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Analytical solutions for density functionally gradient magneto-electro-elastic cantilever beams

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Abstract. The general solution for two-dimensional magneto-electro-elastic media in terms of four harmonic displacement functions is proposed analytically. The expressions of specific solutions of magneto-electro-elastic plane problems with specific body forces are derived. Finally, based on the general solution in the case of distinct eigenvalues and the specific solution for density functionally gradient media, two kinds of beam problems with body forces depending only on the z or x coordinate are solved by the trial-and-error method.

Keywords: general solution; magneto-electro-elastic plane; density functionally gradient media; analytical solutions.

1. Introduction

Due to their excellent properties, composites made of piezoelectric /piezomagnetric materials have found widespread hi-tech applications in many areas such as electronics, microwave, navigation and biology. Accordingly, these materials have been the focus of a considerable amount of research in recent years. In particular, critical information is provided by theoretical analyses and accurate quantitative descriptions of electric, magnetic, and stress fields inside piezoelectric /piezomagnetric composites under working conditions caused by the joint action of mechanical loads, electric fields, and magnetic fields.

In regard to piezoelectric materials, Sosa and Castro (1994) presented the solutions for the cases of concentrated loads and point charge applied at the line boundary of a piezoelectric half-plane. Ding, *et al*. (1997) obtained the general solution of plane problem of piezoelectric media, in which all physical quantities are expressed in three second-order harmonic displacement functions, as well as the solutions for a piezoelectric wedge subjected to concentrated forces and point charge. Ding, *et al*. (1997) derived Green's functions for a two-phase infinite piezoelectric plane. Ding, *et al*. (1997, 1998) derived the Green's functions and fundamental solutions for plane and half-plane piezoelectric problems. Shi

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(2001) studied a piezoelectric cantilever with non-uniform body force while Shi and Chen (2004) obtained a set of analytical solutions for a functionally graded piezoelectric cantilever beam subjected to different loadings, based on a pair of stress and induction functions in the form of polynomials. Shi (2002) obtained the stress and induction functions in the form of polynomials as well as the general solution of a density functionally gradient piezoelectric cantilever by using the Airy stress function method. Yang and Liu (2003) derived the analytical expressions of stress, electric displacement and electric potential for bending of a piezoelectric cantilever beam using the inverse method.

Magneto-electro-elastic materials simultaneously exert piezoelectric, piezomagnetic, and magnetoelectric effects. Pan (2001) and Pan, *et al.* (2002) derived the exact solutions for three-dimensional anisotropy linearly magneto-electro-elastic, simply-supported, and multi-layered rectangular plates under static loads and analytical solutions for free vibrations, respectively. Pan (2002) derived three-dimensional Green's functions in an-isotropic magneto-electro-elastic full space, half space, and bi-materials based on extended Stroh formalism by applying the two-dimensional Fourier transforms. Hou, *et al.* (2003) analyzed the elliptical Hertizan contact of transversely isotropic magneto-electro-elastic bodies with the general solutions in terms of harmonic functions. Wang and Shen (2002) obtained the general solution expressed by five harmonic functions and applied the derived general solution to find the fundamental solution for generalized dislocation and also to derive Green's functions for a semi-infinite magneto-electro-elastic solid. Ding and Jiang (2003) obtained the fundamental solution of an infinite magneto-electro-elastic solid via the method of trial-and-error and derived the boundary integral formulation. Chen, *et al.* (2005, 2003) analyzed the free vibration and bending of non-homogeneous magneto-electro-elastic plates and magneto-electric thermo-elasticity, respectively. Pan, *et al.* (2003) obtained the exact solutions for magneto-electro-elastic laminates in cylindrical bending. Heyliger and Pan (2004) analyzed the static fields in magneto-electro-elastic laminates. Wang, *et al.* (2003) conducted an analysis of multi-layered magneto-electro-elastic plates by the state vector approach.

For the magneto-electro-elastic plane problem, Wang and Shen (2003) presented analytic solutions for the plane problem of a inclusion of arbitrary shape in an entire plane, or within one of the two bonded dissimilar half-plane. Ding and Jiang (2004) obtained the two-dimensional fundamental solution for an infinite magneto-electro-elastic plane on the basis of the general solution, and implemented a boundary element method program to perform the numerical calculations. Jiang and Ding (2004) derived the general solution in the case of distinct eigenvalues in which all physical quantities are expressed in four harmonic displacement functions. They also obtained analytical solutions to various problems with the trial-and-error method, including rectangular beam under uniform tension, electric displacement and magnetic induction, pure shearing and pure bending, a cantilever beam with point forces at the free end, and cantilever beam subjected to uniformly distributed loads.

In this paper, we will consider the magneto-electro-elastic plane problems of density functionally gradient media. First, the specific solutions to plane problems with linearly or non-linearly distributed body forces are derived. In order to eliminate the surface tractions on surfaces and at ends caused by the specific solutions, we superpose the solutions on several kinds of general solutions and the rigid body motion solution obtained with the trial-and-error method to satisfy the boundary displacement conditions. Finally, two kinds of beam problems with body forces depending only on the z or x coordinate are solved.

2. General solution to the plane problem of magneto-electro-elastic solid

For transversely isotropic magneto-electro-elastic bodies, the basic equations have been given in Pan

(2001) (where *xoy* plane denotes the isotropic plane). For plane-strain problems, the displacements u_i, the electric potential Φ, and magnetic potential Ψ are assumed to be independent of *y*. The basic equations for a two-dimensional magneto-electro-elastic solid in the *xoz* coordinates can be simplified as follows

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+f_x=0,\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+f_z=0,$$

