presented in these tables. From the results, it can be observed that, when the slenderness ratio of FG nanobeam decreased (thickness reduces), the frequencies rise. As seen in Table 5-12, by fixing the nonlocal parameter and varying the material distribution parameter results decreasing in the fundamental frequencies, due to increasing in ceramics phase constituent, and hence, stiffness of the beam. However, the increasing of nonlocal parameter causes the decreasing in fundamental frequency, at a constant material graduation index. For the case in hand (simply supported FG nanobeam), changing the nonlocal parameter from 0 to 5 results in a decrease in fundamental frequency parameter of about 22% for the case of with L/h=20, as can be noted from the Tables 5-12. This results indicates that the effect of nonlocal small scale parameter soften the nanobeam. A qualitative effect of nonlocal parameter and material index on the first five dimensionless frequency of simply supported FG nanobeam drawn from Tables 5-12 is

Table 5 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=0)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	9.8797	9.8295	9.8295
	1	9.4238	9.3776	9.3776
	2	9.0257	8.9828	8.9828
	3	8.6741	8.6341	8.6341
	4	8.3607	8.3230	8.3230
	5	8.0789	8.0433	8.0433
50	0	9.8724	9.8631	9.8631
	1	9.4172	9.4097	9.4097
	2	9.0205	9.0135	9.0135
	3	8.6700	8.6636	8.6636
	4	8.3575	8.3514	8.3514
	5	8.0765	8.0707	8.0707
100	0	9.8700	9.8679	9.8679
	1	9.4162	9.4143	9.4143
	2	9.0197	9.0180	9.0180
	3	8.6695	8.6678	8.6678
	4	8.3571	8.3555	8.3555
	5	8.0762	8.0747	8.0747

Table 6 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=0.1)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	9.2129	9.1612	9.1611
	1	8.7879	8.7400	8.7400
	2	8.4166	8.3721	8.3720
	3	8.0887	8.0470	8.0469
	4	7.7964	7.7570	7.7570
	5	7.5336	7.4964	7.4963
50	0	9.2045	9.1924	9.1924
	1	8.7815	8.7698	8.7698
	2	8.4116	8.4006	8.4006
	3	8.0848	8.0744	8.0744
	4	7.7934	7.7835	7.7835
	5	7.5313	7.5219	7.5219
100	0	9.2038	9.1969	9.1969
	1	8.7806	8.7741	8.7741
	2	8.4109	8.4047	8.4047
	3	8.0842	8.0784	8.0784
	4	7.7929	7.7873	7.7873
	5	7.5310	7.5256	7.5256

Table 7 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=0.2)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	8.7200	8.6601	8.6600
	1	8.3175	8.2619	8.2618
	2	7.9661	7.9141	7.9140
	3	7.6557	7.6068	7.6067
	4	7.3791	7.3327	7.3327
	5	7.1303	7.0863	7.0862
50	0	8.7115	8.6895	8.6895
	1	8.3114	8.2900	8.2900
	2	7.9613	7.9410	7.9410
	3	7.6520	7.6327	7.6327
	4	7.3762	7.3577	7.3577
	5	7.1282	7.1104	7.1104
100	0	8.7111	8.6938	8.6938
	1	8.3106	8.2941	8.2941
	2	7.9607	7.9449	7.9449
	3	7.6515	7.6364	7.6364
	4	7.3758	7.3613	7.3613
	5	7.1279	7.1139	7.1139

Table 8 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=0.5)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	7.8061	7.7152	7.7149
	1	7.4458	7.3605	7.3602
	2	7.1312	7.0506	7.0503
	3	6.8533	6.7768	6.7766
	4	6.6057	6.5327	6.5324
	5	6.3830	6.3131	6.3129
50	0	7.7998	7.7413	7.7413
	1	7.4403	7.3854	7.3854
	2	7.1269	7.0745	7.0745
	3	6.8500	6.7998	6.7998
	4	6.6031	6.5548	6.5548
	5	6.3811	6.3345	6.3345
100	0	7.7981	7.7451	7.7451
	1	7.4396	7.3890	7.3890
	2	7.1263	7.0780	7.0779
	3	6.8496	6.8031	6.8031
	4	6.6028	6.5580	6.5580
	5	6.3808	6.3376	6.3376

Table 9 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=1)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	7.0904	6.9680	6.9676
	1	6.7631	6.6477	6.6473
	2	6.4774	6.3678	6.3674
	3	6.2251	6.1206	6.1202
	4	6.0001	5.9000	5.8897
	5	5.7979	5.7018	5.7014
50	0	7.0852	6.9918	6.9917
	1	6.7583	6.6703	6.6703
	2	6.4737	6.3895	6.3895
	3	6.2222	6.1414	6.1414
	4	5.9979	5.9202	5.9201
	5	5.7962	5.7212	5.7211
100	0	7.0833	6.9952	6.9952
	1	6.7577	6.6736	6.6736
	2	6.4731	6.3926	6.3926
	3	6.2217	6.1444	6.1444
	4	5.9976	5.9230	5.9230
	5	5.7960	5.7240	5.7240

Table 10 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=2)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	6.5244	6.3969	6.3964
	1	6.2233	6.1028	6.1024
	2	5.9604	5.8459	5.8455
	3	5.7283	5.6189	5.6185
	4	5.5213	5.4164	5.4161
	5	5.3352	5.2344	5.2340
50	0	6.5189	6.4192	6.4192
	1	6.2191	6.1241	6.1240
	2	5.9571	5.8663	5.8662
	3	5.7257	5.6385	5.6385
	4	5.5193	5.4354	5.4353
	5	5.3338	5.2527	5.2526
100	0	6.5182	6.4224	6.4224
	1	6.2185	6.1272	6.1272
	2	5.9567	5.8692	5.8692
	3	5.7254	5.6413	5.6413
	4	5.5190	5.4381	5.4381
	5	5.3335	5.2553	5.2553

