

Effect of quartic nonlinearity on elastic waves via successive approximation

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Abstract. The theory of nonlinear elastic wave propagation is important in multiple scientific and engineering fields. In this study, we present a comprehensive examination of nonlinear elastic wave profiles through a contemporary approach of successive approximation. This research is related to nonlinear elastic wave models along different types of nonlinearities. Murnaghan potential is used due to the assumption of the hyper-elastic materials. We explore the complication of the governing equations and go through the behaviors of nonlinear waves in one dimension. The comparative aspect of our study is a distinctive feature, as we evaluate and contrast the results obtained using successive approximation along different nonlinearities. Additionally, we present graphical representations of our findings, enhancing the visual comprehension of the wave profiles and their evolution. This study contributes to the nonlinear elastic wave analysis and comparison.

Keywords: Murnaghan's potential; nonlinear elasticity; second harmonic generation; successive approximation method

1. Introduction

The study of elastic wave phenomena in solids hold significant importance in various fields, including Engineering, Physics, Seismology, and Geophysics. The propagation of elastic waves, primarily studied through linearized elasticity theory, has been extensively explored, see for example Achenbach (2012), Ewing *et al.* (1957) and Landau and Lifshitz (2013). However, in many real-world scenarios, deformations within the material are not small, rendering the linear theory inadequate. Therefore, researchers have increasingly turned their attention to nonlinear elasticity theory, as Hooke's law, the cornerstone of linearized theory, is no longer applicable in such cases.

Nonlinear elasticity introduces a stress-strain relationship governed by a potential function, and various models have been proposed to describe this potential, with a comprehensive overview provided by Rushchitsky (2014). This provides a good source of governing equations in different nonlinear models. Due to significant interest in the nonlinear theory of elasticity, many authors have discussed nonlinear elastic wave problems. This provides a good source of governing equations in different nonlinear models. Due to the growing interest in nonlinear elasticity theory, numerous authors have investigated problems related to nonlinear elastic waves.

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Apostol (2003) employed an asymptotic series approach to study one-dimensional nonlinear elastic wave models. Bokhari *et al.* (2007) utilized the symmetry method to find exact solutions for (1+1)-dimensional nonlinear elastic wave models, reducing the governing equations to ordinary differential equations that are easily solvable. For more complex cases, numerical schemes can also be employed. Hussain *et al.* (2023) and Usman and Zaman (2023) conducted Lie symmetry analyses, obtaining invariant solutions and identifying conservation laws for (2+1)-dimensional nonlinear partial differential equations in elasticity.

Mustafa and Masood (2009) delved deeper into these models, offering precise solutions for various interesting cases, particularly when the displacement depends on a single spatial variable in the case of unidirectional wave propagation. Fronk *et al.* (2023) conducted an asymptotic analysis of elastic wave propagation in weakly nonlinear media and materials. Miao and Wang (2023) focused on discussing in-plane nonlinear elastic waves. Zhang *et al.* (2023) examined the dispersion characteristics of elastic waves.

Abderezak *et al.* (2018) studied the role of inhomogeneities in interfacial stresses of hybrid beams. Bensattalah *et al.* (2018) analytically analyzed for the forced vibration of carbon nano-tubes surrounding elastic medium including thermal effect using nonlocal Euler-Bernoulli beam theory. Lata and Kaur (2019a, b) used Fourier transforms to deal with time harmonic deformation in transversely isotropic magneto thermoelastic solids. Kaur *et al.* (2020) focused on plane harmonic wave in rotating media with fractional order heat transfer and explained graphically the effects of viscosity and fractional order parameter by varying different values.

Belarbi *et al.* (2020) conducted a numerical investigation on the static bending and buckling responses of a composite beam reinforced with functionally graded carbon nanotubes, subjected to various types of conditions. Houari *et al.* (2024) explored the shear correction factors for a novel exponential functionally graded porous beam. Ladmek *et al.* (2023) focused on the study of free vibrations in functionally graded carbon nanotubes reinforced composite nanobeams. Sabherwal *et al.* (2024) applied the finite element method to predict the free vibration behavior of sandwich plates.