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_z}{\partial z}=f_e,\quad \frac{\partial B_x}{\partial x}+\frac{\partial B_z}{\partial z}=f_m \tag{1}$$

$$\sigma_x=c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial\Phi}{\partial z}+d_{31}\frac{\partial\Psi}{\partial z},$$

$$\tau_{xz}=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+e_{15}\frac{\partial\Phi}{\partial x}+d_{15}\frac{\partial\Psi}{\partial x},\quad \sigma_z=c_{13}\frac{\partial u}{\partial x}+c_{33}\frac{\partial w}{\partial z}+e_{33}\frac{\partial\Phi}{\partial z}+d_{33}\frac{\partial\Psi}{\partial z},$$

$$D_x=e_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\varepsilon_{11}\frac{\partial\Phi}{\partial x}-g_{11}\frac{\partial\Psi}{\partial x},\quad D_z=e_{31}\frac{\partial u}{\partial x}+e_{33}\frac{\partial w}{\partial z}-\varepsilon_{33}\frac{\partial\Phi}{\partial z}-g_{33}\frac{\partial\Psi}{\partial z},$$

$$B_x=d_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-g_{11}\frac{\partial\Phi}{\partial x}-\mu_{11}\frac{\partial\Psi}{\partial x},\quad B_z=d_{31}\frac{\partial u}{\partial x}+d_{33}\frac{\partial w}{\partial z}-g_{33}\frac{\partial\Phi}{\partial z}-\mu_{33}\frac{\partial\Psi}{\partial z} \tag{2}$$

where $\sigma_i(\tau_{ij})$, u_i, D_i and B_i are the components of stress, displacement, electric displacement and magnetic induction, respectively; Φ and Ψ are the electric potential and magnetic potential, respectively; f_i, f_e and f_m are body force, free charge density and current density, respectively (according to electromagnetic theorem, $f_m = 0$), c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} are the elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants, respectively.

With the strict differential operator theorem presented by Ding, *et al.* (1997), the general solutions for the two-dimensional magneto-electro-elastic solid without body forces in the case of distinct eigenvalues s_j (j = 1~4) have been derived by Ding and Jiang (2004) and Jiang and Ding (2004) and expressed in terms of four harmonic functions as follows

$$u=\sum_{j=1}^{4}\frac{\partial\psi_j}{\partial x},\quad w_m=\sum_{j=1}^{4}s_jk_{mj}\frac{\partial\psi_j}{\partial z_j},\quad \sigma_x=\sum_{j=1}^{4}\omega_{4j}\frac{\partial^2\psi_j}{\partial z_j^2},\quad \sigma_m=\sum_{j=1}^{4}\omega_{mj}\frac{\partial^2\psi_j}{\partial z_j^2},$$

$$\tau_m=\sum_{j=1}^{4}s_j\omega_{mj}\frac{\partial^2\psi_j}{\partial x\partial z_j},\quad (m=1\sim3) \tag{3}$$

where $\omega_{4j} = -\omega_{1j}s_j^2$, and the generalized displacements and stresses are defined as follows

$$w_1=w,\quad w_2=\Phi,\quad w_3=\Psi,\quad \sigma_1=\sigma_z,\quad \sigma_2=D_z,\quad \sigma_3=B_z,$$

$$\tau_1=\tau_{xz},\quad \tau_2=D_x,\quad \tau_3=B_x \tag{4}$$

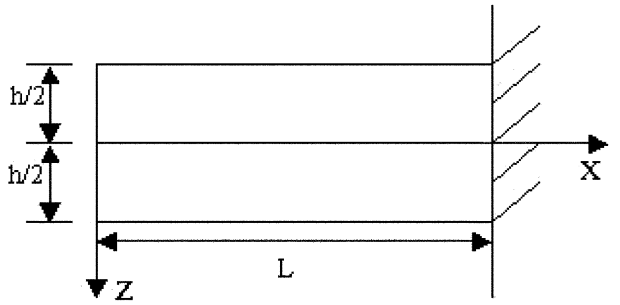

Fig. 1 The geometry and coordinate system of acantilever beam

The functions ψ_j satisfy the following equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^2}\right)\psi_j = 0, \qquad (j = 1\sim4) \tag{5}$$

where $z_j = s_j z$ (j = 1~4) and s_j^2 are the four roots of the equation (we take Re (s_j) > 0)

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0 \tag{6}$$

where a_n (n = 1~5), k_{mj} and ω_{mj} (m = 1~3, j = 1~4) in Eqs. (6) and (3) are the same as those derived by Hou, *et al.* (2003) and Ding and Jiang (2003).

In calculating k_{mj}, ω_{mj} in Eq. (3) and a_n in Eq. (6), taking $d_{15} = d_{31} = d_{33} = 0$, $g_{11} = g_{33} = 0$, $\mu_{11} = 0$ and $\mu_{33} = 1$, and changing j from 1 to 3, m from 1 to 2, respectively, we obtain $a_5 = 0$, $s_4 = 0$. It can be seen that the degenerated general solution corresponds with that of plane problem of piezoelectric media given by Ding, *et al.* (1997). Furthermore, The solution can also be degenerated into that of problem of an orthotrpic beam as in Jiang and Ding (2005).

The harmonic polynol listed in Appendix A can be chosen as harmonic functions simply by replacing z with z_j. Six analytical solutions for a cantilever beam without body forces and with surface loads are obtained by using the harmonic polynomials, as presented in the following section.

In the following sections, we consider the magneto-electro-elastic cantilever beam shown in Fig. 1. At the fixed end ($x = L$) of the beam, the conditions of electric potential and magnetic potential are $\Phi = \Phi_0$ and $\Psi = \Psi_0$ at two appointed points, i.e., $\Phi(L, z') = \Phi_0$ and $\Psi(L, z'') = \Psi_0$, respectively. As both of electric potential and magnetic potential can be superposed on arbitrary constants without inducing any difference in the generalized stresses, the conditions of electric potential and magnetic potential are not presented.

