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The effect of different functionalities of FGM and FGPM layers on free vibration analysis of the FG circular plates integrated with piezoelectric layers

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Abstract. The present paper deals with the free vibration analysis of the functionally graded solid and annular circular plates with two functionally graded piezoelectric layers at top and bottom subjected to an electric field. Classical plate theory (CPT) is used for description of the all deformation components based on a symmetric distribution. All the mechanical and electrical properties except Poisson's ratio can vary continuously along the thickness direction of the plate. The properties of plate core can vary from metal at bottom to ceramic at top. The effect of non homogeneous index of functionally graded and functionally graded piezoelectric sections can be considered on the results of the system. 1st and 2nd modes of natural frequencies of the system have been evaluated for both solid and annular circular plates, individually.

Keywords: circular plate; free vibration; functionally graded piezoelectric material; natural frequency

1. Introduction

Intelligent or smart materials have been discovered in India. They have found that these materials can absorb tiny particles when heated (Gautschi 2002, Tichý *et al.* 2010). Due to this phenomenon, the new introduced materials have been named "Ceylon Magnet". Quartz has been known as first applied intelligent materials. In 1880, Pierre and Jacques Curie have presented scientific definition of smart materials. Since that time, these materials have been named piezoelectric materials. Word "piezoelectric" have been extracted from Greece word "piezen" that means pressure.

Although, these materials primarily have been known as a material with absorption property, since 1880, sensorial applications has been highlighted and increased as a main and fundamental application of these materials especially in industrial instruments. Investigation on the relation between applied loads and electric potential in a piezoelectric structure such as plate, shell and disk can be considered as an important subject in the context of piezoelectric structures. Furthermore, usage the material with variable properties (functionally graded piezoelectric materials) can improve and develop design condition in order to attaining the best optimization of structure. Using the material with variable properties can propose optional design for designers

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and engineers.

The properties of functionally graded materials can vary continuously along the axis of coordinate system. These materials have been created for the first time in laboratory by a Japanese group of the material scientists (Yamanouchi *et al.* 1990). For many advantageous properties, these materials can be used in the vigorous environments with abruptly gradient of the pressure and temperature.

A comprehensive study of literature can carefully justify the necessity of the proposed subject and novelty of this study.

Nonlinear analysis of functionally graded plates and shallow shells has been studied by Woo and Meguid (2001). They considered the effect of geometric nonlinearity on the results of the problem. Chen *et al*. (2004) studied the free vibration analysis of simply supported, fluid-filled cylindrically orthotropic functionally graded cylindrical shells with arbitrary thickness. Predefined functions with unknown amplitudes have been employed in order to substitute in governing differential equations. An exponential function of natural frequency and time dependency has been considered in predefined functions.

Prakash and Ganapathi (2006) analyzed flexural vibration and thermo elastic stability of the functionally graded circular plates using the finite element approach. Varying along the thickness direction and uniform distribution has been considered for temperature distribution. Fundamental governing equations have been derived by using the Lagrange's equation of the motion.

Huang and Shen (2006) considered the vibration analysis and dynamic responses of functionally graded square plates bonded with piezoelectric layers under thermal loads. The nonlinear formulations were based on the higher-order shear deformation and general von Karman plate theories. Thermal buckling and free vibration analyses of a functionally graded cylindrical shell with temperature-dependent material properties were presented by Kadoli and Ganesan (2006). Clamped-clamped boundary conditions were imposed on the boundaries. Bhangale et al. (2006) performed the linear thermoelastic buckling and free vibration analyses for a functionally graded truncated conical shells.

The linear and nonlinear vibration analyses of a three-layer coating-FGM-substrate cylindrical panel with general boundary conditions were performed by Liew *et al.* (2006). They assumed that cylindrical panel subjected to a temperature gradient along the thickness direction. The natural frequencies have been calculated in terms of various parameters such as non homogeneous index, temperature rising and mode of vibration. GhannadPour and Alinia (2006) investigated the large deflection analysis of a rectangular FG plate based on the Von Karman theory for simulation of the large deflection. The solution was obtained using minimization of the total potential energy with respect to unknown parameters. The solution has directed authors to investigate the effect of non homogeneity on the stresses and deformations.

Nonlinear thermal bending analysis of FGM plates under combination of thermal and electrical loads has been studied by Hui-Shen Shen (2007).

Efraim and Eisenberger (2007) analyzed the free vibration analysis of the annular FGM plates. They considered three displacement components along the radial, circumferential and transverse directions and two rotational components around the radial and circumferential directions. Allahverdizadeh *et al*. (2008 a, b) investigated the nonlinear free and forced vibration of thin circular FG plates. For the assumed structure, they performed a study for studying the vibration amplitude and thermal effect. The analysis was assumed to be axisymmetric and solution was derived based on a semi-analytical approach. Non-linear shooting and Adomian decomposition methods have been proposed to determine the large deflection of a cantilever beam under arbitrary loading conditions by Banerjee *et al*. (2008).

Ebrahimi and Rastgo (2008) investigated on the free vibration analysis of the circular plates made of functionally graded materials. The power function is employed for simulation of the material properties distribution along the thickness direction. Plate was composed of a FG layer and two FGP layers at top and bottom of that. The obtained results were verified by those obtained results from three dimensional finite element analyses. They showed that natural frequencies of the structure tend to an asymptotic value for non homogeneous indexes greater than 10.

Li *et al.* (2008) studied the three dimensional vibration analysis the functionally graded plates. They used various theories of plate analysis such as Classical Plate Theory (CPT), First order Shear Deformation Theory (FSDT), Third order Shear Deformation Theory (TSDT), and compared the previous results with the mentioned plate theories. The Hamilton's principle has been used for derivation of the final relations and results. A set of complete and orthogonal Chebyshev polynomial functions have been employed. Selection of these functions has some advantages such as high convergence rate and stability.