Table 11 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=5)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	6.0025	5.9174	5.9172
	1	5.7256	5.6454	5.6452
	2	5.4837	5.4077	5.4075
	3	5.2702	5.1977	5.1975
	4	5.0797	5.0105	5.0102
	5	4.9086	4.8421	4.8419
50	0	5.9990	5.9389	5.9389
	1	5.7218	5.6659	5.6659
	2	5.4808	5.4274	5.4273
	3	5.2679	5.2166	5.2166
	4	5.0780	5.0287	5.0287
	5	4.9072	4.8597	4.8597
100	0	5.9970	5.9420	5.9420
	1	5.7212	5.6689	5.6689
	2	5.4803	5.4302	5.4302
	3	5.2675	5.2194	5.2194
	4	5.0777	5.0313	5.0313
	5	4.9071	4.8622	4.8622

Table 12 The variation of non-dimensional fundamental natural frequencies for different nonlocal parameter and slenderness ratios of simply supported beams (b=1000 nm, L=10,000 nm, h=100 nm, p=10)

L/h	$\mu*10^{-12}$	FEM Eltaher *et al.* (2012)	Present	
			DTM	Analytical
20	0	5.7058	5.6522	5.6521
	1	5.4425	5.3923	5.3922
	2	5.2126	5.1653	5.1652
	3	5.0096	4.9648	4.9647
	4	4.8286	4.7859	4.7858
	5	4.6659	4.6250	4.6250
50	0	5.7001	5.6729	5.6729
	1	5.4389	5.4121	5.4121
	2	5.2098	5.1843	5.1843
	3	5.0074	4.9830	4.9830
	4	4.8269	4.8035	4.8034
	5	4.6646	4.6420	4.6420
100	0	5.7005	5.6759	5.6759
	1	5.4384	5.4150	5.4150
	2	5.2094	5.1870	5.1870
	3	5.0071	4.9856	4.9856
	4	4.8267	4.8060	4.8060
	5	4.6644	4.6445	4.6445

presented in Figs. 2, 3 and 4. Figs. 2, 3 and 4 demonstrate the variation of first five fundamental frequencies of FG nanobeam with varying of the material distribution and nonlocality parameter at L/h=20, 50 and 100 respectively. As can be noted, the first five dimensionless frequency of simply supported FG nanobeam decrease acutely as the material index parameter increases from 0 to 10. It can be observed that, the 1st and 2nd frequency reduce with high rate where the power exponent in range from 0 to 5 than that where power exponent in range between 5 and 10. While the 3rd, 4th and 5th frequencies reduce have high rate in range from 0 to 2. Also increasing nonlocal parameter from 0 to 5 results in a decrease in all first five fundamental frequency parameters of the FG nanobeam. The 1st frequency decreases as the nonlocality parameter increased with the same trend. Whereas, in higher frequencies the effect of the nonlocality parameter is more obvious when increase from 0 to $3*10^{-12}$ than that nonlocality parameter in interval between $3*10^{-12}$ and $5*10^{-12}$. Furthermore, the power exponent is effective only in the range 0-5 for 3rd, 4th

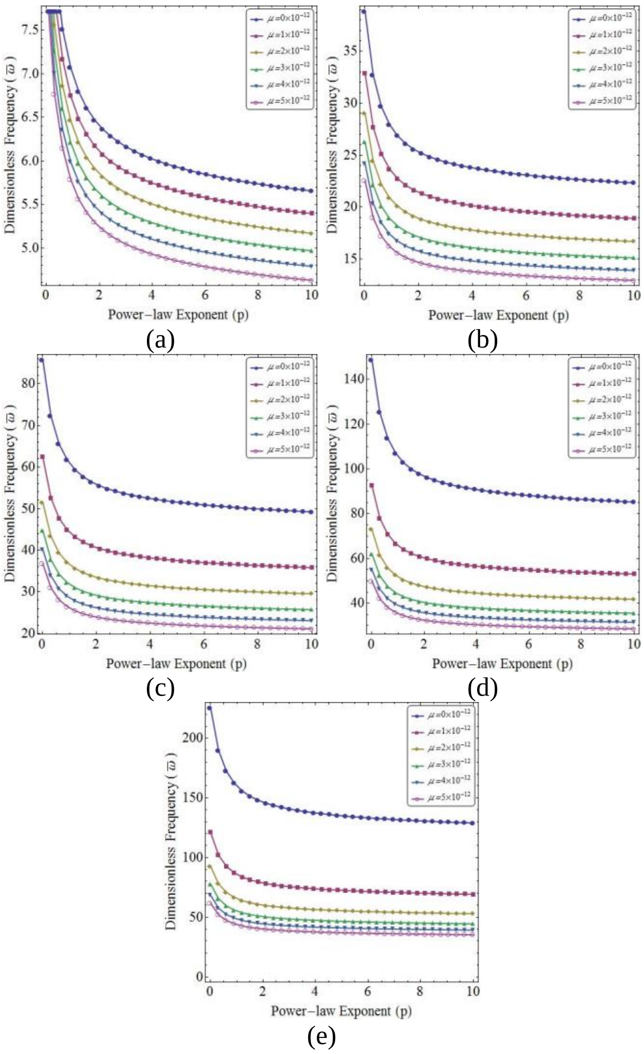


Fig. 2 The variation of the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th dimensionless frequency of simply supported FG nanobeam with material graduation for different nonlocality parameter (L/h=20)