Namayandeh *et al.* (2020) analytically solved the temperature and thermal stress distributions in a hollow circular cylinder composed of anisotropic and isotropic materials via Bessel's functions. Garg *et al.* (2022, 2023a, b) utilized numerical methods including finite element method and refined zig-zag theory, to analyze the bending behavior of power law sigmoidal sandwich functionally graded beams and plates, respectively, considering the thermal conditions. Mohammadimehr (2022) analyzed Buckling and bending of a sandwich beam based on nonlocal stress-strain elasticity theory with porous core and functionally graded face sheets. Othman *et al.* (2023) analyzed sandwich plate, constructed with orthotropic and isotropic composite materials, to obtain the static and harmonic behavior.

Yu *et al.* (2023) performed a comprehensive investigation into the thermally modulated Klein tunneling within elastic wave systems. Chen *et al.* (2023) focused on an inverse scattering problem using time domain elastic wave equations. Zhang *et al.* (2023a, b) studied an impedance matching method for a one-dimensional elastic-wave system. Shi *et al.* (2023) utilized local radial basis function collocation method to simulate the in-plane elastic wave propagation in composite plate. Bai *et al.* (2023) systematically summarized the advances in experimental observation methods, establishment of nonlinear constitutive relations and derivation of wave equations.

Yang *et al.* (2023) studied elastic wave propagation via numerical simulations. Sang *et al.* (2023) employed machine learning (ML) for analysis of elastic wave propagations. Gartsev (2023) developed several automated procedures and algorithms to increase the robustness of the extraction of the measured parameters. Shojaei *et al.* (2023) described application of peri-dynamics to the

propagation of elastic waves in unbounded domains. Delory *et al.* (2023) worked on guided elastic wave propagation in a highly-stretched eco-flex plate, a nearly incompressible elastomer. In the context of linear elasticity theory, deformations (denoted as u_i in an elastic body are assumed to be small, and strain is defined as Achenbach (2012)

$$\varepsilon_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}. \quad (1)$$

Hooke's law postulates that the stress tensor is directly proportional to the strain, i.e.,

$$\tau_{ij} = C_{ijkl} \varepsilon_{kl}. \quad (2)$$

In the case of isotropic materials, fourth-order tensor C_{ijkl} takes the following form

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \quad (3)$$

This leads to the simplified Hooke's law as follows

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \quad (4)$$

where λ and μ are constants representing Lamé's elastic parameters. The equation of motion, known as the Cauchy stress equation, is expressed as

$$\tau_{ij,j} + \rho f_i = \rho \ddot{u}_i. \quad (5)$$

By using Eq. (1) and Eq. (4), we can write it as

$$\rho \ddot{u}_i = \mu u_{ij,j} + \rho f_i + (\lambda + \mu) u_{j,ji}. \quad (6)$$

In the case of nonlinearity, the deformations are no longer small, and the nonlinear strain is given by Rushchitsky (2014)

$$\varepsilon_{ij} = \frac{u_{j,i} + (u_{i,j} + u_{k,i}u_{k,j})}{2}. \quad (7)$$

Since, the strain displacement relation given in Eq. (7) due to the nonlinear term $u_{k,i}u_{k,j}$ does not satisfy Hooke's law, to address this nonlinearity, different potential functions have been introduced to establish a relationship between stress and strain. In this paper, we focus on Murnaghan's potential, defined in Rushchitsky (2014), to give a relationship between stress and strain as

$$W(\varepsilon_{ik}) = \frac{1}{3} a \varepsilon_{ik} \varepsilon_{im} \varepsilon_{km} + b (\varepsilon_{ik})^2 \varepsilon_{mm} + \frac{1}{2} \lambda (\varepsilon_{mm})^2 + \mu (\varepsilon_{ik})^2 + \frac{1}{3} c (\varepsilon_{mm})^3. \quad (8)$$

Here, μ and λ are Lamé's coefficients, a, b and c are Murnaghan constants. Additionally, stress can be calculated using the following expression (Rushchitsky 2014)

$$\tau_{nm} = \frac{\partial W}{\partial u_{m,n}}. \quad (9)$$

In the absence of body forces, the Cauchy stress equation of motion is written as

$$\tau_{ij,j} = \rho \ddot{u}_i. \quad (10)$$

Here, the displacement vector is denoted as u_i , stress tensor is $\tau_{ij,j}$, and the density of the material is ρ .