Alinia and GhannadPour (2009) investigated the large deflection analysis of a rectangular FG plate with logarithmic distribution of material properties. Khoshgoftar *et al.* (2009) investigated thermo elastic analysis of a FGP cylinder under pressure. It was assumed that all mechanical and electrical properties except Poisson ratio vary continuously along the thickness direction based on a power function. Malekzadeh and Vosoughi (2009) investigated the large amplitude vibration of composite beams on the nonlinear elastic foundation. The foundation was supposed that has cubic nonlinearity with shearing layer.

Sarfaraz Khabbaz *et al.* (2009) investigated the nonlinear analysis of FG plates under pressure based on the higher-order shear deformation theory. The first and higher order shear deformation theories were employed to investigate the large deflection of FG plate. The effect of the thickness and non homogeneous index were investigated on the distribution of the displacements and stresses. Some application and analysis of functionally graded and functionally graded piezoelectric materials can be considered in the recent publications (Soufyane 2009, Olfatnia *et al*. 2010, Zyłkaa and Janus 2010, Rahimi *et al.* 2011, Arefi and Rahimi 2011, 2012a, b, c, d, Arefi *et al.* 2011, 2012). Hosseinzadeh and Ahmadian (2010) developed static and dynamic instability of a functionally graded microbeam bonded with piezoelectric layers due to the electric actuation. The results of this research can be used in micro electro mechanical systems, especially in micro switches

Due to usage of piezoelectric structures with constant or variable properties in serious conditions, studying the free vibration characteristics of the functionally graded plates integrated with piezoelectric layers is important for technical applications. The effect of non homogeneous index can be considered as the most important results of this study. Studying the effect of various values of piezoelectric thickness and different boundary conditions is another important result of this paper.

2. Formulation

This section presents the basic equations for free vibration analysis of the functionally graded solid and annular circular plates with two functionally graded piezoelectric layers. The classic plate theory (CPT) is employed for description of the time-dependent displacements. In the classical plate theory, the displacement of every layer is expressed by two terms including the displacement of mid-plane and rotation about the mid-plane (Ugural 1981, Ebrahimi and Rastgo 2008, Arefi and Rahimi 2012b, Allahverdizadeh *et al.* 2008). Therefore, three time-dependent displacement components can be defined as

$$\begin{cases} u(r,z,t) = u_0(r,t) - z\dfrac{\partial w_0(r,t)}{\partial r} \\ w(r,z,t) = w_0(r,t) \end{cases} \tag{1}$$

where, u_0, w_0 are displacement components of the plate mid-plane ($z = 0$) and $\vec{u} = (u, 0, w)$ is symmetric displacement vector. The components of strains can be obtained using (Lai 1999)

$$\{\varepsilon\} = \tfrac{1}{2}\left\{\nabla\vec{\mathbf{u}} + \nabla^T\vec{\mathbf{u}}\right\} \tag{2}$$

where, $\vec{u} = (u, 0, w)$ is the vector of displacement as defined in Eq. (1). Due to symmetric distribution of loading, material properties and boundary conditions, the problem must be regarded as a symmetric problem and then $\varepsilon_{r\theta}$ and consequently $\sigma_{r\theta}$ must be zero. The components of strains can be obtained using Eqs. (1) and (2) as follows (Allahverdizadeh *et al.* 2008)

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_0}{\partial r} - z\frac{\partial^2 w_0}{\partial r^2} \\ \varepsilon_{\theta\theta} &= \frac{u_0}{r} - \frac{z}{r}\frac{\partial w_0}{\partial r} \\ \varepsilon_{r\theta} &= 0 \end{aligned} \tag{3}$$

As shown in Fig 1, the plate is containing the piezoelectric layers. Therefore, the constitutive equations for this structure are (Khoshgoftar *et al.* 2009, Arefi and Rahimi 2011, Qian *et al.* 2008)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \tag{4}$$

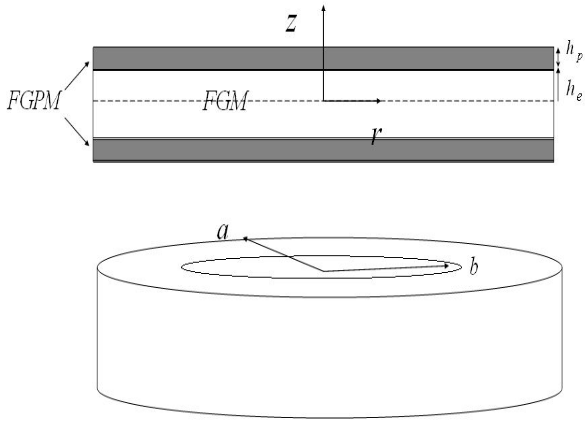

Fig 1 The schematic figure of a functionally graded circular plate bonded with functionally graded piezoelectric layers

in which, σ_{ij} and ε_{kl} are the stress and strain components, E_k is electric field, C_{ijkl} and e_{ijk} are the stiffness and piezoelectric coefficients. Electric field E_k is obtained by a potential function $\phi(r,z,t)$ as follows (Khoshgoftar *et al.* 2009, Arefi and Rahimi 2011, Ebrahimi and Rastgo 2008)

$$\phi = \phi(r,z,t) \rightarrow \begin{cases} E_r = -\dfrac{\partial \phi(r,z,t)}{\partial r} \\ E_\theta = 0 \\ E_z = -\dfrac{\partial \phi(r,z,t)}{\partial z} \end{cases} \tag{5}$$