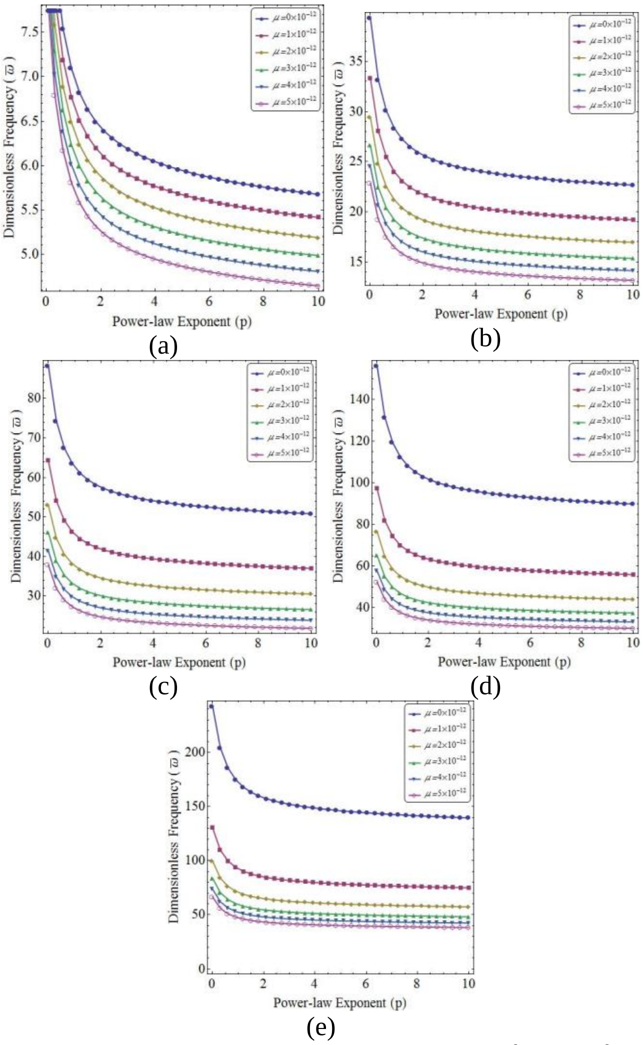


Fig. 3 The variation of the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th dimensionless frequency of simply supported FG nanobeam with material graduation for different nonlocality parameter (L/h=50)

and 5th frequencies at a constant nonlocal parameter. Figs. 5, 6 and 7 demonstrate the variation of mode number with changing of the nonlocality parameter at slenderness ratios (l/h=20, 50 and 100) of FG nanobeam with simply-supported edge conditions and different material distribution respectively. As presented, the influence of nonlocality parameter on the nondimensional frequency increased as the growing in mode number. Also, it can be deduced that, the influence of nonlocality parameter on the frequencies unaffected with the material distribution. In order to investigate the effects of different boundary conditions on FG nanobeam vibration characteristics the nondimensional frequencies of nanobeams with different edge conditions (clamped- clamped, clamped-simply supported and clamped-free) are tabulated in Tables 13-15. According to the obtained results using differential transform method (DTM), The effects of slenderness ratio, material exponent and nonlocal parameter on the first two dimensionless frequencies of the FG nanobeam for different boundary conditions are presented in Tables 13-15. As it can be seen from Tables, For all boundary conditions(C-C, C-S, C-F), the results show a decreasing of about 42% in first two natural frequencies, at a constant slenderness ratio and nonlocal parameter.

From Table 13, by increasing the nonlocal parameter from 0 to $5*10^{-12}$, at a fixed slenderness ratio and material graduation parameter, the first nondimensional frequency decreases about 20-22%. Also, it can be noticed that, for a constant nonlocal parameter and material graduation parameter, by changing slenderness ratio from 20 to 100, the 1st and 2st nondimensional frequencies increases about 1.6 % and 3.6% respectively.

The effect of nonlocal and material distribution parameter and slenderness ratio on the frequencies of clamped- simply supported FG nanobeam, is illustrated in Table 14. It is observed that, by increasing the nonlocal parameter from 0 to $5*10^{-12}$, the first natural frequency decreases by about 20-22% at a fixed slenderness ratio and material graduation parameter. Also, it can be noticed that, when the slenderness ratio increases, for a constant nonlocal

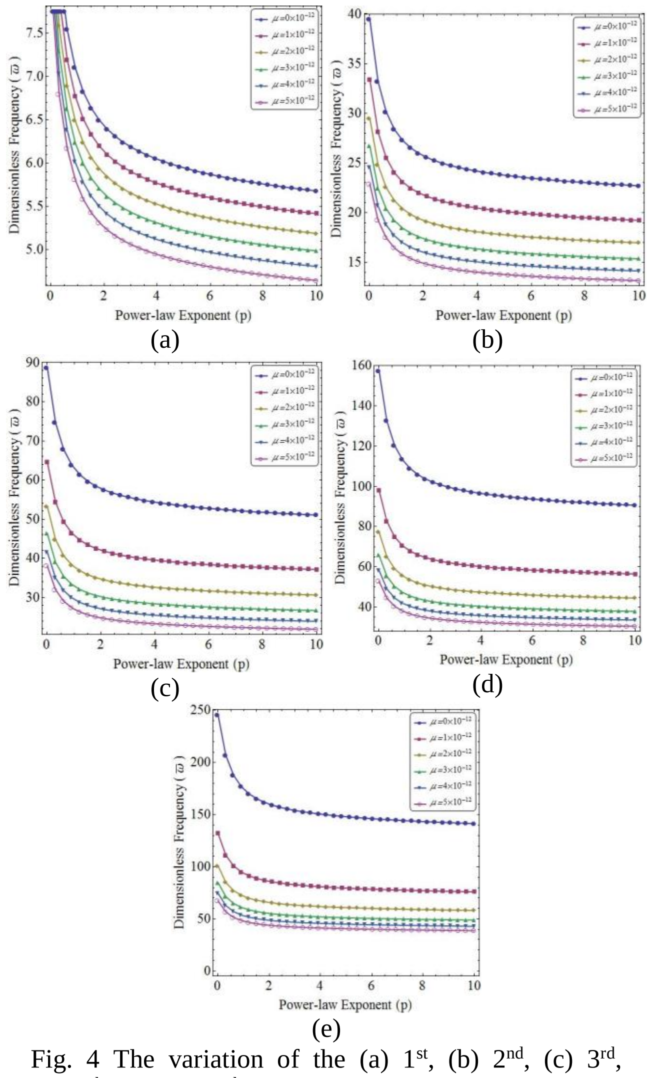


Fig. 4 The variation of the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th and (e) 5th dimensionless frequency of simply supported FG nanobeam with material graduation for different nonlocality parameter (L/h=100)