3. Three solutions for cantilever beam without body forces and with surface loads

3.1. Cantilever beam under uniform loads on the upper and lower surfaces

We introduce the displacement function as follows

$$\psi_j = (x^2 - z_j^2)A_{2j} + \left(x^2 z_j - \frac{1}{3}z_j^3\right)B_{3j} + B_{5j}\left(x^2 z_j - 2x^2 z_j^3 + \frac{1}{5}z_j^5\right) \tag{7}$$

where A_{2j}, B_{3j} and B_{5j} ($j = 1\sim4$) are unknown constants to be determined.
Substituting Eq. (7) into Eqs. (3) leads to

$$u = \sum_{j=1}^{4} 2xA_{2j} + \sum_{j=1}^{4} [2xz_jB_{3j} + (4x^3z_j - 4xz_j^3)B_{5j}] \tag{8a}$$

$$w_m = \sum_{j=1}^{4} s_jk_{mj}(-2z_jA_{2j}) + \sum_{j=1}^{4} s_jk_{mj}[(x^2 - z_j^2)B_{3j} + (x^3 - 6x^2z_j^2 + z_j^4)B_{5j}] \tag{8b}$$

$$\sigma_m = \sum_{j=1}^{4} \omega_{mj}(-2A_{2j}) + \sum_{j=1}^{4} \omega_{mj}[-2z_jB_{3j} + (-12x^2z_j + 4z_j^3)B_{5j}] \tag{8c}$$

$$\tau_m = \sum_{j=1}^{4} s_j\omega_{mj}[2xB_{3j} + (4x^3 - 12xz_j^2)B_{5j}], \quad (m = 1\sim3) \tag{8d}$$

$$\sigma_x = \sum_{j=1}^{4} \omega_{4j}(-2A_{2j}) + \sum_{j=1}^{4} \omega_{4j}[-2z_jB_{3j} + (-12x^2z_j + 4z_j^3)B_{5j}] \tag{8e}$$

The boundary conditions are

$$z = \pm(h/2): \sigma_m = \beta_m \pm C_m, (m = 1 \sim 3), \quad \tau_{xz} = 0 \tag{9a}$$

$$x = 0: \int_{-h/2}^{+h/2} \sigma_x dz = 0, \int_{-h/2}^{+h/2} \sigma_x z dz = 0, \int_{-h/2}^{+h/2} \tau_m dz = 0, \ (m = 1 \sim 3) \tag{9b}$$

$$(x = L, z = 0): \quad u = 0, \quad w = 0, \quad \partial w/\partial x = 0 \tag{9c}$$

$\beta_m = \frac{1}{2}\left[\sigma_m\left(z = +\frac{h}{2}\right) + \sigma_m\left(z = -\frac{h}{2}\right)\right]$, and $C_m = \frac{1}{2}\left[\sigma_m\left(z = +\frac{h}{2}\right) - \sigma_m\left(z = -\frac{h}{2}\right)\right]$.

Substituting Eqs. (8c)-(8e) into Eqs. (9a,b) , we arrive at

$$\sum_{j=1}^{4} \omega_{mj}A_{2j} = -\beta_m/2, \qquad \sum_{j=1}^{4} \omega_{mj}\left(-hs_jB_{3j} + \frac{1}{2}h^3s_j^3B_{5j}\right) = C_m, \quad (m = 1 \sim 3) \tag{10}$$

$$\sum_{j=1}^{4} s_j\omega_{mj}B_{5j} = 0, \quad (m = 1 \sim 3), \quad \sum_{j=1}^{4} \omega_{4j}A_{2j} = 0 \tag{11}$$

$$\sum_{j=1}^{4} s_j\omega_{1j}(2B_{3j} - 3h^2s_j^2B_{5j}) = 0 \tag{12}$$

$$\sum_{j=1}^{4} s_j\omega_{4j}(-10B_{3j} + 3h^2s_j^2B_{5j}) = 0 \tag{13}$$

The unknown constants A_{2j}, B_{3j} and B_{5j} ($j = 1\sim4$) can be determined from the twelve equations Eqs. (10)-(13). To satisfy the boundary conditions given in Eqs. (9c), the solution above should be superposed on the rigid body displacements solutions as follows

$$u_1 = u_0 + \omega_0 z, \quad w_1 = w_0 - \omega_0 x \tag{14}$$

where

$$u_0 = -2L\sum_{j=1}^{4} A_{2j}, \quad \omega_0 = 2L\sum_{j=1}^{4} s_j k_{1j}(B_{3j} + 2L^2 B_{5j})$$

$$w_0 = L^2\sum_{j=1}^{4} s_j k_{1j}(B_{3j} + 3L^2 B_{5j}) \tag{15}$$

3.2. Cantilever beam with axial force N and bending moment M at the free end

We constitute the displacement function as follows

$$\psi_j = (x^2 - z_j^2)A_{2j} + \left(x^2 z_j - \frac{1}{3}z_j^3\right)B_{3j} \qquad (j = 1\sim 4) \tag{16}$$

where A_{2j} and B_{3j} are unknown constants to be determined.
Substituting Eq. (16) into Eqs. (3) leads to

$$u = \sum_{j=1}^{4} 2xA_{2j} + \sum_{j=1}^{4} 2xz_j B_{3j} \tag{17a}$$

$$w_m = \sum_{j=1}^{4} s_j k_{mj}[-2z_j A_{2j} + (x - z_j^2)B_{3j}], \quad (m = 1\sim 3) \tag{17b}$$

$$\sigma_m = \sum_{j=1}^{4} \omega_{mj}(-2A_{2j} - 2z_j B_{3j}), \ \tau_m = 2x\sum_{j=1}^{4} s_j \omega_{mj} B_{3j}, \quad (m = 1\sim 3) \tag{17c}$$

$$\sigma_x = \sum_{j=1}^{4} \omega_{4j}(-2A_{2j} - 2z_j B_{3j}) \tag{17d}$$

The boundary conditions are

$$z = \pm(h/2): \quad \sigma_m = 0, \ (m = 1\sim 3), \quad \tau_{xz} = 0 \tag{18a}$$

$$x = 0: \ \int_{-h/2}^{+h/2}\sigma_x dz = N, \ \int_{-h/2}^{+h/2}\sigma_x z dz = M, \ \int_{-h/2}^{+h/2}\tau_m dz = 0, \quad (m = 1\sim 3) \tag{18b}$$

$$(x = L, z = 0): \quad u = 0, \quad w = 0, \quad \partial w/\partial x = 0 \tag{18c}$$

the axial force N is positive in tension and the bending moment M acts clockwise.
Substituting Eqs. (17c) and (17d) into Eqs. (18a,b), we have

$$\sum_{j=1}^{4} \omega_{mj} A_{2j} = 0, \quad \sum_{j=1}^{4} s_j \omega_{mj} B_{3j} = 0, \quad (m = 1\sim 3) \tag{19}$$

$$-\frac{h^3}{6}\sum_{j=1}^{4} s_j \omega_{4j} B_{3j} = M, \quad -2h\sum_{j=1}^{4} \omega_{4j} A_{2j} = N \tag{20}$$