Due to symmetric distribution of the electric potential, the circumferential component of electric field E_θ is zero. The electric displacement D_i in the electromechanical system is defined using the linear composition of the strain and electric field as (Qian *et al.* 2008, Khoshgoftar *et al.* 2009, Arefi and Rahimi 2012)

$$D_i = e_{ijk}\varepsilon_{jk} + \eta_{ik}E_k \tag{6}$$

where, η_{ik} are the dielectric coefficients. Due to small ratio of the plate thickness with respect to the radius of the plate, the normal stress σ_{zz} and shear stresses $\sigma_{rz}, \sigma_{\theta z}$ are negligible. The constitutive equations based on the plane stress condition for elastic sections of the plate (FG) $-h_e \le z \le h_e$ are expressed as

$$\begin{cases} \sigma_{rr} = C^e{}_{rrrr}\varepsilon_{rr} + C^e{}_{rr\theta\theta}\varepsilon_{\theta\theta} \\ \sigma_{\theta\theta} = C^e{}_{\theta\theta rr}\varepsilon_{rr} + C^e{}_{\theta\theta\theta\theta}\varepsilon_{\theta\theta} \end{cases} \tag{7}$$

The constitute equations for piezoelectric sections of the plate (FGP) $h_e \le |z| \le h_e + h_p$ are

$$\begin{cases} \sigma_{rr} = C^p{}_{rrrr}\varepsilon_{rr} + C^p{}_{rr\theta\theta}\varepsilon_{\theta\theta} - e_{rrr}E_r - e_{rrz}E_z \\ \sigma_{\theta\theta} = C^p{}_{\theta\theta rr}\varepsilon_{rr} + C^p{}_{\theta\theta\theta\theta}\varepsilon_{\theta\theta} - e_{\theta\theta r}E_r - e_{\theta\theta z}E_z \end{cases} \tag{8}$$

The electric displacement equations for piezoelectric sections of the plate (FGP) $h_e \le |z| \le h_e + h_p$ are

$$\begin{cases} D_r = e_{rrr}\varepsilon_{rr} + e_{r\theta\theta}\varepsilon_{\theta\theta} + \eta_{rr}E_r + \eta_{rz}E_z \\ D_z = e_{zrr}\varepsilon_{rr} + e_{z\theta\theta}\varepsilon_{\theta\theta} + \eta_{zr}E_r + \eta_{zz}E_z \end{cases} \tag{9}$$

Using Eqs. (3), (5), (7), (8) and (9), the potential energy per unit volume of the plate $\bar{u}_P$ can be evaluated analytically as follows (Arefi and rahimi 2011)

$$\begin{aligned} \bar{u}_P &= \tfrac{1}{2}\{\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} - \mathbf{E}^T\mathbf{D}\} \rightarrow \\ \bar{u}_P &= \tfrac{1}{2}\{\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} - D_r E_r - D_z E_z\} \end{aligned} \tag{10}$$

After derivation of potential energy of the system, we can use the Hamilton principle to compose total energy of the system. Total energy of the system is including potential energy, kinetic energy and energy due to external works. By introducing potential energy by $\bar{u}_p$, kinetic energy by $\bar{u}_k$ and energy due to external works by $\bar{u}_w$, Hamilton principle can be written for evaluation of total energy of the system, U that is including the strain energy, kinetic energy and energy due to external works as follows

$$\Upsilon = \bar{u}_k - \bar{u}_p$$
$$U = \iint_A \int_{-(h_e+h_p)}^{(h_e+h_p)} \Upsilon dz dA + \iint_A p(r) w dA \tag{11}$$

In which $\bar{u}_k$ and $\bar{u}_p$ are kinetic and potential energy, respectively and can be introduced as follows

$$\bar{u}_k = \frac{1}{2}\rho(z)\{\dot{u}^2(r,z,t) + \dot{w}^2(r,z,t)\}$$
$$\bar{u}_p = \frac{1}{2}\{\varepsilon^T\sigma - E^T D\} = \frac{1}{2}\{\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} - D_r E_r - D_z E_z\} \tag{12}$$

where, $\rho(z)$ is variable density of used material.

The energy equation must be divided for two different sections of the plate as follows

$$U = \iint_A \int_{-h_e}^{h_e} \frac{1}{2}\rho_f(z)\{\dot{u}^2(r,z,t) + \dot{w}^2(r,z,t)\} dz dA + 2\iint_A \int_{h_e}^{h_e+h_p} \frac{1}{2}\rho_p(z)\{\dot{u}^2(r,z,t) + \dot{w}^2(r,z,t)\} dz dA$$
$$- \iint_A \int_{-h_e}^{h_e} \frac{1}{2}\{\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta}\} dz dA - 2\iint_A \int_{h_e}^{h_e+h_p} \frac{1}{2}\{\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} - D_r E_r - D_z E_z\} dz dA \tag{13}$$
$$+ \iint_A p(r) w dA$$

The functionally graded annular circular plate is assumed to be under fixed supports at outer and inner radii. Therefore the displacements and the slope of that at outer and inner edges are zero. Furthermore it is assumed that electric potential is zero at outer and inner edges. The procedure of solution can be continued with assumption of three fields for displacements and electric potential. The appropriate power function can be employed for description of deformation and electric potential in an annular circular palte as follows (Ugural 1981)

$$u_0(r,t) = e^{i\omega t}[1-(\frac{r}{a})^2]^2 \times [1-(\frac{r}{b})^2]^2 \sum_{p=0} (U_p r^p)$$
$$w(r,t) = e^{i\omega t}[1-(\frac{r}{a})^2]^2 \times [1-(\frac{r}{b})^2]^2 \sum_{p=0} (W_p r^p) \tag{14}$$
$$\phi(r,z,t) = e^{i\omega t} f(z)[1-(\frac{r}{a})^2]^2 \times [1-(\frac{r}{b})^2]^2 \sum_{p=0} (\Phi_p r^p)$$

where, U_p, W_p and Φ_p describes the amplitudes of the displacement components and electric potential, n defines the number of the required terms for definition of the three assumed fields.