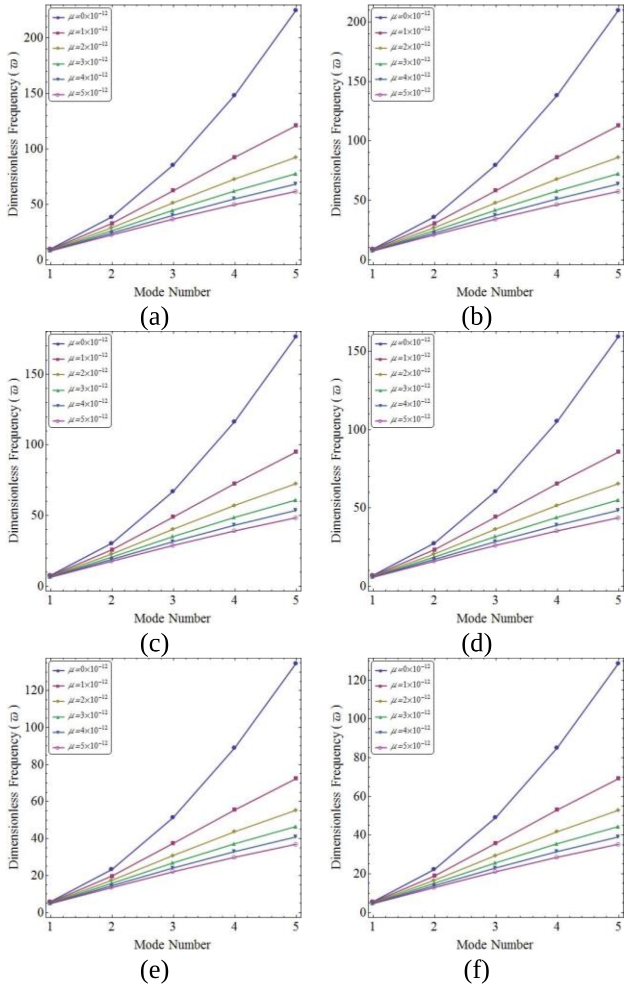


Fig. 5 The effect of nonlocality parameter on dimensionless frequency of FG nanobeam for various mode numbers and with different material graduation indexes (p=(a) 0, (b) 0.1, (c) 0.5, (d) 1, (e) 5, (f) 10) (L/h=20)

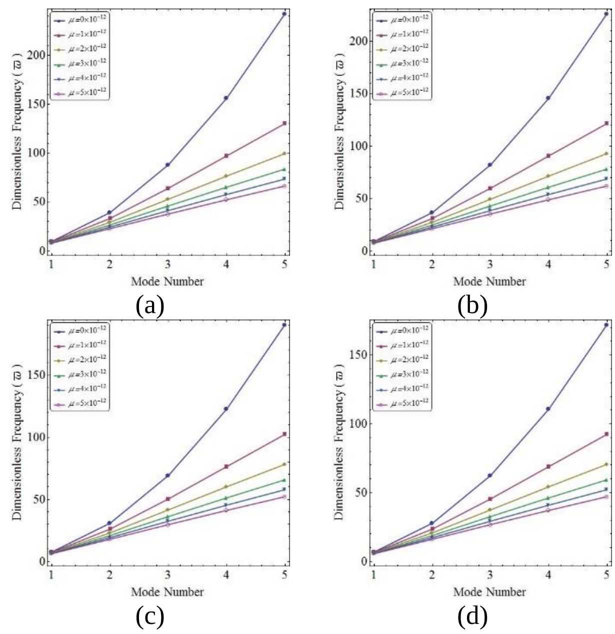
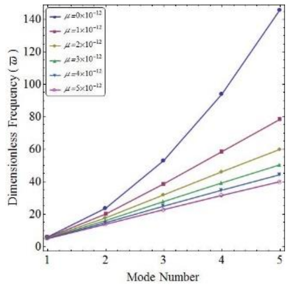
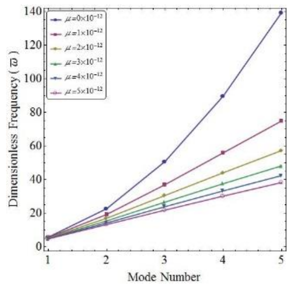


Fig. 6 The effect of nonlocality parameter on dimensionless frequency of FG nanobeam for various mode numbers and with different material graduation indexes (p = (a) 0, (b) 0.1, (c) 0.5, (d) 1, (e) 5, (f) 10) (L/h=50)

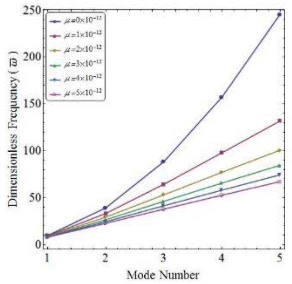


(e)

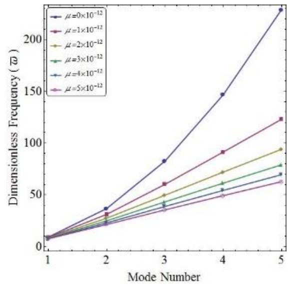


(f)