In the realm of literature, many methods such as analytical and Lie symmetry methods have been

applied. These methodologies produce solutions. The main aim of attention of this paper is comparative analysis of solution profiles in one dimension varying the type of nonlinearity. In this article, we settle the nonlinearity using the Murnaghan potential. We employ a successive approximation method to obtain approximate series solutions. This approach offers the benefit of circumventing the need for any physical parametric assumptions in the given problem. The successive approximation method is used to solve problems regarding nonlinear elastic waves. We also present the solution profile representation of the results via MATLAB.

The rest of the paper is organized as follows. Section 2 includes the derivation of the problem. In section 3, the successive approximation method is briefly described and applied to find solutions. In the same section MATLAB is used for the graphical profile representation of the solutions. In section 4, the results are discussed in details. Finally, in the last section 4 the conclusion is given in a summary form.

2. Derivation of nonlinear elastic wave models

In this section we derive the elastic wave models using Murnaghan's potential.

2.1 (1+1)-dimensional quadratically nonlinear elastic wave model

Consider the motion in one spatial variable, in which $(x_1, t) = (x, t)$ and $u_1 = u$. The nonlinear strain Eq. (7) has the following components

$$\begin{aligned}\epsilon_{11} &= u_x + \frac{1}{2}u_x^2, \epsilon_{22} = 0, \epsilon_{33} = 0, \epsilon_{12} = \epsilon_{21} = 0, \\ \epsilon_{13} &= \epsilon_{31} = 0, \epsilon_{23} = \epsilon_{32} = 0.\end{aligned}\quad (11)$$

Therefore, Eq. (8) becomes

$$W = \left(\frac{\lambda}{2} + \mu\right)u_x^2 + \left(\frac{\lambda}{2} + \mu + \frac{a}{3} + b + \frac{c}{3}\right)u_x^3.\quad (12)$$

By using Eq. (9) we have

$$\tau_{11} = 3\left(\frac{\lambda}{2} + \mu + \frac{a}{3} + b + \frac{c}{3}\right)u_x^2 + (\lambda + 2\mu)u_x.\quad (13)$$

From Eq. (10) and Eq. (13), we have the (1+1)-dimensional quadratically nonlinear elastic wave equation such as

$$u_{tt} - N_0 u_{xx} = N_1 u_x u_{xx}.\quad (14)$$

Here, $N_0 = \frac{\lambda+2\mu}{\rho}$ and $N_1 = \frac{3(\lambda+2\mu)+2(a+3b+c)}{\rho}$ are elastic parameters.

2.2 (1+1)-dimensional cubically nonlinear elastic wave model

Consider the motion in one spatial variable, in which $(x_1, t) = (x, t)$ and $u_1 = u$. The nonlinear strain Eq. (7) has the following components

$$\epsilon_{11} = u_x + \frac{1}{2}u_x^2, \epsilon_{22} = 0, \epsilon_{33} = 0, \epsilon_{12} = \epsilon_{21} = 0,$$

$$\epsilon_{13} = \epsilon_{31} = 0, \epsilon_{23} = \epsilon_{32} = 0. \tag{15}$$

Therefore, Eq. (8) becomes

$$W = \left((a + 3b + c + \frac{\lambda}{4} + \frac{\mu}{2}) u_x^2 + (\frac{2}{3}a + 2b + \frac{2}{3}c + 2\mu + \lambda) u_x + 2\mu + \lambda \right) \frac{u_x^2}{2}. \tag{16}$$