Harmonic vibration is considered by employing the time-dependent exponential function $e^{i\omega t}$. ω denotes the frequency of excited harmonic vibration. $f(z)$ guarantees this assumption that the electric potential at top and bottom of two piezoelectric layers must be zero. Therefore this function is (Ebrahimi and Rastgo 2008)

$$\phi(z=h_e)=\phi(z=h_e+h_p)=0\rightarrow f(z)=(1-\{\frac{2z-2h_e-h_p}{h_p}\}^2) \tag{15}$$

By substitution of strain-displacement and electric field-electric potential equations in the behavioral equations, we will have

$$\begin{cases}\sigma_{rr}=C^p{}_{rrrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+C^p{}_{rr\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})+e_{rrr}\frac{\partial\phi(r,z,t)}{\partial r}+e_{rrz}\frac{\partial\phi(r,z,t)}{\partial z}\\ \sigma_{\theta\theta}=C^p{}_{\theta\theta rr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+C^p{}_{\theta\theta\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})+e_{\theta\theta r}\frac{\partial\phi(r,z,t)}{\partial r}+e_{\theta\theta z}\frac{\partial\phi(r,z,t)}{\partial z}\\ D_r=e_{rrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+e_{r\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})-\eta_{rr}\frac{\partial\phi(r,z,t)}{\partial r}-\eta_{rz}\frac{\partial\phi(r,z,t)}{\partial z}\\ D_z=e_{zrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+e_{z\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})-\eta_{zr}\frac{\partial\phi(r,z,t)}{\partial r}-\eta_{zz}\frac{\partial\phi(r,z,t)}{\partial z}\end{cases} \tag{16}$$

Substitution of stress, strain, electric displacement and electric field equations in Eq. (12) presents final form of energy equation as follows

$$\begin{aligned}\bar{u}_k&=\tfrac{1}{2}\rho(z)\{[\frac{\partial u_0(r,t)}{\partial t}-z\frac{\partial}{\partial t}(\frac{\partial w_0(r,t)}{\partial r})]^2+[\frac{\partial w_0(r,t)}{\partial t}]^2\}\\ \bar{u}_p&=\tfrac{1}{2}\{[C^p{}_{rrrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+C^p{}_{rr\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})+e_{rrr}\frac{\partial\phi(r,z,t)}{\partial r}+e_{rrz}\frac{\partial\phi(r,z,t)}{\partial z}](\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})\\ &+[C^p{}_{\theta\theta rr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+C^p{}_{\theta\theta\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})+e_{\theta\theta r}\frac{\partial\phi(r,z,t)}{\partial r}+e_{\theta\theta z}\frac{\partial\phi(r,z,t)}{\partial z}](\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})\\ &+[e_{rrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+e_{r\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})-\eta_{rr}\frac{\partial\phi(r,z,t)}{\partial r}-\eta_{rz}\frac{\partial\phi(r,z,t)}{\partial z}]\frac{\partial\phi(r,z,t)}{\partial r}\\ &+[e_{zrr}(\frac{\partial u_0}{\partial r}-z\frac{\partial^2 w_0}{\partial r^2})+e_{z\theta\theta}(\frac{u_0}{r}-\frac{z}{r}\frac{\partial w_0}{\partial r})-\eta_{zr}\frac{\partial\phi(r,z,t)}{\partial r}-\eta_{zz}\frac{\partial\phi(r,z,t)}{\partial z}]\frac{\partial\phi(r,z,t)}{\partial z}\}\end{aligned} \tag{17}$$

Energy per unit volume of the structure that derived using Eq. (17) can be integrated for FGM and FGPM sections individually using Eq. (13). The total energy is evaluated in terms of the displacements amplitude U_p, W_p and electric potential Φ_p and natural frequency ω by substitution of displacement and electric potential fields from Eq. (14) into Eqs. (17) as follows

$$U=U(U_p,W_p,\Phi_p) \tag{18}$$

The solution of the system can be obtained by minimizing the energy equation (Eq. (18)) with respect to three amplitudes U_p, W_p, Φ_p. Rearranging the minimized energy equation yields

$$([\mathrm{K}]-\omega^2[M])\overrightarrow{U}=0 \tag{19}$$

In which $\overrightarrow{U} = \{U_p \quad W_p \quad \Phi_p\}^T$ is symbolic vector of displacements and electric potential.

Assumption of zero solution for $\overrightarrow{U}$ tends to a trivial solution. In order to avoid this trivial solution and for evaluation of the nontrivial solution, we must consider coefficient of $\overrightarrow{U}$. This consideration yields the natural frequencies of the system as follows

$$[K]\text{-}\omega^2[M] = 0 \tag{20}$$

This minimizing tends to a system of algebraic equations (Eq. (20)). The algebraic equation can be solved and then final solutions of the system may be obtained analytically. The algebraic equations can be solved by using the various analytical or numerical methods based on a computer program or package. Within these methods, solution by using the mathematical software is preferred.