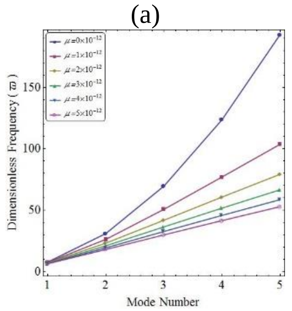
Fig. 6 Continued



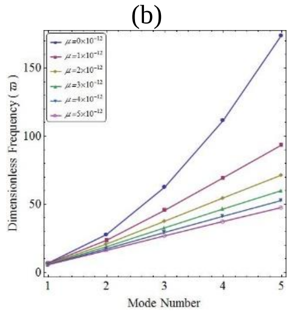
(a)



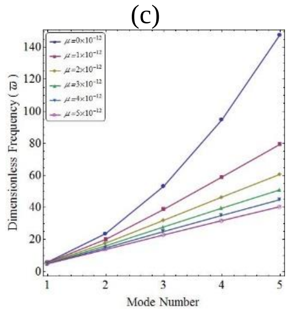
(b)



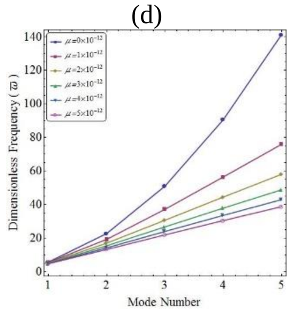
(c)



(d)



(e)



(f)

Fig. 7 The effect of nonlocality parameter on dimensionless frequency of FG nanobeam for various mode numbers and with different material graduation indexes (p = (a) 0, (b) 0.1, (c) 0.5, (d) 1, (e) 5, (f) 10) (L/h=100)

parameter and material graduation parameter, the first two nondimensional frequencies increases about 1 % and 2% respectively.

From Table 15 for a clamped-free FG nanobeam, by increasing the nonlocal parameter from 0 to $5*10^{-12}$, the first natural frequency increases by about 2-2.5% at a fixed slenderness ratio and material graduation parameter. Whereas at the same condition, the second frequency decreases about 24%. It can be noticed that, for a constant nonlocal parameter and material graduation parameter, changing slenderness

Table 13 The variation of the first two nondimensional frequencies for different material distributions, nonlocal parameter and slenderness ratio for C-C beam

		Power-law exponent (p)									
L/h	$\mu*10^{-12}$	0		1		2		5		10	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
20	0	22.0110	59.4339	15.6058	42.1497	14.3190	38.6493	13.2349	35.6863	12.6397	34.0739
	1	20.7725	49.2094	14.7275	34.8917	13.5136	31.9991	12.4911	29.5522	11.9294	28.2179
	2	19.7169	42.8781	13.9788	30.4015	12.8270	27.8831	11.8569	25.7535	11.3239	24.5911
	3	18.8039	38.4798	13.3314	27.2825	12.2332	25.0235	11.3084	23.1135	10.8001	22.0705
	4	18.0047	35.2001	12.7647	24.9568	11.7134	22.8910	10.8281	21.1445	10.3414	20.1904
	5	17.2977	32.6347	12.2633	23.1378	11.2535	21.2229	10.4032	19.6042	9.9356	18.7196
50	0	22.3140	61.3074	15.8184	43.4619	14.5218	39.8948	13.4335	36.8982	12.8315	35.2433
	1	21.0540	50.6936	14.9251	35.9371	13.7018	32.9886	12.6751	30.5118	12.1071	29.1435
	2	19.9811	44.1509	14.1646	31.2987	13.0037	28.7311	12.0293	26.5745	11.4902	25.3828
	3	19.0540	39.6131	13.5073	28.0818	12.4003	25.7782	11.4712	23.8435	10.9572	22.7743
	4	18.2428	36.2323	12.9322	25.6851	11.8724	23.5782	10.9829	21.8087	10.4908	20.8308
	5	17.5256	33.5892	12.4238	23.8114	11.4057	21.8583	10.5512	20.2179	10.0784	19.3114
100	0	22.3584	61.5900	15.8496	43.6605	14.5515	40.0837	13.4626	37.0825	12.8596	35.4212
	1	21.0952	50.9182	14.9541	36.0952	13.7294	33.1384	12.7021	30.6575	12.1331	29.2841
	2	20.0198	44.3435	14.1918	31.4345	13.0295	28.8596	12.0546	26.6991	11.5147	25.5031
	3	19.0906	39.7847	13.5331	28.2028	12.4248	25.8926	11.4951	23.9543	10.9802	22.8813
	4	18.2777	36.3887	12.9568	25.7954	11.8957	23.6825	11.0056	21.9096	10.5127	20.9282
	5	17.5590	33.7339	12.4473	23.9134	11.4280	21.9547	10.5729	20.3112	10.0993	19.4013

Table 14 The variation of the first two nondimensional frequencies for different material distributions, nonlocal parameter and slenderness ratio for C-S beam