The unknown constants A_{2j} and B_{3j} can be determined from Eqs. (19) and (20). To satisfy the boundary conditions Eqs. (18c), the solution above should be superposed on the rigid body displacements solutions as follows

$$u_1 = u_0 + \omega_0 z \ , \quad w_1 = w_0 - \omega_0 x \tag{21}$$

where

$$u_0 = -2L\sum_{j=1}^{4} A_{2j}, \quad \omega_0 = 2L\sum_{j=1}^{4} s_j k_{1j} B_{3j}, \quad w_0 = L^2\sum_{j=1}^{4} s_j k_{1j} B_{3j} \tag{22}$$

3.3. Cantilever beam with tangential loads in form of polynomial of x to the fourth power on the upper and lower surfaces

We introduce the displacement function as follows

$$\psi_j = xz_j B_{2j} + (x^3 z_j - xz_j^3)B_{4j} + \left(x^5 z_j - \frac{10}{3}x^3 z_j^3 + xz_j^5\right)B_{6j} + (x^7 z_j - 7x^5 z_j^3 + 7x^3 z_j^5 - xz_j^7)B_{8j}$$
$$+\left(x^2 z_j - \frac{1}{3}z_j^3\right)B_{3j} + \left(x^4 z_j - 2x^2 z_j^3 + \frac{1}{5}z_j^5\right)B_{5j} + \left(x^6 z_j - 5x^4 z_j^3 + 3x^2 z_j^5 - \frac{1}{7}z_j^7\right)B_{7j}, \ (j = 1\sim 4) \tag{23}$$

where B_{2j}, B_{4j}, B_{6j}, B_{8j}, B_{3j}, B_{5j} and B_{7j} are unknown constants to be determined.
Substituting Eq. (23) into Eqs. (3) leads to

$$u = \sum_{j=1}^{4}[z_j B_{2j} + (3x^2 z_j - z_j^3)B_{4j} + (5x^4 z_j - 10x^2 z_j^3 + z_j^5)B_{6j} + (7x^6 z_j - 35x^4 z_j^3$$
$$+ 21x^2 z_j^5 - z_j^7)B_{8j} + 2xz_j B_{3j} + (4x^3 z_j - 4xz_j^3)B_{5j} + (6x^5 z_j - 20x^3 z_j^3 + 6xz_j^5)B_{7j}] \tag{24a}$$

$$w_m = \sum_{j=1}^{4} s_j k_{mj}[xB_{2j} + (x^3 - 3xz_j^2)B_{4j} + (x^5 - 10x^3 z_j^2 + 5xz_j^4)B_{6j} + (x^7 - 21x^5 z_j^2 + 35x^3 z_j^4$$
$$- 7xz_j^6)B_{8j} + (x^2 - z_j^2)B_{3j} + (x^4 - 6x^2 z_j^2 + z_j^4)B_{5j} + (x^6 - 15x^4 z_j^2 + 15x^2 z_j^4 - z_j^6)B_{7j}]$$
$$(m = 1 \sim 3) \tag{24b}$$

$$\sigma_m = \sum_{j=1}^{4}\omega_{mj}[-6xz_j B_{4j} + 20(-x^3 z_j + xz_j^3)B_{6j} + (-42x^5 z_j + 140x^3 z_j^3 - 42xz_j^5)B_{8j}$$
$$-2z_j B_{3j} + (-12x^2 z_j + 4z_j^3)B_{5j} + (-30x^4 z_j + 60x^2 z_j^3 - 6z_j^5)B_{7j}], (m = 1 \sim 3) \tag{24c}$$

$$\tau_m = \sum_{j=1}^{4} s_j \omega_{mj}[B_{2j} + (3x^2 - 3z_j^2)B_{4j} + (5x^4 - 30x^2 z_j^2 + 5z_j^4)B_{6j} + (7x^6 - 105x^4 z_j^2$$
$$+ 105x^2 z_j^4 - 7z_j^6)B_{8j} + 2xB_{3j} + (4x^3 - 12xz_j^2)B_{5j} + (6x^5 - 60x^3 z_j^2 + 30xz_j^4)B_{7j}]$$
$$(m = 1 \sim 3) \tag{24d}$$

$$\sigma_x = \sum_{j=1}^{4}\omega_{4j}[-6xz_j B_{4j} + 20(-x^3 z_j + xz_j^3)B_{6j} + (-42x^5 z_j + 140x^3 z_j^3 - 42xz_j^5)B_{8j}$$
$$-2z_j B_{3j} + (-12x^2 z_j + 4z_j^3)B_{5j} + (-30x^4 z_j + 60x^2 z_j^3 - 6z_j^5)B_{7j}] \tag{24e}$$

The boundary conditions are

$$z = \pm(h/2): \quad \sigma_m = 0, \ (m = 1\sim3), \quad \tau_{xz} = T_1x^1 + T_2x^2 + T_3x^3 + T_4x^4 \tag{25a}$$

$$x = 0: \int_{-h/2}^{+h/2} \sigma_x dz = 0, \int_{-h/2}^{+h/2} \sigma_x z dz = 0, \int_{-h/2}^{+h/2} \tau_m dz = 0, (m = 1 \sim 3) \tag{25b}$$

$$(x = L, z = 0): \quad u = 0, \quad w = 0, \quad \partial w/\partial x = 0 \tag{25c}$$