3. Material properties

Before solution of the problem, it is appropriate to define the material properties for the FG and FGP layers. For FG layer, it is assumed that the bottom of the plate is steel and top of that is ceramic. Therefore the distribution of the material properties for FG layer is (Ebrahimi and Rastgo 2008)

$$E(z) = (E_c - E_m)(\frac{1}{2} + \frac{z}{2h_e})^{n_1} + E_m \qquad -h_e \le z \le h_e \tag{21}$$

where, $E(z = -h_e) = E_m$, $E(z = h_e) = E_c$, $2h_e$ is thickness of elastic solid section of the plate and n_1 is the non-homogeneous index of ceramic-metal section of the plate. The distribution of the mechanical and electrical properties for the two FGP layers can be supposed as a power function along the thickness direction as follows (Khoshgoftar *et al*. 2009)

$$E(z) = E_i(\frac{|z|}{h_e})^{n_2} \qquad h_e < |z| \le h_e + h_p \tag{22}$$

where, E_i represents the value of the all mechanical and electrical components at $|z| = h_e$ and h_e is thickness of the piezoelectric section. n_2 denotes the index of variable properties along the thickness direction in piezoelectric section of the plate. By these assumptions (Eqs. (21) and (22)), the energy equation can be obtained using Eq. (13). By minimization of the energy equation using Eq. (18), a set of equations can be derived. The solution of the equations can be evaluated using the Maple software.

The variable material properties of the FGP plate are selected as follows

$$Elastic\ solid\ section: \ C^e{}_{rrrr} = C^e{}_{\theta\theta\theta\theta} = \frac{E_1(z)}{1-v^2}, C^e{}_{rr\theta\theta} = C^e{}_{\theta\theta rr} = \frac{vE_1(z)}{1-v^2}$$

$$E_1(z) = (E_c - E_m)(\frac{1}{2} + \frac{z}{2h_e})^{n_1} + E_m \quad -h_e \le z \le h_e$$

$$piezoelectric\ \ section: C^{p}{}_{rrrr} = C^{p}{}_{\theta\theta\theta\theta} = \frac{E_2(z)}{1-\nu^2}, C^{p}{}_{rr\theta\theta} = C^{p}{}_{\theta\theta rr} = \frac{\nu E_2(z)}{1-\nu^2},$$

$$E_2(z) = E_{h_e}\left(\frac{|z|}{h_e}\right)^{n_2} \qquad h_e \le |z| \le h_e + h_p$$

$$e_{rrr} = e_1(z), e_{r\theta\theta} = e_{\theta\theta r} = e_{rrz} = e_{zrr} = e_{z\theta\theta} = e_{\theta\theta z} = e_2(z),$$

$$\eta_{rr} = \eta_{zz} = \eta_{1h_e}(z), \eta_{rz} = \eta_{zr} = \eta_{2h_e}(z)$$

$$e_1(z) = e_{1h_e}\left(\frac{|z|}{h_e}\right)^{n_2}, e_2(z) = e_{2h_e}\left(\frac{|z|}{h_e}\right)^{n_2}, \eta_{1h_e}(z) = \eta_{1h_e}\left(\frac{|z|}{h_e}\right)^{n_2}, \eta_{2h_e}(z) = \eta_{2h_e}\left(\frac{|z|}{h_e}\right)^{n_2} \quad (23)$$

$$\text{for } h_e \le |z| \le h_e + h_p$$

$$E_c = 3.8\times10^{11}pa, E_m = 2\times10^{11}pa, E_{h_e} = 7.6\times10^{10}pa, \rho_c = 3800\, kg/m^3, \rho_m = 2700\, kg/m^3, \rho_p = 7500\, kg/m^3$$

$$e_{1h_e} = 0.35VmN^{-1}, e_{2h_e} = -0.16VmN^{-1}, \eta_{1h_e} = 9.03\times10^{-11}mN^{-1}, \eta_{2h_e} = 5.62\times10^{-11}mN^{-1}$$

The necessary numerical values for the circular plate can be considered as follows

$$b = 0.03m, \quad a = 0.15m \quad, h_e = 0.01m \quad, h_p = 0.002m \quad, \nu = 0.3$$

4. Numerical results

In this section, the effect of different values of non homogeneous index can be considered on the behavior of the system. The obtained results can be classified to two classes. The first class of results considers the effect of different functionalities of FGM core while the second class considers the effect of different functionalities of FGPM integrated layers. All obtained results can be presented for two costume geometries i.e., solid circle and annular circular plates.

4.1 Solid circle

It is assumed that outer surface of the plate is constrained. For this model, we can employ following harmonic distribution for displacement and electric potential fields

$$u_0(r,t) = e^{i\omega t}[1-(\frac{r}{a})^2]^2 \sum_{p=0} (U_p r^p)$$

$$w(r,t) = e^{i\omega t}[1-(\frac{r}{a})^2]^2 \sum_{p=0} (W_p r^p) \quad (22)$$

$$\phi(r,z,t) = e^{i\omega t} f(z)[1-(\frac{r}{a})^2]^2 \sum_{p=0} (\Phi_p r^p)$$

The effect of non homogeneity can be considered on the natural frequencies of the system. These considerations present useful and applicable results about application of different non homogeneous indexes in a structure made of functionally graded materials bonded with piezoelectric materials. Shown in Figs 2, 3 are the 1st and 2nd natural frequencies of the solid circular plate in terms of different values of non homogeneous index of FGM core (n_1)

The vibration behavior of functionally graded piezoelectric solid circle can be validated by considering the appropriate reference (Ebrahimi and Rastgoo 2008). By comparing the Figs. 2 and 3 in this paper with same figures in the reference, you can conclude that by increasing the non homogeneous index of FGM core, natural frequencies of the system tend to an asymptotic value.