		Power−law exponent (p)									
L/h	$\mu*10^{-12}$	0		1		2		5		10	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
20	0	15.2719	48.6974	10.8268	34.5262	9.9370	31.6740	9.1888	29.2670	8.7764	27.9486
	1	14.4625	40.7629	10.2529	28.8998	9.4104	26.5142	8.7021	24.5016	8.3115	23.3983
	2	13.7674	35.7625	9.7600	25.3542	8.9582	23.2622	8.2840	21.4974	7.9122	20.5295
	3	13.1622	32.2475	9.3310	22.8620	8.5644	20.9760	7.9200	19.3852	7.5645	18.5124
	4	12.6293	29.6042	8.9532	20.9880	8.2177	19.2568	7.5995	17.7967	7.2584	16.9955
	5	12.1556	27.5235	8.6173	19.5127	7.9095	17.9034	7.3145	16.5461	6.9862	15.8012
50	0	15.3945	49.7555	10.9130	35.2717	10.0190	32.3795	9.2688	29.9512	8.8535	28.6086
	1	14.5771	41.6242	10.3335	29.5072	9.4869	27.0880	8.7766	25.0570	8.3835	23.9338
	2	13.8753	36.5089	9.8360	25.8809	9.0302	23.7592	8.3541	21.9779	7.9799	20.9928
	3	13.2645	32.9163	9.4030	23.3341	8.6327	21.4212	7.9864	19.8153	7.6287	18.9272
	4	12.7269	30.2161	9.0219	21.4199	8.2829	19.6640	7.6628	18.1899	7.3195	17.3746
	5	12.2490	28.0911	8.6831	19.9136	7.9718	18.2812	7.3750	16.9108	7.0447	16.1528
100	0	15.4123	49.9132	10.9255	35.3828	10.0309	32.4848	9.2804	30.0535	8.8647	28.7073
	1	14.5937	41.7525	10.3452	29.5977	9.4980	27.1736	8.7875	25.1399	8.3939	24.0139
	2	13.8909	36.6201	9.8470	25.9594	9.0407	23.8333	8.3643	22.0497	7.9897	21.0620
	3	13.2794	33.0159	9.4135	23.4044	8.6427	21.4876	7.9961	19.8796	7.6380	18.9891
	4	12.7411	30.3073	9.0319	21.4843	8.2923	19.7248	7.6719	18.2486	7.3283	17.4313
	5	12.2626	28.1757	8.6927	19.9733	7.9809	18.3375	7.3838	16.9652	7.0531	16.2053

Table 15 The variation of the first two nondimensional frequencies for different material distributions, nonlocal parameter and slenderness ratio for C-F beam

L/h	$\mu * 10^{-12}$	Power-law exponent (p) 0		1		2		5		10	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
20	0	3.5172	21.914	2.4933	15.5351	2.2892	14.2613	2.1180	13.1916	2.0231	12.6002
	1	3.5323	20.5407	2.5040	14.5612	2.2990	13.3668	2.1271	12.3635	2.0318	11.8090
	2	3.5478	19.3596	2.5150	13.7239	2.3091	12.5979	2.1364	11.6518	2.0407	11.1292
	3	3.5637	18.3286	2.5262	12.9931	2.3194	11.9268	2.1459	11.0308	2.0498	10.5359
	4	3.5800	17.4173	2.5378	12.3471	2.3300	11.3336	2.1557	10.4819	2.0592	10.0116
	5	3.5967	16.6035	2.5496	11.7701	2.3409	10.8039	2.1658	9.9918	2.0688	9.5434
50	0	3.5162	22.0151	2.4925	15.6061	2.2885	14.3281	2.1173	13.2560	2.0225	12.6622
	1	3.5314	20.6572	2.5033	14.6435	2.2984	13.4442	2.1265	12.4381	2.0313	11.8810
	2	3.5470	19.4854	2.5144	13.8129	2.3085	12.6816	2.1359	11.7325	2.0402	11.2069
	3	3.5630	18.4594	2.5258	13.0856	2.3190	12.0138	2.1455	11.1146	2.0494	10.6167
	4	3.5795	17.5503	2.5374	12.4411	2.3296	11.4221	2.1554	10.5672	2.0589	10.0938
	5	3.5963	16.7367	2.5493	11.8644	2.3406	10.8926	2.1656	10.0772	2.0686	9.6258
100	0	3.5160	22.0296	2.4924	15.6164	2.2884	14.3378	2.1172	13.2653	2.0224	12.6712
	1	3.5313	20.6740	2.5032	14.6554	2.2983	13.4554	2.1264	12.4489	2.0312	11.8914
	2	3.5469	19.5036	2.5143	13.8257	2.3085	12.6937	2.1358	11.7441	2.0402	11.2182
	3	3.5629	18.4784	2.5257	13.0990	2.3189	12.0264	2.1455	11.1268	2.0494	10.6284
	4	3.5794	17.5696	2.5373	12.4547	2.3296	11.4349	2.1554	10.5795	2.0588	10.1057
	5	3.5963	16.7560	2.5493	11.8780	2.3406	10.9054	2.1655	10.0896	2.0685	9.6377

ratio has no considerable effect on the nondimensional frequencies of FG nanobeam.

5. Conclusions

Vibration analysis of FG nanobeams based on Timoshenko beam theory and Eringen nonlocal constitutive equations with various boundary conditions is investigated. The Navier-based analytical model and a semi analytical differential transform method are employed in solving the governing equations derived through Hamilton's principle. The good agreement between the results of this article and those available in literature validated the presented approach. Finally, through some parametric study and numerical examples, the effect of different parameters are investigated. The effects of small scale parameter, material property gradient index, mode number, slenderness ratio and boundary conditions on fundamental frequencies of FG nanobeams are investigated. Numerical results demonstrate that the small scale effects play an important role on the vibrational behavior of the FG nanobeam. Thus, the nonlocality effects should be reflected in the study of dynamic behavior of nanostructures. Also, it is observed that the power-law index has an important effect on the vibration responses of FG nanobeam, and the dynamic behavior can be enhanced by selecting appropriate values of the power-law index.

References

Asghari, M., M.T. Ahmadian, M.H. Kahrobaiyan, M. Rahaeifard (2010), "On the size-dependent behavior of functionally graded micro-beams", *Mater. Des.*, **31**(5), 2324-2329.

Alshorbagy, A.E., M.A. Eltaher, F.F. Mahmoud (2011), "Free vibration characteristics of a functionally graded beam by finite element method", *Appl. Math. Model.*, **35**(1), 412-425.

Aghelinejad, M., Zare, K., Ebrahimi, F. and Rastgoo, A. (2011), "Nonlinear thermomechanical post-buckling analysis of thin functionally graded annular plates based on Von-Karman's Plate Theory", *Mech. Adv. Mater. Struct.*, **18**(5), 319-326.

Aifantis, E.C. (1984), "On the microstructural origin of certain inelastic models", *J. Eng. Mater. Technol.*, **106**(4), 326-330.