Substituting Eqs. (24c)-(24e) into Eqs. (25a,b), we have

$$\sum_{j=1}^{4} \omega_{mj}\left(-s_j h B_{3j} + \frac{1}{2}s_j^3 h^3 B_{5j} - \frac{3}{16}h^5 s_j^5 B_{7j}\right) = 0, \quad (m = 1 \sim 3) \tag{26}$$

$$\sum_{j=1}^{4} \omega_{mj}\left(-3s_j h B_{4j} + \frac{5}{2}s_j^3 h^3 B_{6j} - \frac{21}{16}s_j^5 h^5 B_{8j}\right) = 0, \quad (m = 1 \sim 3) \tag{27}$$

$$\sum_{j=1}^{4} \omega_{mj}\left(-6hs_j B_{5j} + \frac{15}{2}h^3 s_j^3 B_{7j}\right) = 0, \quad (m = 1 \sim 3) \tag{28}$$

$$\sum_{j=1}^{4} \omega_{mj}\left(-10s_j h B_{6j} + \frac{35}{2}s_j^3 h^3 B_{8j}\right) = 0, \quad (m = 1 \sim 3) \tag{29}$$

$$\sum_{j=1}^{4} s_j \omega_{mj} B_{7j} = 0, \quad \sum_{j=1}^{4} s_j \omega_{mj} B_{8j} = 0, \quad (m = 1 \sim 3) \tag{30}$$

$$\sum_{j=1}^{4} s_j \omega_{1j}\left(B_{2j} - \frac{3}{4}s_j^2 h^2 B_{4j} + \frac{5}{16}s_j^4 h^4 B_{6j} - \frac{7}{64}s_j^6 h^6 B_{8j}\right) = 0 \tag{31}$$

$$\sum_{j=1}^{4} s_j \omega_{1j}\left(2B_{3j} - 3h^2 s_j^2 B_{5j} + \frac{15}{8}s_j^4 h^4 B_{7j}\right) = T_1 \tag{32}$$

$$\sum_{j=1}^{4} s_j \omega_{1j}\left(3B_{4j} - \frac{15}{2}s_j^2 h^2 B_{6j} + \frac{105}{16}s_j^4 h^4 B_{8j}\right) = T_2 \tag{33}$$

$$\sum_{j=1}^{4} s_j \omega_{1j}(4B_{5j} - 15s_j^2 h^2 B_{7j}) = T_3 \tag{34}$$

$$\sum_{j=1}^{4} s_j \omega_{1j}\left(5B_{6j} - \frac{105}{16}s_j^2 h^2 B_{8j}\right) = T_4 \tag{35}$$

$$\sum_{j=1}^{4} s_j \omega_{mj}\left(B_{2j} - \frac{1}{4}s_j^2 h^2 B_{4j} + \frac{1}{16}s_j^4 h^4 B_{6j} - \frac{1}{64}s_j^6 h^6 B_{8j}\right) = 0, \quad (m = 1\sim3) \tag{36}$$

$$\sum_{j=1}^{4} s_j \omega_{4j}\left(-\frac{1}{3}B_{3j} + \frac{1}{10}s_j^2 h^2 B_{5j} - \frac{3}{112}s_j^4 h^4 B_{7j}\right) = 0 \tag{37}$$

From Eq. (24b) and the third of Eqs. (25c), we have

$$\sum_{j=1}^{4} s_j k_{1j}(B_{2j} + 3L^2 B_{4j} + 5L^4 B_{6j} + 7L^6 B_{8j}) = 0 \tag{38}$$

The unknown constants B_{2j}, B_{4j}, B_{6j}, B_{8j}, B_{3j}, B_{5j} and B_{7j} can then be determined from Eqs. (26)-(38). To satisfy the left boundary conditions of Eqs. (25c), the solution above should be superposed on the rigid body displacements solution as follows.

$$u_1 = \omega_0 z, \qquad w_1 = w_0 - \omega_0 x \tag{39}$$

where

$$\omega_0 = 2L\sum_{j=1}^{4} s_j k_{1j}(B_{3j} + 2L^2 B_{5j} + 3L^4 B_{7j}),$$

$$w_0 = L\sum_{j=1}^{4} s_j k_{1j}(LB_{3j} + 3L^3 B_{5j} + 5L^5 B_{7j} - B_{2j} - L^2 B_{4j} - L^4 B_{6j} - L^6 B_{8j}) \tag{40}$$

4. The specific solutions to beam with non-uniform body force

4.1. The body forces depends only on the z coordinate

$$f_x = -Q(z), \quad f_z = -P(z), \quad f_e = 0 \tag{41}$$

where $Q(z)$ and $P(z)$ are two arbitrary functions of z.

It is then apparent that Eqs. (1) and (2) have the specific solution of generalized displacement as follows.

$$u^* = \frac{1}{c_{44}}G(z), \quad w^* = \frac{\Delta_1}{\Delta}F(z), \quad \Phi^* = \frac{\Delta_2}{\Delta}F(z), \quad \Psi^* = \frac{\Delta_3}{\Delta}F(z) \tag{42}$$

where

$$G(\xi) = \int_0^{\xi}(\xi-\eta)Q(\eta)d\eta, \quad F(\xi) = \int_0^{\xi}(\xi-\eta)P(\eta)d\eta \tag{43}$$

$$\Delta_1 = \varepsilon_{33}\mu_{33} - g_{33}^2, \quad \Delta_2 = e_{33}\mu_{33} - g_{33}d_{33}, \quad \Delta_3 = d_{33}\varepsilon_{33} - e_{33}g_{33} \tag{44a}$$

$$\Delta = \begin{vmatrix} c_{33} & e_{33} & d_{33} \\ e_{33} & -\varepsilon_{33} & -g_{33} \\ d_{33} & -g_{33} & -\mu_{33} \end{vmatrix} \tag{44b}$$

Substituting Eq. (42) into Eqs. (2) leads to the specific solution of generalized stress

$$\sigma_x^* = kF'(z),\ \sigma_z^* = F'(z),\ D_z^* = 0,\ B_z^* = 0$$
$$\tau_{xz}^* = G'(z),\ D_x^* = \frac{e_{15}}{c_{44}}G'(z),\ B_x^* = \frac{d_{15}}{c_{44}}G'(z) \tag{45}$$

where $k = (c_{13}\Delta_1 + \varepsilon_{31}\Delta_2 + d_{31}\Delta_3)/\Delta$.

From Eqs. (45), we find that the beam has uniformly distributed generalized stresses τ_{xz}^*, σ_z^*, D_z^* and B_z^* on surfaces ($z = \pm h/2$) along length, and distributed loads τ_{xz}^*, σ_x^*, D_x^* and B_x^* along height at the two ends ($x = 0, L$).