By using Eq. (9) we have

$$\tau_{11} = 2 \left((a + 3b + c + \frac{\lambda}{4} + \frac{\mu}{2}) u_x^2 + (\frac{1}{2}a + \frac{3}{2}b + \frac{1}{2}c + \frac{3}{2}\mu + \frac{3}{4}\lambda) u_x + 2\mu + \frac{\lambda}{2} \right) u_x. \tag{17}$$

From Eq. (10) and Eq. (17), we have the (1+1)-dimensional cubically nonlinear elastic wave equation such as

$$u_{tt} - N_0 u_{xx} = N_1 u_x u_{xx} + N_2 u_x^2 u_{xx}. \tag{18}$$

Here, $N_0 = \frac{\lambda+2\mu}{\rho}$, $N_1 = \frac{3(\lambda+2\mu)+2(a+3b+c)}{\rho}$ and $N_2 = \frac{\frac{3}{2}(\lambda+2\mu)+6(a+3b+c)}{\rho}$ are elastic parameters.

2.3 (1+1)-dimensional quartically nonlinear elastic wave model

Now, Consider the motion in one spatial variable, in which $(x_1, t) = (x, t)$ and $u_1 = u$. Therefore, nonlinear strain Eq. (7) has the following components

$$\begin{aligned} \epsilon_{11} &= u_x + \frac{1}{2}u_x^2, \epsilon_{22} = 0, \epsilon_{33} = 0, \epsilon_{12} = \epsilon_{21} = 0, \\ \epsilon_{13} &= \epsilon_{31} = 0, \epsilon_{23} = \epsilon_{32} = 0. \end{aligned} \tag{19}$$

So, the, Eq. (8) becomes

$$W = \left((\frac{a}{2} + \frac{3}{2}b + \frac{c}{2}) u_x^3 + (a + 3b + c + \frac{\lambda}{4} + \frac{\mu}{2}) u_x^2 + (\frac{2}{3}a + 2b + \frac{2}{3}c + 2\mu + \lambda) u_x + 2\mu + \lambda \right) \frac{u_x^2}{2}. \tag{20}$$

By using Eq. (9) we have

$$\tau_{11} = \frac{5}{4} \left((a + 3b + c) u_x^3 + (\frac{8}{5}a + \frac{24}{5}b + \frac{8}{5}c + \frac{2}{5}\lambda + \frac{4}{5}\mu) u_x^2 + (\frac{4}{5}a + \frac{12}{5}b + \frac{4}{5}c + \frac{12}{5}\mu + \frac{6}{5}\lambda) u_x + \frac{8}{5}\mu + \frac{4}{5}\lambda \right) u_x. \tag{21}$$

From Eq. (10) and Eq. (21), we have the (1+1)-dimensional quartically nonlinear elastic wave equation such as

$$u_{tt} - N_0 u_{xx} = N_1 u_x u_{xx} + N_2 u_x^2 u_{xx} + N_3 u_x^3 u_{xx}. \tag{22}$$

Here, $N_0 = \frac{\lambda+2\mu}{\rho}$, $N_1 = \frac{3(\lambda+2\mu)+2(a+3b+c)}{\rho}$, $N_2 = \frac{\frac{3}{2}(\lambda+2\mu)+6(a+3b+c)}{\rho}$ and $N_3 = \frac{5(a+3b+c)}{\rho}$ are elastic parameters. We use the following Table 1 for computing the values for elastic parameters N_0 , N_1 , N_2 , and N_3 (Rushchitsky 2014).

We solve these nonlinear partial differential equations by the successive approximation method

Table 1 The standard values of constants in system international (SI) used in the analysis

Material	$\lambda \times 10^{-10}$	$\mu \times 10^{-10}$	$a \times 10^{-11}$	$b \times 10^{-11}$	$c \times 10^{-11}$	$\rho \times 10^{-4}$
Aluminum	5.2	2.7	-0.65	-2.05	-3.7	0.27
Copper	10.7	4.8	-2.8	-1.72	-2.4	0.893
Molybdenum	15.7	1.1	-0.26	-2.83	-3.72	1.02
Polystyrene	0.369	0.114	-0.108	-0.0785	-0.0981	0.105
Steel	9.4	7.9	-3.25	-3.1	-8.0	0.78
Tungsten	7.5	7.3	-1.08	-1.43	-9.08	1.89