Baughman, R.H., Zakhidov, A.A. and de Heer, W.A. (2002), "Carbon nanotubes--the route toward applications", *Science*, **297**(5582), 787-792.

Civalek, O., C. Demir, B. Akgoz (2010), "Free vibration and bending analyses of cantilever microtubules based on nonlocal continuum model", *Math. Comput. Appl.*, **15**(2), 289-298.

Ebrahimi, F. (2013), "Analytical investigation on vibrations and dynamic response of functionally graded plate integrated with piezoelectric layers in thermal environment", *Mech. Adv. Mater. Struct.*, **20**(10), 854-870.

Ebrahimi, F. and A. Rastgoo (2008a), "Free vibration analysis of smart annular FGM plates integrated with piezoelectric layers", *Smart Mater. Struct.*, **17**(1), 015044.

Ebrahimi, F. and A. Rastgoo (2008b), "An analytical study on the free vibration of smart circular thin FGM plate based on classical plate theory", *Thin-Wall. Struct.*, **46**(12), 1402-1408.

Ebrahimi, F., Naei, M.H. and Rastgoo, A. (2009), "Geometrically nonlinear vibration analysis of piezoelectrically actuated FGM plate with an initial large deformation", *J. Mech. Sci. Technol.*, **23**(8), 2107-2124.

Ebrahimi, F. and Rastgoo, A. (2011), "Nonlinear vibration analysis of piezo-thermo-electrically actuated functionally graded circular plates", *Arch. Appl. Mech.*, **81**(3), 361-383.

Ebrahimi, F. and Rastgoo, A. (2009), "Nonlinear vibration of smart circular functionally graded plates coupled with piezoelectric layers", *Int. J. Mech. Mater. Des.*, **5**(2), 157-165.

Ebrahimi, F. and Barati, M.R. (2016a), "Temperature distribution effects on buckling behavior of smart heterogeneous nanosize plates based on nonlocal four-variable refined plate theory", *Int. J. Smart Nano Mater.*, **7**(3) 1-25.

Ebrahimi, F. and Barati, M.R. (2016b), "Vibration analysis of smart piezoelectrically actuated nanobeams subjected to magneto-electrical field in thermal environment", *J. Vib. Control*, 1077546316646239.

Ebrahimi, F. and Barati, M.R. (2016c), "Size-dependent thermal stability analysis of graded piezomagnetic nanoplates on elastic medium subjected to various thermal environments", *Appl. Phys. A*, **122**(10), 910.

Ebrahimi, F. and Barati, M.R. (2016d), "Static stability analysis of smart magneto-electro-elastic heterogeneous nanoplates embedded in an elastic medium based on a four-variable refined plate theory", *Smart Mater. Struct.*, **25**(10), 105014.

Ebrahimi, F. and Barati, M.R. (2016e), "Buckling analysis of piezoelectrically actuated smart nanoscale plates subjected to magnetic field", *J. Intel. Mater. Syst. Struct.*, 1045389X16672569.

Ebrahimi, F., Barati, M.R. and Dabbagh, A. (2016), "A nonlocal strain gradient theory for wave propagation analysis in temperature-dependent inhomogeneous nanoplates", *Int. J. Eng. Sci.*, **107**, 169-182.

Ebrahimi, F. and Dabbagh, A. (2016), "On flexural wave propagation responses of smart FG magneto-electro-elastic nanoplates via nonlocal strain gradient theory", *Compos.*

Struct., **162**, 281-293.

Ebrahimi, F. and Hosseini, S.H.S. (2016a), "Thermal effects on nonlinear vibration behavior of viscoelastic nanosize plates", *J. Therm. Stress.*, **39**(5), 606-625.

Ebrahimi, F. and Hosseini, S.H.S. (2016b), "Double nanoplate-based NEMS under hydrostatic and electrostatic actuations", *Eur. Phys. J. Plus*, **131**(5), 1-19.

Ebrahimi, F. and Barati, M.R. (2016f), "A nonlocal higher-order shear deformation beam theory for vibration analysis of size-dependent functionally graded nanobeams", *Arab. J. Sci. Eng.*, **41**(5), 1679-1690.

Ebrahimi, F. and Barati, M.R. (2016g), "Vibration analysis of nonlocal beams made of functionally graded material in thermal environment", *Eur. Phys. J. Plus*, **131**(8), 279.

Ebrahimi, F. and Barati, M.R. (2016h), "Dynamic modeling of a thermos-piezo-electrically actuated nanosize beam subjected to a magnetic field", *Appl. Phys. A*, **122**(4), 1-18.

Ebrahimi, F. and Barati, M.R. (2016i), "A unified formulation for dynamic analysis of nonlocal heterogeneous nanobeams in hygro-thermal environment", *Appl. Phys. A*, **122**(9), 792.

Ebrahimi, F. and Barati, M.R. (2016j), "A nonlocal higher-order refined magneto-electro-viscoelastic beam model for dynamic analysis of smart nanostructures", *Int. J. Eng. Sci.*, **107**, 183-196.

Ebrahimi, F. and Barati, M.R. (2016k), "Hygrothermal effects on vibration characteristics of viscoelastic FG nanobeams based on nonlocal strain gradient theory", *Compos. Struct.*, **159**, 433-444.

Ebrahimi, F. and Barati, M.R. (2016l), "Buckling analysis of nonlocal third-order shear deformable functionally graded piezoelectric nanobeams embedded in elastic medium", *J. Brazil. Soc. Mech. Sci. Eng.*, 1-16.

Ebrahimi, F. and Barati, M.R. (2016m), "Magnetic field effects on buckling behavior of smart size-dependent graded nanoscale beams", *Eur. Phys. J. Plus*, **131**(7), 1-14.