4.2. The body forces depends only on the x coordinate

$$f_x = -Q(x),\quad f_z = -P(x),\quad f_e = 0 \tag{46}$$

where $Q(x)$ and $P(x)$ are two arbitrary functions of x.
It is easy to verify that Eqs. (1) and (2) have the specific solution as follows

$$u^* = \frac{1}{c_{11}}G(x),\quad w^* = \frac{S_1}{S}F(x),\quad \Phi^* = \frac{S_2}{S}F(x),\quad \Psi^* = \frac{S_3}{S}F(x) \tag{47}$$

where $G(x)$ and $F(x)$ are expressed as Eq. (43), and the coefficients are

$$S_1 = \varepsilon_{11}\mu_{11} - g_{11}^2,\quad S_2 = e_{15}\mu_{11} - d_{15}g_{11},\quad S_3 = d_{15}\varepsilon_{11} - e_{15}g_{11} \tag{48a}$$

$$S = \begin{vmatrix} c_{44} & e_{15} & d_{15} \\ e_{15} & -\varepsilon_{11} & -g_{11} \\ d_{15} & -g_{11} & -\mu_{11} \end{vmatrix} \tag{48b}$$

Substituting Eq. (47) into Eqs. (2) leads to the specific solution of generalized stress

$$\sigma_x^* = G'(x),\quad \sigma_x^* = \frac{c_{13}}{c_{11}}G'(x),\quad D_z^* = \frac{e_{31}}{c_{11}}G'(x),\quad B_z^* = \frac{d_{31}}{c_{11}}G'(x),$$
$$\tau_{xz}^* = F'(x),\quad D_x^* = 0,\quad B_x^* = 0 \tag{49}$$

From Eqs. (49), we find that the beam has distributed generalized loads τ_{xz}^*, σ_z^*, D_z^* and B_z^* on surfaces (z = $\pm h/2$) along length, and uniformly distributed loads τ_{xz}^*, σ_x^*, D_x^* and B_x^* along the height at the two ends (x = 0, L).

5. The analytical solutions to density functionally gradient cantilever beams

For magneto-electro-elastic plane problem, the solution to Eqs. (1) and (2) should be expressed by the superposition principle:

$$u = u_0 + u^*, \quad w_m = w_{m0} + w_m^*, \qquad (m = 1\sim3) \tag{50a}$$

$$\sigma_x = \sigma_{x0} + \sigma_x^*, \quad \sigma_m = \sigma_{m0} + \sigma_m^*, \quad \tau_m = \tau_{m0} + \tau_m^*, \qquad (m = 1\sim3) \tag{50b}$$

where u_0, w_{m0} ($w_{10} = w_0$, $w_{20} = \Phi_0$, $w_{30} = \Psi_0$), σ_{x0}, σ_{m0} and τ_{m0} are the general solutions expressed as Eqs. (3) for beams without body forces, and u^*, w_m^*, σ_x^*, σ_m^* and τ_x^* are the specific solutions expressed as Eqs. (42) and (45) or (47) and (49) for beams with body forces.

In the next sections, we consider two kinds of cantilever beam of functionally gradient material (FGM) as shown in Fig. 1. The boundary conditions are

$$z = \pm h/2: \ \sigma_m = 0, \quad (m = 1\sim3), \quad \tau_{xz} = 0 \tag{51a}$$

$$x = 0: \int_{-h/2}^{h/2} \sigma_x dz = 0, \quad \int_{-h/2}^{h/2} \sigma_x z dz = 0, \quad \int_{-h/2}^{h/2} \tau_m dz = 0, \qquad (m = 1\sim3) \tag{51b}$$

$$(x = L, z = 0): u = 0, \quad w = 0, \quad \partial w/\partial x = 0 \tag{51c}$$

5.1. The solution to cantilever beam with body forces depending only on the z coordinate

$$f_x = 0, \quad f_z = \rho g, \quad \rho = d_0 e^{\lambda z} \tag{52}$$

where d_0 and λ are material constants, ρ is the density, and g is the acceleration of gravity.

Substituting Eq. (52) into Eq. (43) leads to

$$F(z) = -d_0 g\left(-\frac{z}{\lambda} + \frac{e^{\lambda z}}{\lambda^2} - \frac{1}{\lambda^2}\right), \quad G(z) = 0 \tag{53}$$

The corresponding specific solution can be obtained by substituting Eq. (53) into Eqs. (42) and (45)

$$u^* = 0, \quad w_m^0 = -\frac{\Delta_m}{\Delta} d_0 g\left(-\frac{z}{\lambda} + \frac{e^{\lambda z}}{\lambda^2} - \frac{1}{\lambda^2}\right), \quad (m = 1\sim3) \tag{54}$$

$$\sigma_x^* = -kd_0 g\left(-\frac{1}{\lambda} + \frac{e^{\lambda z}}{\lambda}\right), \quad \sigma_z^* = -d_0 g\left(-\frac{1}{\lambda} + \frac{e^{\lambda z}}{\lambda}\right), \quad \tau_{xz}^* = 0 \tag{55a}$$

$$D_x^* = D_z^* = 0, \quad B_x^* = B_z^* = 0 \tag{55b}$$

It is apparently that the boundary displacement conditions (51c) at the fixed end ($x = L$) have been satisfied with Eq. (54). At the same time, we find that the specific solution Eq. (55a) may cause normal surface tractions ($z = \pm h/2$)

$$\sigma_x^* = P_0 = -d_0 g\left(-\frac{1}{\lambda} + \frac{e^{\pm\lambda h/2}}{\lambda}\right) = -\frac{d_0 g}{\lambda}\left(-1 + \cosh\frac{\lambda h}{2}\right) \pm \left(-\frac{d_0 g}{\lambda}\sinh\frac{\lambda h}{2}\right) \tag{56}$$

To satisfy the surface tractions conditions (51a), we need only superpose the specific solution (54) and (55) on the analytical solution (8) for a cantilever beam without body forces and under uniform loads on upper and lower surfaces, where $\beta_1 = \frac{d_0 g}{\lambda}\left(-1+\cosh\frac{\lambda h}{2}\right)$, $\beta_2 = 0$, $\beta_3 = 0$, $C_1 = \frac{d_0 g}{\lambda}\sinh\frac{\lambda h}{2}$, $C_2 = 0$, $C_3 = 0$. Then, in order to satisfy the tractions conditions (51b) at the free end ($x = 0$), the specific solution should be superposed on the analytical solution (17),

with $N = -kd_0 g\left(\frac{h}{\lambda} - \frac{2}{\lambda^2}\sinh\frac{\lambda h}{2}\right)$, $M = kd_0 g\left(\frac{h}{\lambda^2}\cosh\frac{\lambda h}{2} - \frac{2}{\lambda^3}\sinh\frac{\lambda h}{2}\right)$.