Table 2 The standard values of elastic parameters in the case of polystyrene and steel

Parametric values	Polystyrene	Steel
N_0	0.000005685714286	0.00003230769231
N_1	0.00001621600000	0.00009165384615
N_2	0.000006005142858	0.00003265384615
N_3	-0.000002102857142	-0.00001317307692

using the numeral values of elastic parameters given in the following Table 2 for comparison purposes.

3. Solutions using the successive approximation method

The main points of the method are given below:

- Step 1: Write the model in an operator form as

$$\mathcal{L}[u] = \mathfrak{N}[u], \quad (23)$$

where, \mathcal{L} is the linear variant and \mathfrak{N} is nonlinear part collectively taken on right hand side.

- Step 2: This method is convenient to use if the solution of the closed problem is known. However, the closed problem related to Eq. (23) can be written as

$$\mathcal{L}[u] = 0. \quad (24)$$

- Step 3: The parameter ϵ is introduced such that for $\epsilon = 0$, it coincides with Eq. (24) and for $\epsilon = 1$, it coincides with Eq. (23) i.e.,

$$\mathcal{L}[u] = \epsilon \mathfrak{N}[u]. \quad (24)$$

- Step 4: According to the method, the solution $u(x, t, \epsilon)$ of Eq. (25) is sought in the form of a convergent series (Rushchitsky 2014)

$$u(x, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n u^n(x, t), \quad (26)$$

where, $u^0(x, t)$ is the solution of the linear Eq. (24).

- Step 5: When Eq. (26) is substituted into Eq. (25), the recurrent equation for any approximation is obtained as

$$\mathcal{L}[u^{n+1}] = \mathfrak{N}[u^n]. \quad (27)$$

For known (found) approximations $u^n(x, t)$, the solution of Eq. (23) is represented as the infinite sum of approximations

$$u(x, t) = u(x, t, \epsilon = 1) = u^0(x, t) + u^1(x, t) + u^2(x, t) + \dots \tag{28}$$

By inserting the values of the components into Eq. (28), we obtain the series solution. For concrete problems, it is not easy to find a closed-form solution, therefore, a truncated number of solution terms of this series is used for numerical purposes.

3.1 Quartically nonlinear elastic wave solution

It is convenient to rewrite Eq. (22) as

$$u_{tt} - N_0 u_{xx} = N_1 u_x u_{xx} + N_2 u_x^2 u_{xx} + N_3 u_x^3 u_{xx}. \tag{29}$$

The closed problem related to Eq. (29) is written as

$$u_{tt} - N_0 u_{xx} = 0, \tag{30}$$

along with the known solution as

$$u^0(x, t) = u_0 \cos(kx - \omega t). \tag{31}$$

The parameter ϵ is introduced such that for $\epsilon = 0$, it coincides with Eq. (30) and for $\epsilon = 1$, it coincides with Eq. (29) i.e.,

$$u_{tt} - N_0 u_{xx} = \epsilon(N_1 u_x u_{xx} + N_2 u_x^2 u_{xx} + N_3 u_x^3 u_{xx}). \tag{32}$$

According to the method, the solution $u(x, t, \epsilon)$ of Eq. (32) is sought in the form of a convergent series

$$u(x, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n u^n(x, t), \tag{33}$$

When Eq. (33) is substituted into Eq. (32), the recurrent equation for any approximation is obtained as

$$u_{tt}^{(n)} - N_0 u_{xx}^{(n)} = \left(N_1 u_x^{(n-1)} + N_2 (u_x^{(n-1)})^2 + N_3 (u_x^{(n-1)})^3 \right) u_{xx}^{(n-1)} \quad n = 1, 2, 3, \dots \tag{34}$$