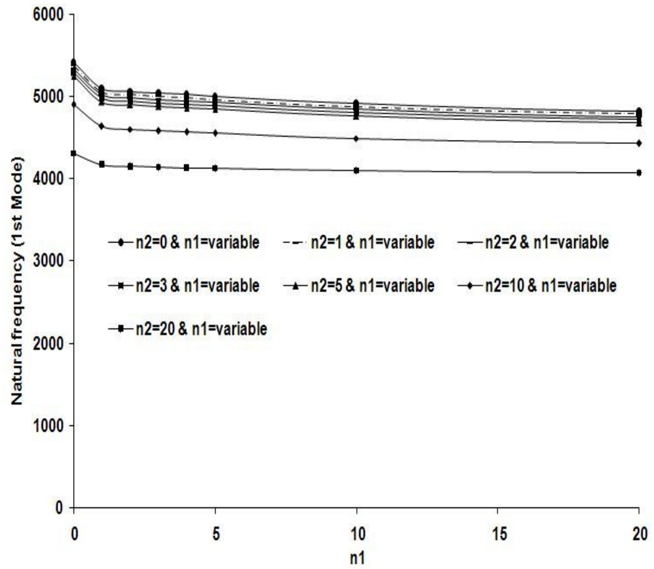

Fig. 2 1st natural frequency in terms of different values of non homogeneous index of FGM core

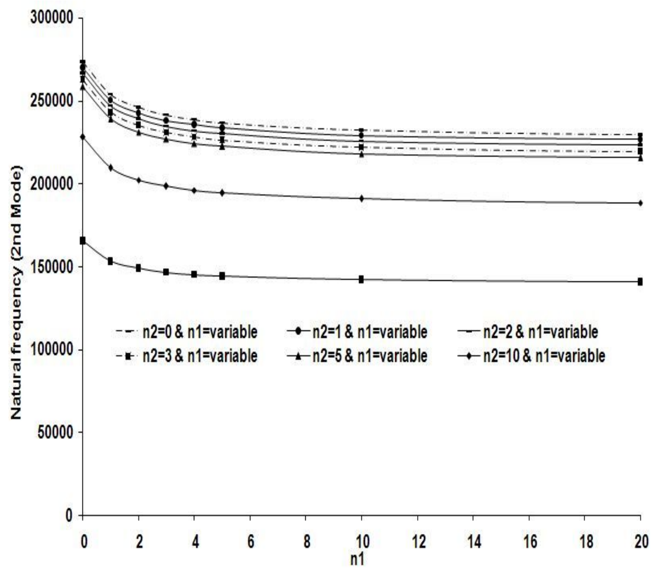

Fig. 3 2nd natural frequency in terms of different values of non homogeneous index of FGM core

Shown in Figs 4, 5 are the 1st and 2nd natural frequencies of the solid circular plate in terms of different values of non homogeneous index of FGPM layers (n_2).

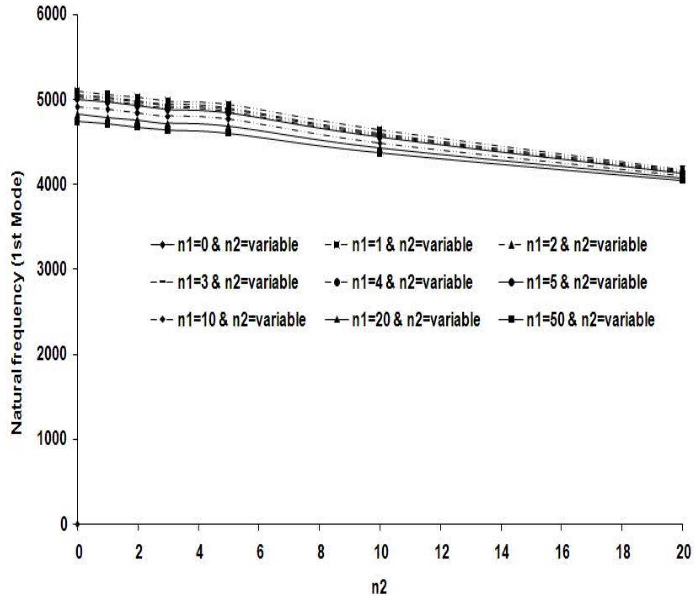

Fig. 4 1st natural frequency in terms of different values of non homogeneous index of FGPM layers

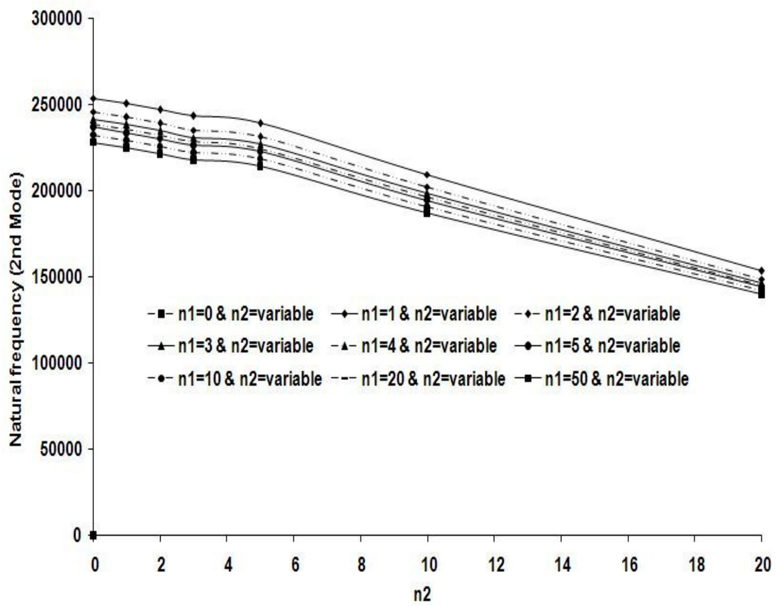

Fig. 5 2nd natural frequency in terms of different values of non homogeneous index of FGPM layers