5.2. The solution to cantilever beam with body forces depending only on the x coordinate

$$f_x = 0, \quad f_z = \rho g, \quad \rho = \sum_{n=0}^{3} c_n (x/L)^n \tag{57}$$

where $c_n (n = 0, 1, 2, 3)$ are material constants.

From Eq. (43), we have

$$F(x) = -\sum_{n=0}^{3} \frac{c_n g}{(n+1)(n+2)L^n} x^{n+2}, \quad G(x) = 0 \tag{58}$$

The corresponding specific solution can be obtained by substituting Eq. (58) into Eqs. (47) and (49)

$$u^* = 0, \quad w_m^* = -\frac{S_m}{S}\sum_{n=0}^{3} \frac{c_n g}{(n+1)(n+2)L^n} x^{n+2}, \quad (m = 1\sim3) \tag{59}$$

$$\tau_{xz} = -\sum_{n=0}^{3} \frac{c_n g}{(n+1)L^n} x^{n+1}, \quad \sigma_x^* = \sigma_x^* = 0, \quad D_x^* = D_z^* = 0, \quad B_x^* = B_z^* = 0 \tag{60}$$

It is apparently that the specific stress solution Eqs. (60) satisfies the traction boundary conditions (51b) at the free end ($x = 0$) automatically, and may cause the fourth power of x tangential tractions on the two surfaces ($z = \pm h/2$).

$$\tau_{xz}^* = \sum_{n=0}^{3} H_n(x) = -\sum_{n=0}^{3} \frac{c_n g}{(n+1)L^n} x^{n+1} \tag{61}$$

To satisfy the surface tractions conditions (51a), we should superpose the specific solution (59) and (60) on the analytical solution for a cantilever beam without body forces and under distributed surface loads ($\sigma_z = 0$, $D_z = 0$, $B_z = 0$, $\tau_{xz} = -\sum_{n=0}^{3} H_n x$), i.e., the solution (24) with $T_4 = \frac{c_3 g}{4L^3}$, $T_3 = \frac{c_2 g}{3L^2}$, $T_2 = \frac{c_1 g}{2L}$ and $T_1 = c_0 g$, respectively.

To satisfy the displacement conditions (51c) at the fixed end ($x = L$), we should superpose the specific solution (59) on the rigid body displacements solutions as follows.

$$u_1 = \omega_0 z, \quad w_1 = w_0 - \omega_0 x \tag{62}$$

where

$$\omega_0 = -\frac{S_1}{S}\sum_{n=0}^{3}\frac{c_n L g}{n+1}, \quad w_0 = -\frac{S_1}{S}\sum_{n=0}^{3}\frac{c_n L^2 g}{n+2} \tag{63}$$

6. Examples

In order to demonstrate the advantage of FGM, the weight of the beam $\rho_0 hLg$ is assumed to be a constant, i.e.,

$$\rho_0 Lh = \int_0^L\int_{-\frac{h}{2}}^{+\frac{h}{2}}\rho dz dx \tag{64}$$

where ρ_0 is the average density. Substituting Eq. (52) and (57) into Eq. (64), we obtain $\rho_0 = \frac{2d_0}{\lambda_h}\mathrm{sihn}\frac{\lambda h}{2}$ for the first kind of beam (density depend only on the z coordinate) and $\rho_0 = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4}$ for the second kind (density only on the x coordinate), respectively. When $\lambda = 0$ or $c_1 = c_2 = c_3 = 0$, the beam is homogeneous.

Based on the above equations, all the displacements, stresses, electric and magnetic quantities at any inner or boundary point of the cantilever beam can be obtained. In the calculation, we set $L = 150$ mm, $h = 6$ mm and $\rho_0 = 7800$ kg/m^3. Assume that a piezoelectric cantilever beam and an orthotropic cantilever beam have the same constants as those of magneto-electro-elastic beam shown in Table 1 and the same geometric dimensions and boundary conditions.

The deflections (w^1, w^2) of the beam at point ($x = z = 0$) are listed in Table 2 for comparison, and are caused by two kinds of functionally graded density depending only on the z coordinate or on x coordinate for six cases, respectively. When the density depends on the z coordinate: (case 1) $\lambda = 500$/m, (case 2) $\lambda = 400$/m, (case 3) $\lambda = 300$/m, (case 4) $\lambda = 200$/m, (case 5) $\lambda = 100$/m and (case 6) $\lambda = 0$; when the density depends on the x coordinate: (case 1) $c_0 = \rho_0$, $c_1 = c_2 = c_3 = 0$ (case 2) $c_1 = 2\rho_0$, $c_0 = c_2 = c_3 = 0$, (case 3) $c_2 = 3\rho_0$, $c_0 = c_1 = c_3 = 0$, (case 4) $c_3 = 4\rho_0$, $c_0 = c_1 = c_2 = 0$, (case 5) $c_0 = c_1 = c_2 = c_3 = 12/25\rho_0$ and (case 6) $c_0 = 1/4\rho_0$, $c_1 = 1/2\rho_0$, $c_2 = 3/4\rho_0$, $c_3 = \rho_0$. Meanwhile, the stress σ_x^1 and σ_x^2 stand for the point ($x = L/2$, $z = -h/2$) corresponding to the two kinds of density depending only on z coordinate or on the x coordinate for the six cases given above, respectively. In Table 2, "M", "P" and "E" denote magneto-electro-elastic, piezoelectric and orthotropic material beam, respectively. The x coordinate of the center of gravity (x_0, z_0) is listed in Table 3 for different cases, where x_0^1 and x_0^2 denote the

Table 1 Material properties (Pan 2002)

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	c_{66}
1.66×10^{11}	7.7×10^{10}	7.8×10^{10}	1.62×10^{11}	4.3×10^{10}	4.45×10^{10}
e_{31}	e_{33}	e_{15}	d_{31}	d_{33}	d_{15}
-4.4	18.6	11.6	580.3	699.7	550
ε_{11}	ε_{33}	g_{11}	g_{33}	μ_{11}	μ_{33}
1.12×10^{-8}	1.26×10^{-8}	5.0×10^{-12}	3.0×10^{-12}	5×10^{-6}	10×10^{-6}

Unit: c - N/m^2, e - C/m^2, d - N/Am, ε - C/Vm, μ - Ns^2/C^2, g - Ns/VC.