For $n = 1$, we obtain the first approximation from Eq. (34) as

$$u^{(1)}(x, t) = \left(\frac{N_1 u_0^2 k^2}{8N_0} \right) x \cos 2(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{8N_0} \right) x \sin(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{24N_0} \right) x \sin 3(kx - \omega t) + \left(\frac{7N_3 u_0^4 k^4}{32N_0} \right) x \cos 2(kx - \omega t) + \left(\frac{N_3 u_0^4 k^4}{16N_0} \right) x \cos 4(kx - \omega t) \dots \tag{35}$$

The successive approximate solution to the quartically nonlinear elastic wave Eq. (22) is as follows

$$u(x, t) = u_0 \cos(kx - \omega t) + \left(\frac{N_1 u_0^2 k^2}{8N_0} \right) x \cos 2(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{8N_0} \right) x \sin(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{24N_0} \right) x \sin 3(kx - \omega t) + \left(\frac{7N_3 u_0^4 k^4}{32N_0} \right) x \cos 2(kx - \omega t) + \left(\frac{N_3 u_0^4 k^4}{16N_0} \right) x \cos 4(kx - \omega t) \dots \tag{36}$$

We plot the surface profile for the solution of quartically nonlinear elastic wave model given in Eq. (36) as shown in Fig. 1 as follows

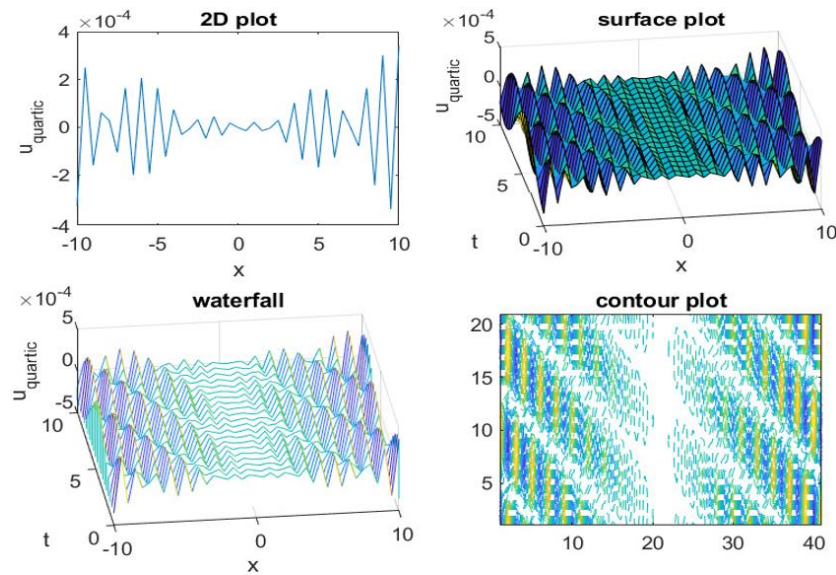


Fig. 1 Quartically nonlinear elastic wave profile assuming $t = 0.5$ and other typical parametric values

3.2 Cubically nonlinear elastic wave solution

In the same fashion, the successive approximate solution to the cubically nonlinear elastic wave Eq. (18) is as follows

$$u(x, t) = u_0 \cos(kx - \omega t) + \left(\frac{N_1 u_0^2 k^2}{8N_0}\right) x \cos 2(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{8N_0}\right) x \sin(kx - \omega t) + \left(\frac{N_2 u_0^3 k^3}{24N_0}\right) x \sin 3(kx - \omega t) \dots \quad (37)$$

3.3 Quadratically nonlinear elastic wave solution

Similarly, the successive approximate solution to the quadratically nonlinear elastic wave Eq. (14) is as follows

$$u(x, t) = u_0 \cos(kx - \omega t) + \left(\frac{N_1 u_0^2 k^2}{8N_0}\right) x \cos 2(kx - \omega t) \dots \quad (38)$$

4. Discussions

Quartically, cubically and quadratically nonlinear elastic wave solutions expressed in Eq. (36), Eq. (37) and Eq. (38) respectively have always been a topic of interest, primarily in terms of their theoretical validation of second harmonic generation. Initially, the wave slightly deviates from a linear harmonic wave. As the wave propagates over distance or time, the first harmonic gradually combines with the second harmonic, ultimately giving rise to a modulated wave. Gradually, the influence of the second harmonic becomes more pronounced and dominates the waveform.