4.2 Annular circle

For an annular circular plate model, we can employ Eq. (14) for harmonic distribution of displacements and electric potential. The effect of non homogeneous indexes (FGM and FGPM) can be studied on the natural frequencies of the plate. For this purpose, the 1st and 2nd modes of natural frequencies of the FGM circular plates are evaluated. Figs. 6 and 7 show the 1st and 2nd natural frequencies of annular circular plate in terms of variable non homogeneous index of FGM core. This distribution indicates that the value of natural frequencies tend to asymptotic value for large values of non homogeneous index (Ebrahimi and Rastgo 2008).

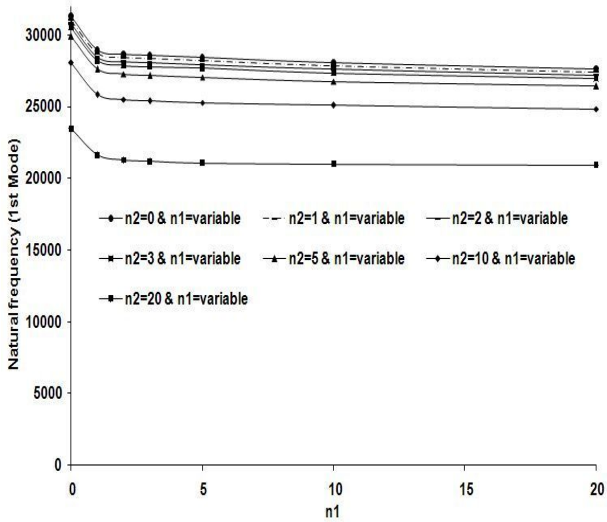

Fig. 6 1st natural frequency in terms of different values of non homogeneous index of FGM core

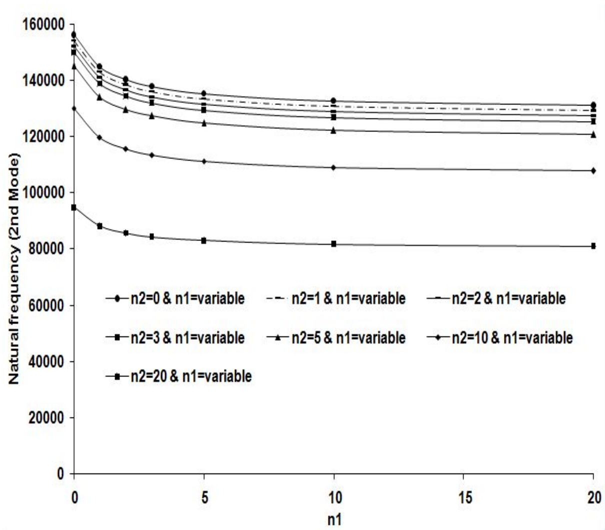

Fig. 7 2nd natural frequency in terms of different values of non homogeneous index of FGM core

Figs. 8 and 9 show the 1st and 2nd natural frequencies of annular circular plate in terms of variable non homogeneous index of FGPM layers.

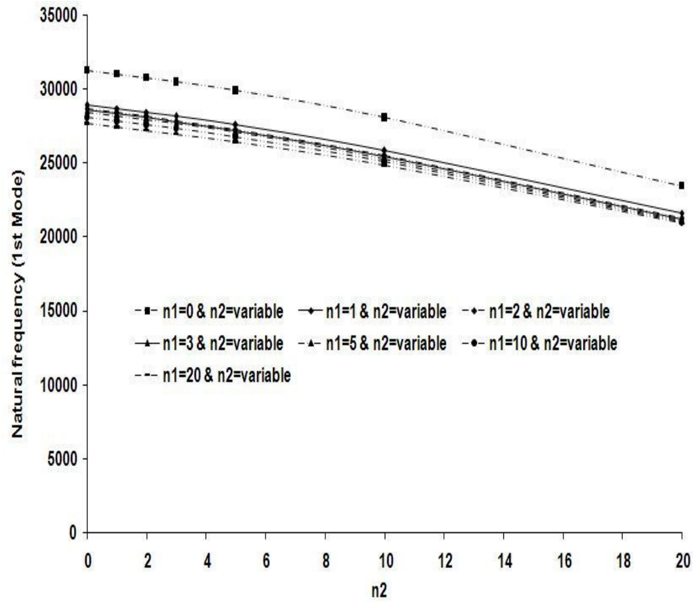

Fig. 8 1st natural frequency in terms of different values of non homogeneous index of FGPM core

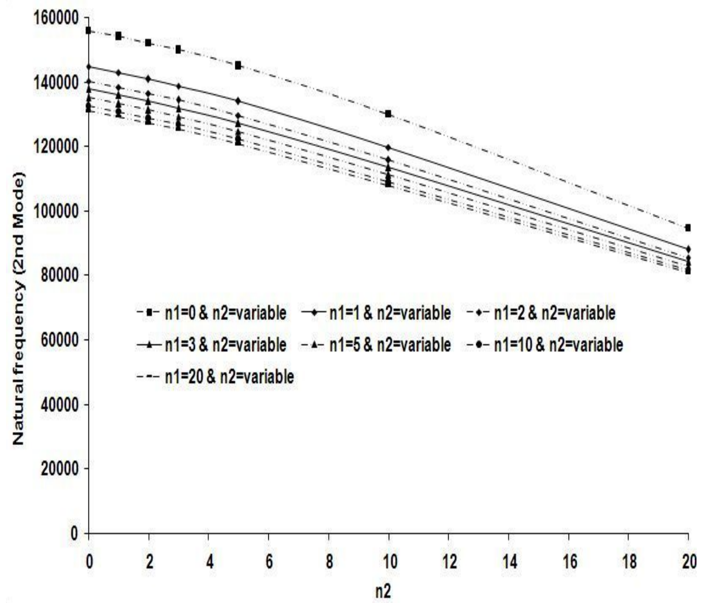

Fig. 9 2nd natural frequency in terms of different values of non homogeneous index of FGPM core