Ebrahimi, F. and Barati, M.R. (2016n), "Buckling analysis of smart size-dependent higher order magneto-electro-thermo-elastic functionally graded nanosize beams", *J. Mech.*, 1-11.

Ebrahimi, F. and Barati, M.R. (2017), "A nonlocal strain gradient refined beam model for buckling analysis of size-dependent shear-deformable curved FG nanobeams", *Compos. Struct.*, **159**, 174-182.

Ebrahimi, F., Ghadiri, M., Salari, E., Hoseini, S. A.H. and Shaghaghi, G.R. (2015a), "Application of the differential transformation method for nonlocal vibration analysis of functionally graded nanobeams", *J. Mech. Sci. Technol.*, **29**(3), 1207-1215.

Ebrahimi, F. and Salari, E. (2015a), "Thermal buckling and free vibration analysis of size dependent Timoshenko FG nanobeams in thermal environments", *Compos. Struct.*, **128**, 363-380.

Ebrahimi, F. and Salari, E. (2015b), "Size-dependent free flexural vibrational behavior of functionally graded nanobeams using semi-analytical differential transform method", *Compos. Part B: Eng.*, **79**, 156-169.

Ebrahimi, F. and Salari, E. (2015c), "Nonlocal thermo-mechanical vibration analysis of functionally graded nanobeams in thermal environment", *Acta Astronautica*, **113**, 29-50.

Ebrahimi, F., Salari, E. and Hosseini, S.A.H. (2015b), "Thermomechanical vibration behavior of FG nanobeams subjected to linear and nonlinear temperature distributions", *J. Therm. Stress.*, **38**(12), 1360-1386.

Ebrahimi, F. and Barati, M.R. (2016o), "An exact solution for buckling analysis of embedded piezoelectro-magnetically actuated nanoscale beams", *Adv. Nano Res.*, **4**(2), 65-84.

Ebrahimi, F. and Barati, M.R. (2016p), "Electromechanical buckling behavior of smart piezoelectrically actuated higher-order size-dependent graded nanoscale beams in thermal environment", *Int. J. Smart Nano Mater.*, **7**(2) 1-22.

Ebrahimi, F. and Barati, M.R. (2016q), "Small scale effects on hygro-thermo-mechanical vibration of temperature dependent nonhomogeneous nanoscale beams", *Mech. Adv. Mater. Struct.*, 1-13.

Ebrahimi, F., Salari, E. and Hosseini, S.A.H. (2016), "In-plane thermal loading effects on vibrational characteristics of functionally graded nanobeams", *Meccanica*, **51**(4), 951-977.

Eltaher M.A., S.A. Emam, F.F. Mahmoud (2012), "Free vibration analysis of functionally graded size-dependent nanobeams", *Appl. Math. Comput.*, **218**, 7406-7420.

Eltaher M.A., A.E. Alshorbagy, F.F. Mahmoud (2013), "Determination of neutral axis position and its effect on natural frequencies of functionally graded macro/ nanobeams", *Compos. Struct*., **99**, 193-201.

Eltaher M.A., S.A. Emam, F.F. Mahmoud (2013b), "Static and stability analysis of nonlocal functionally graded nanobeams", *Compos. Struct*., **96**, 82-88.

Eringen, A.C. (1972b), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710.

Iijima, S. (1991), "Helical microtubules of graphitic carbon", *Nature*, **354**(6348), 56-58.

Ke, L.L. and Y.S. Wang (2011), "Size effect on dynamic stability of functionally graded microbeams based on a modified couple stress theory", *Compos. Struct.*, **93**(2), 342-350.

Lee, Z., Ophus, C., Fischer, L.M., Nelson-Fitzpatrick, N., Westra, K.L., Evoy, S. and Mitlin, D. (2006), "Metallic NEMS components fabricated from nanocomposite Al-Mo films", *Nanotechnology*, **17**(12), 3063-3070.

Maranganti, R. and Sharma, P. (2007), "Length scales at which classical elasticity breaks down for various materials", *Phys. Rev. Lett.*, **98**(19), 195504.

Mindlin, R.D. (1964), "Micro-structure in linear elasticity", *Arch. Ration. Mech. Anal.*, **16**(1), 51-78.

Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int. J. Eng. Sci.*, **45**(2), 288-307.

Simsek, M. and H.H. Yurtcu (2013), "Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory", *Compos. Struct*., **97**, 378-386.

Tauchert, T.R. (1974), *Energy principles in structural mechanics*, New York : McGraw-Hill.

Thai, H.T. (2012), "A nonlocal beam theory for bending, buckling, and vibration of nanobeams", *Int. J. Eng. Sci.*, **52**, 56-64.

Wang, X. and Cai, H. (2006), "Effects of initial stress on non-coaxial resonance of multi-wall carbon nanotubes", *Acta Materialia*, **54**(8), 2067-2074.

Wang, Q. (2005), "Wave propagation in carbon nanotubes via nonlocal continuum mechanics", *J. Appl. Phys.*, **98**(12), 124301.

Wang, Q. and Varadan, V.K. (2006), "Vibration of carbon nanotubes studied using nonlocal continuum mechanics", *Smart Mater. Struct.*, **15**(2), 659.

Witvrouw, A. and Mehta, A. (2005), "The use of functionally graded poly-SiGe layers for MEMS applications", *Funct. Graded Mater. VIII*, **492-493**, 255-260.

Zhang, Y.Q., Liu, G.R. and Wang, J.S. (2004), "Small-scale effects on buckling of multiwalled carbon nanotubes under axial compression", *Phys. Rev. B*, **70**(20), 205430.

Zhu, H., Wang, J. and Karihaloo, B. (2009), "Effects of surface and initial stresses on the bending stiffness of trilayer plates and nanofilms", *J. Mech. Mater. Struct*., **4**(3), 589-604.

Zhou, J.K. (1986), "Differential transformation and its applications for electrical circuits", Huazhong University Press, Wuhan, China.

CY