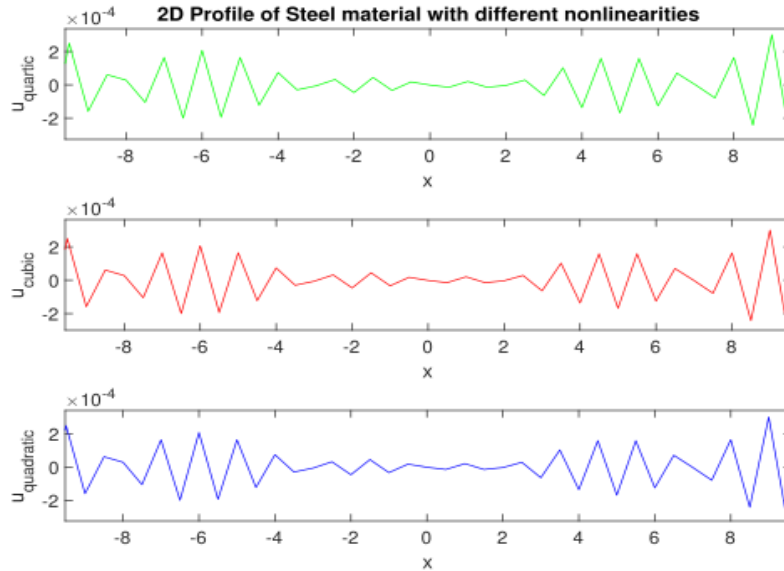


Fig. 2 2D profile of steel material with quartic, cubic and quadratic nonlinearities

The quartically nonlinear wave solution is different from the cubically nonlinear wave solution with the addition of an even function of double and fourth angles. The cubically nonlinear wave solution is different from the quadratically nonlinear wave solution with the addition of odd functions of single and double angle. These additional terms increase the amplitude successively. The variation in amplitude can be observed in the solution profiles. With the addition of terms, the wave profile moves upward graphically. The solution profiles also depend upon the initial amplitude, so they can be modified through parameters.

The following Fig. 2 illustrate 2D profile of the steel material having same parameters but different types of nonlinearity. The quartic, cubic and quadratic nonlinearities are represented by green, red and blue color respectively in the materials. To see the variation in profiles we zoom in the graphs via subplots.

The concept of second harmonic generation can be effectively illustrated. This confirms that a harmonic wave passing through a quartically, cubically and quadratically nonlinear medium indeed generates its own second harmonic. It is frequently mentioned that the amplitudes of the first and second harmonics are not directly proportional. When the amplitudes are suitable, it signifies a scenario where the second harmonic becomes dominant. Therefore, an analysis of the variability of the coefficient is of significant interest. In the comparison of nonlinear elastic waves, the stiffness of a material is determined by various factors, including the material's characteristics and its reaction to deformation. Steel and polystyrene are two different materials with distinct characteristics.

We also compare graphically the quartic-cubic and quartic-quadratic profiles as shown in the following Fig. 3.

The coefficient encloses some parameters dependent on the choice of material, such as the initial amplitude u_0 , wavenumber k , angular frequency ω , phase velocity N_0 , Murnaghan's five constants, density ρ inside N_1 , N_2 , and N_3 . For the vast majority of real materials, including metals, alloys, polymers, and others the standard numerals are given in the SI system. Steel is stiffer than polystyrene as shown in the profile in Fig. 4.