4.2.1 Amplitude of electric potential and radial displacement

The free vibration analysis can be completed by evaluating the amplitude of electric potential and radial displacement in terms of varying the non homogeneous index of FGM core. Figs. 10 and 11 show the amplitude of electric potential and radial displacement in terms of varying non homogeneous index. These results indicate that with increasing the non homogeneous index, both amplitudes tend to an asymptotic value.

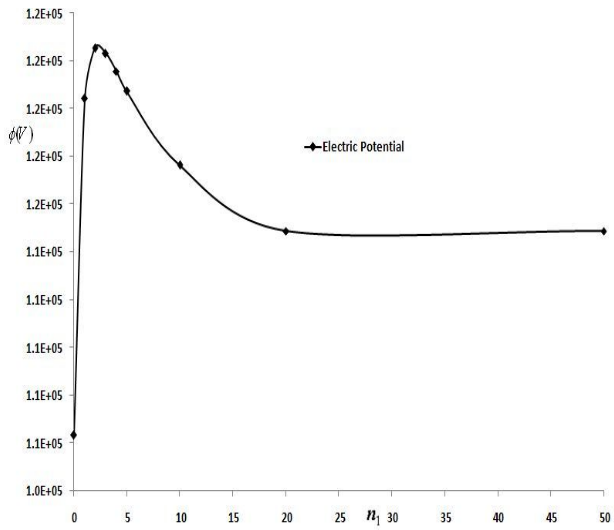

Fig. 10 The amplitude of electric potential in terms of different values of non homogeneous index of FGM core (n_1)

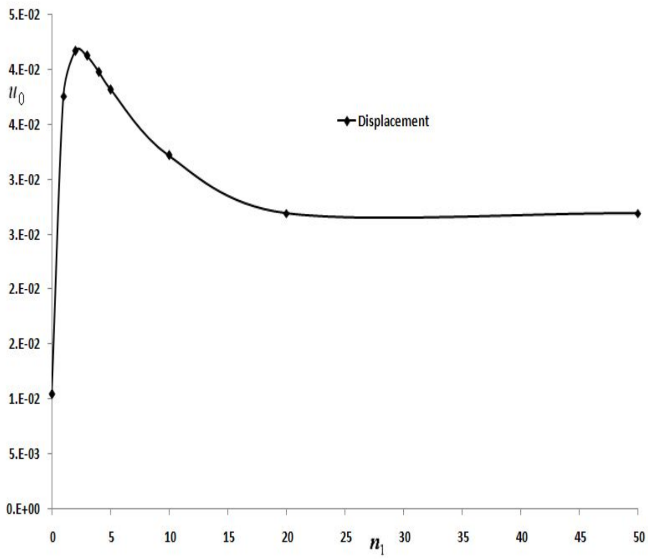

Fig. 11 The amplitude of displacement in terms of different values of non homogeneous index of FGM core (n_1)

5. Conclusions

The free vibration analysis of the functionally graded circular plates integrated with piezoelectric layers performed in the present study. Two custom types of circular plates have been considered for this analysis. Two natural frequencies of the plate in terms of various values of non homogeneous index have been evaluated for solid and annular circular plate, individually. The main results of this study can be presented as follows:

1. Investigation of the effect of varying the non homogeneous index of employed functionally graded material indicates that the natural frequencies of the system decreases monotonically with increasing the value of non homogeneous index. This decreasing is valid for both types of assumed boundary conditions and models. The values of natural frequencies tend to an asymptotic value for high values of non homogeneous index of FGM section. This result is in accordance with literature (Ebrahimi and Rastgo 2008).
2. The effect of variable non homogeneous index of FGPM section considered as another important result of this study. From the obtained results, it can be concluded that the natural frequencies of the both used models of the plates (solid and annular circular plates) decreases with increasing the non homogeneous index of piezoelectric layer, monotonically.
3. The effect of constrains can be studied on the vibration characteristics of the plate. The obtained results indicate that adding constraints to the plate increases natural frequency of 1st mode of vibration. Unlike, natural frequency of 2nd mode of vibration decreases while constraints add to structure.

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CC

Nomenclature

C_{ijkl} stiffness coefficient

$C^{e}{}_{ijkl}$ stiffness coefficient for FG layer

$C^{p}{}_{ijkl}$ stiffness coefficient for FGP layer

D_i electric displacement

e_{ijk} piezoelectric coefficient

E_k electric field components

$2h_e$ thickness of FG layer

h_p thickness of FGP layer

a,b outer and inner radii

p applied pressure

$E(z)$ distribution of material properties

n_1 non-homogeneous index of FGM section

r,z components of coordinate system

u,w displacement components at a general point

u_0,w_0 displacement components at mid-plane

ε_{ij} strain components

σ_{ij} stress components

$\bar{u}$ energy per unit volume

U total energy of system

η_{ik} dielectric coefficient

U_n,W_n,Φ_n amplitude of assumed function

ϕ electric potential

p the number of terms for displacement and electric field

n_2 non-homogeneous index of FGPM section