Table 2 Deflection w and stress σ_x of cantilever beam with body force

ρ case	(1)	(2)	(3)	(4)	(5)	(6)
w^1 (M)	0.1085E-4m	0.1085E-4m	0.1085E-4m	0.1085E-4m	0.1085E-4m	0.1085E-4m
w^1 (P)	0.1145E-4m	0.1145E-4m	0.1145E-4m	0.1145E-4m	0.1145E-4m	0.1145E-4m
w^1 (E)	0.1253E-4m	0.1253E-4m	0.1253E-4m	0.1253E-4m	0.1253E-4m	0.1253E-4m
w^2 (M)	0.1085E-4m	0.5775E-5m	0.3602E-5m	0.2463E-5m	0.7466E-5m	0.5673E-5m
w^2 (P)	0.1145E-4m	0.6099E-5m	0.3795E-5m	0.2594E-5m	0.7881E-5m	0.5986E-5m
w^2 (E)	0.1253E-4m	0.6663E-5m	0.4148E-5m	0.2830E-5m	0.8618E-5m	0.6544E-5m
σ_x^1 (M)	0.2148E+6 Pa	0.2148E+6 Pa	0.2148E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa
σ_x^1 (P)	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2148E+6 Pa
σ_x^1 (E)	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa	0.2149E+6 Pa
σ_x^2 (M)	0.2149E+6 Pa	0.7154E+5 Pa	0.2678E+5 Pa	0.1069E+5 Pa	0.1259E+6 Pa	0.7097E+5 Pa
σ_x^2 (P)	0.2149E+6 Pa	0.7175E+5 Pa	0.2680E+5 Pa	0.1077E+5 Pa	0.1259E+6 Pa	0.7105E+5 Pa
σ_x^2 (E)	0.2149E+6 Pa	0.7157E+5Pa	0.2680E+5Pa	0.1070E+5Pa	0.1259E+6Pa	0.7099E+5Pa

Table 3 The x coordinate of the center of gravity of the cantilever beam

ρ case	(1)	(2)	(3)	(4)	(5)	(6)
x_0^1/L	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
x_0^2/L	0.5000	0.6667	0.7500	0.8000	0.6160	0.6792

two kinds of density depending only on the z coordinate or on the x coordinate, respectively.

From the results listed in Table 2 and 3, we arrive at the following:

(1) when the density depends only on the z coordinate, all the results of deflections and stresses of six considered cases correspond with those of a homogeneous beam, because the center of gravity remains at the same distance from the fixed end ($x = L$). The deflection of the piezoelectric beam is larger than that of magneto-electro-elastic one, while the deflection of orthotropic beam is the largest. However, the stresses exhibit no noticeable differences.

(2) When the density depends only on x coordinate, all the results of deflections and stresses are different for the six considered cases and for different material, that is, the longer the distance between the center of gravity and the fixed end ($x = L$), the larger deflections and stresses. For the fourth case, the deflection caused by body force is the smallest, and is nearly twenty-three percent of the homogeneous beam, and the stress is only five percent of the homogeneous beam. For the same case, the deflection of piezoelectric beam is larger than that of magneto-electro-elastic beam, while the deflection of the orthotropic beam is the largest. However, the differences between the stresses are slight.

7. Conclusions

From Table 3, we can qualitative analyse the law of the deflections and stresses varying with x for different cases and material. Using the equations and the analytical solutions, we can make a quantitative analysis of the deflections and stresses. The analytical solutions to density functionally gradient magneto-electro-elastic cantilever beams derived in this paper by the superposition principal and the trial-and-error method are more explicit and convenient than those by the stress method.

Numerical results show that adopting certain value of in-homogeneity parameters c_n can optimise the mechanical-electric responses. This will be of particular importance in modern engineering design.

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Appendix A

Harmonic polynomials for the plane problems can be written in the following form

$$\varphi_n^m(x,z) = x^{n-m}z^m + \sum_{i=1}^{\left[\frac{n-m}{2}\right]} (-1)^i \frac{(n-m)(n-m-1)\ldots(n-m-2i+1)}{(2i+m)!} x^{n-2i-m} z^{2i+m}$$

$$(m = 0,1;\ n = 1, 2,\ldots) \tag{A1}$$

where $\left[\frac{n-m}{2}\right]$ denotes the largest integer $\le \frac{n-m}{2}$. From Eq. (A1), the first seventeen harmonic polynomials can be written as follows

$$\varphi_0^0(x,z) = 1,\ \varphi_1^0(x,z) = x,\ \varphi_1^1(x,z) = z,\ \varphi_2^0(x,z) = x^2 - z^2$$

$$\varphi_2^1(x,z) = xz,\ \varphi_3^0(x,z) = x^3 - 3xz^2,\ \varphi_3^1(x,z) = x^2 z - \frac{1}{3}z^3$$

$$\varphi_4^0(x,z) = x^4 - 6x^2z^2 + z^4,\ \varphi_4^1(x,z) = x^3 z - xz^3$$

$$\varphi_5^0(x,z) = x^5 - 10x^3z^2 + 5xz^4,\ \varphi_5^1(x,z) = x^4 z - 2x^2z^3 + \frac{1}{5}z^5$$

$$\varphi_6^0(x,z) = x^6 - 15x^4z^2 + 15x^2z^4 - z^6,\ \varphi_6^1(x,z) = x^5 z - \frac{10}{3}x^3z^3 + xz^5$$

$$\varphi_7^0(x,z) = x^7 - 21x^5z^2 + 35x^3z^4 - 7xz^6,\ \varphi_7^1(x,z) = x^6 z - 5x^4z^3 + 3x^2z^5 - \frac{1}{7}z^7$$

$$\varphi_8^0(x,z) = x^8 - 28x^6z^2 + 70x^4z^4 - 28x^2z^6 + z^8$$

$$\varphi_8^1(x,z) = x^7 z - 7x^5z^3 + 7x^3z^5 - xz^7 \tag{A2}$$

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