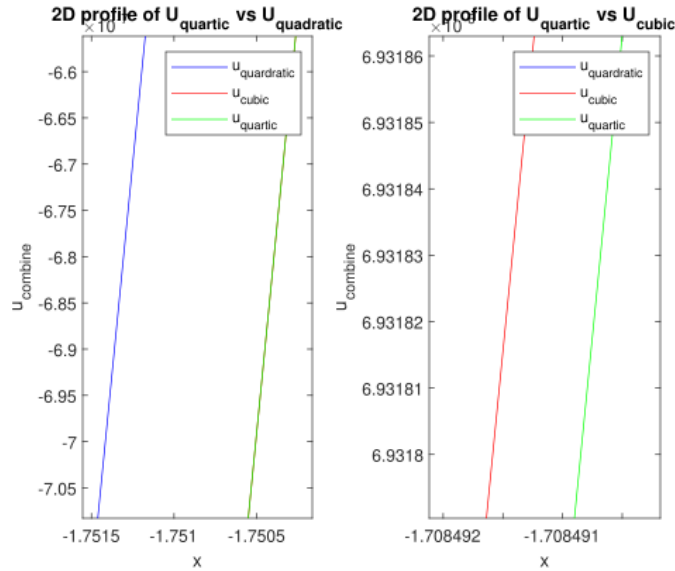


Fig. 3 Zoom in combined 2D plot of nonlinear elastic wave profiles for comparison of nonlinearities

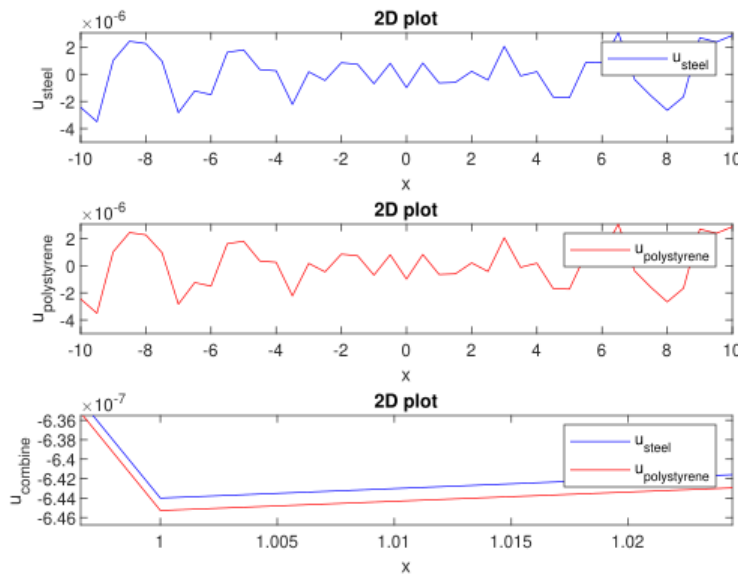


Fig. 4 Quartically nonlinear elastic wave profile comparison of steel and polystyrene parametric values

In this profile we compare quartic type of nonlinearity for both steel and polystyrene materials. Steel has a high modulus of elasticity, which indicates that it can withstand large stress and strain before getting permanent deformation. It possesses a relatively high Young’s modulus, which quantifies the stiffness of a material during linear elastic deformation. The stiffness of Steel makes it a good material for applications requiring strength and rigidity, such as structural engineering, material sciences and design engineering etc. Polystyrene, in contrast, is a form of plastic due to its lesser rigidity. Therefore, polystyrene materials are easily molded in contrast with steel materials.

5. Conclusions

In this study, we solve the quartically nonlinear elastic wave via successive approximation. A comprehensive comparison of the various types of nonlinearities is considered. Through a detailed analysis of various models, we have valuable insight into the behavior of nonlinear elastic waves in different contexts. It provides a concrete foundation for understanding the complicated interactions between nonlinearities, material, parameters and wave propagation. It also shed light on the complex nature of nonlinear elastic waves and their dependence on factors such as potential functions, Lamé's coefficients, and Murnaghan constants. By using the successive approximation method, we obtained approximate solutions and gain a deeper understanding of the wave nature. Further, this research highlighted the importance of nonlinear elastic wave theory in addressing real world phenomena where deformations or cracks are not smaller. Such phenomena have practical implications in fields such as Material sciences, Engineering, Physics, Seismology, and Geophysics, where the behavior of materials under various conditions must be accurately characterized.

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